Hyperbolic geometry, discrete groups, and manifolds

Alexis Marchand (IMPAN)

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Course webpage. Link to USOS

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Language. English

Short description. *Hyperbolic geometry* was created in the nineteenth century as an example of non-Euclidean geometry and has come to play a fundamental role in contemporary geometry and topology. The structure of discrete groups of hyperbolic isometries is very rich, and deeply connected with hyperbolic manifolds.

This course will start by introducing hyperbolic space, its different models, and its geometry, before moving on to discrete groups of isometries, in particular in the hyperbolic plane, but also in higher dimensions, with a view towards the geometry of hyperbolic surfaces and manifolds.

Basic requirements. Undergraduate-level group theory and topology. Exposure to fundamental groups and covering spaces is helpful but not required; these notions will be recalled in the course.

Full description. The following plan is tentative; each item is meant to correspond to roughly one week (i.e. one lecture and one exercise session).

I. The hyperbolic plane

- I.1) Historical background & definition of the Poincaré half-plane
- I.2) Background on Riemannian geometry
- I.3) The Poincaré disc, Klein disc, and hyperboloid models

II. $PSL_2(\mathbb{R})$ and hyperbolic geometry

- II.1) Projective geometry, Möbius transformations, & the cross-ratio
- II.2) Geodesics in the hyperbolic plane
- II.3) Action of $PSL_2(\mathbb{R})$ and classification of isometries

III. Fuchsian groups

- III.1) Discrete groups and their action
- III.2) Fundamental domains
- III.3) Poincaré's polygon theorem

IV. Surfaces and their geometric structures

IV.1) Background on fundamental groups and covering spaces

IV.2) Geometric structures on surfaces

IV.3) Teichmüller space and the Fenchel–Nielsen parameters

V. Higher-dimensional hyperbolic manifolds

V.1) The hyperbolic space in higher dimensions and its isometries

V.2) Elementary groups, Kazhdan–Margulis, & thick-thin decompositions

V.3) Outlook: hyperbolisation & Mostow rigidity

Bibliography.

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[5] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, third edition, Graduate Texts in Mathematics, 149, Springer, Cham, [2019] ©2019; MR4221225