Advanced Analysis

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Contents

Notation 0.0.1. *We shall write* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$.

1 Topological vector spaces

1.1 Generalities

Definition 1.1.1 (Topological vector space)**.** *A* topological vector space *is a Hausdorff space* E *that is also a* $\mathbb{K}$*-vector space such that the maps* $(x, y) \in E^2 \longmapsto x+y \in E$ *and* $(\lambda, x) \in \mathbb{K}\times E \longmapsto \lambda x \in E$ *are both continuous.*

Example 1.1.2. *Normed spaces are topological vector spaces.*

Remark 1.1.3. *Let* E *be a topological vector space.*

(i) *For* $x \in E$*, the* translation $\tau_x : y \in E \longmapsto x + y \in E$ *is a homeomorphism (with inverse* τ_{-x}*).*

(ii) *For* $\lambda \in \mathbb{K}^*$*, the* dilatation $h_\lambda : y \in E \longmapsto \lambda y \in E$ *is a homeomorphism (with inverse* $h_{\lambda^{-1}}$*).*

Corollary 1.1.4. *Let* E *be a topological vector space.*

(i) *The neighbourhoods of* $x \in E$ *are exactly the translations of those of* 0.

(ii) *For* $\lambda \in \mathbb{K}^*$*, a subset* $V \subseteq E$ *is a neighbourhood of* 0 iff λV *is a neighbourhood of* 0.

Proposition 1.1.5. *Let* E *be a topological vector space. Then any neighbourhood* V *of* 0 *in* E *is* absorbing*, i.e.*

$$\forall x \in E,\ \exists r > 0,\ \forall \lambda \in \mathbb{K},\ |\lambda| < r \Longrightarrow \lambda x \in V.$$

Proof. Choose $x \in E$ and consider $\psi_x : \lambda \in \mathbb{K} \longmapsto \lambda x \in E$. The map ψ_x is continuous, so $\psi_x^{-1}(V)$ is a neighbourhood of 0 in $\mathbb{K}$, i.e. there exists $r > 0$ s.t. $0 \in B_{\mathbb{K}}(r) \subseteq \psi_x^{-1}(V)$. In other words, $\psi_x(B_{\mathbb{K}}(r)) \subseteq V$, which was to be proved. □

Definition 1.1.6 (Bounded subsets)**.** *Let* E *be a topological vector space. A subset* $A \subseteq E$ *is said to be* bounded *if for every neighbourhood* V *of* 0 *in* E*, there exists* $r > 0$ *s.t.* $|\lambda| < r \Longrightarrow \lambda A \subseteq V$.

Corollary 1.1.7. *In topological vector spaces, singletons are bounded.*

Proposition 1.1.8. *Let* E, F *be two topological vector spaces and* $f : E \to F$ *be a linear map. Then* f *is continuous* iff f *is continuous at* 0.

Notation 1.1.9. *If* E, F *are two topological spaces, we shall write* $\mathcal{L}(E, F)$ *for the set of all continuous linear maps from* E *to* F*. This is a* $\mathbb{K}$*-vector space, which we would like to equip with the structure of a topological vector space.*

1.2 Completeness

Vocabulary 1.2.1. *A complete normed space is called a* Banach space*.*

Example 1.2.2.

(i) *If K is a compact topological space, then the space $\mathcal{C}(K)$ of all continuous maps from K to $\mathbb{K}$ is a Banach space, equipped with the supremum norm.*

(ii) *If X is a σ-finite measured space and $p \in [1, +\infty]$, then the space $L^p(X)$ is a Banach space.*

Theorem 1.2.3 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space.*

(i) *If $(\mathcal{O}_n)_{n\in\mathbb{N}}$ is a countable family of dense open subsets of X, then $\bigcup_{n\in\mathbb{N}} \mathcal{O}_n$ is dense in X.*

(ii) *If $(F_n)_{n\in\mathbb{N}}$ is a countable family of closed subsets of X with empty interior, then $\bigcap_{n\in\mathbb{N}} F_n$ has an empty interior.*

Definition 1.2.4 (Metric vector space)**.** *A* metric vector space *E is a topological vector space whose topology is defined by a translation-invariant metric, i.e. a metric d s.t. there exists a map $D : E \to \mathbb{R}_+$ s.t. $\forall (x, y) \in E$, $d(x, y) = D(x - y)$ (note that D is not necessarily homogeneous).*

Theorem 1.2.5. *Let E be a complete metric vector space, let F be a topological vector space. For any set $\Phi \subseteq \mathcal{L}(E, F)$, the following assertions are equivalent:*

(i) *For all $x \in E$, $\{\varphi(x),\ \varphi \in \Phi\}$ is bounded in F.*

(ii) *Φ is* equicontinuous*, i.e. for any open subset $W \subseteq F$, there exists an open subset $V \subseteq E$ s.t. $\forall \varphi \in \Phi$, $\varphi(V) \subseteq W$.*

(iii) *Φ is* equicontinuous at 0*, i.e. for any neighbourhood W of 0 in F, there exists a neighbourhood V of 0 in E s.t. $\forall \varphi \in \Phi$, $\varphi(V) \subseteq W$.*

Proof. (i) $\Leftarrow$ (ii) $\Leftrightarrow$ (iii) Clear. (i) $\Rightarrow$ (iii) Let W be a neighbourhood of 0 in F. As $(x, y) \mapsto x - y$ is continuous, there exists C neighbourhood of 0 in F s.t. $C - C = \{c - c',\ (c, c') \in C^2\} \subseteq W$. Likewise, there exists U neighbourhood of 0 in F s.t. $U + U \subseteq C$. Let us show that $\overline{U} \subseteq C$: for $x \in \overline{U}$, $x - U$ is a neighbourhood of x, so it meets U, i.e. there exists $y \in U \cap (x - U)$; therefore, there exists $z \in U$ s.t. $x = y + z \in U + U \subseteq C$. Hence, we get $\overline{U} - \overline{U} \subseteq W$. Now, define:

$$X = \bigcap_{\varphi\in\Phi} \varphi^{-1}\left(\overline{U}\right).$$

The set X is closed in E. By assumption, for all $x \in E$, there exists $n \in \mathbb{N}^*$ s.t. $\frac{1}{n}\{\varphi(x),\ \varphi \in \Phi\} \subseteq \overline{U}$, i.e. $x \in nX$. Therefore:

$$E = \bigcup_{n\in\mathbb{N}^*} nX.$$

By the Baire Category Theorem, there exists $n_0 \in \mathbb{N}^*$ s.t. n_0X has nonempty interior. But $X = \frac{1}{n_0}(n_0X)$, so X has a nonempty interior. Thus, there exists $x \in X$ and V neighbourhood of 0 in E s.t. $x + V \subseteq X$. In other words: $\forall \varphi \in \Phi$, $\varphi(x + V) \subseteq \overline{U}$, so $\forall \varphi \in \Phi$, $\varphi(V) \subseteq \varphi(V - V) = \varphi(x + V) - \varphi(x + V) \subseteq \overline{U} - \overline{U} \subseteq W$. □

Corollary 1.2.6 (Uniform Boundedness Principle / Banach-Steinhaus Theorem)**.** *Let E be a Banach space and let F be a normed space. For any set $\Phi \subseteq \mathcal{L}(E, F)$, the following assertions are equivalent:*

(i) *For all $x \in E$, $\{\varphi(x),\ \varphi \in \Phi\}$ is bounded in F.*

(ii) *Φ is equicontinuous.*

(iii) *Φ is equicontinuous at* 0*.*

(iv) $\{\|\varphi\|\,,\ \varphi \in \Phi\}$ *is bounded in* $\mathbb{R}$.

Remark 1.2.7. *There are two ways to apply the Banach-Steinhaus Theorem:*

(i) *If we have a sequence* $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{L}(E,F)^{\mathbb{N}}$ *and a* $\varphi \in \mathcal{L}(E,F)$ *s.t.* $\forall x \in E$, $\varphi_n(x) \to \varphi(x)$, *then the sequence* $(\|\varphi_n\|)_{n\in\mathbb{N}}$ *is bounded, which leads to:*

$$\forall x \in E,\ \|\varphi(x)\| \leq \left(\liminf_{n\to+\infty} \|\varphi_n\|\right) \|x\|\,.$$

Hence, φ *is linear continuous.*

(ii) *If we have a sequence* $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{L}(E,F)^{\mathbb{N}}$ *s.t.* $\|\varphi_n\| \to +\infty$, *then there exists* $x \in E$ *s.t.* $(\varphi_n(x))_{n\in\mathbb{N}}$ *is unbounded. This is actually true for every* x *in a dense* G_δ.

Theorem 1.2.8 (Open Mapping Theorem / Banach–Schauder Theorem)**.** *Let* E, F *be two complete metric vector spaces and* $T : E \to F$ *be a linear continuous map.*

(i) *If* T *is onto, then for any* V *neighbourhood of* 0 *in* E, $T(V)$ *is a neighbourhood of* 0 *in* F.

(ii) *If* T *is bijective, then it is a homeomorphism.*

Proof. It is enough to prove (i). Suppose that T is onto and choose $r > 0$. We need only prove that $\exists s > 0$, $TB_E(r) \supseteq B_F(s)$, where $B_E(r) = \{x \in E,\ D(x) < r\}$. *First step.* Since $B_E(r)$ is absorbing (by Proposition 1.1.5), we have $E = \bigcup_{n\in\mathbb{N}^*} nB_E(r)$. And T is onto, so:

$$F = \bigcup_{n\in\mathbb{N}^*} T\left(nB_E(r)\right) = \bigcup_{n\in\mathbb{N}^*} \overline{T\left(nB_E(r)\right)} = \bigcup_{n\in\mathbb{N}^*} n\overline{TB_E(r)}.$$

By the Baire Category Theorem, there exists $n_0 \in \mathbb{N}^*$ s.t. $n_0\overline{TB_E(r)}$ has nonempty interior. Therefore, $\overline{TB_E(r)}$ has nonempty interior. *Second step.* Let $a \in \overset{\circ}{\overline{TB_E\left(\frac{r}{2}\right)}}$ and let U be a neighbourhood of a in $\overline{TB_E\left(\frac{r}{2}\right)}$. Then $V = U - U$ is a neighbourhood of 0 in F, and $V \subseteq \overline{TB_E(r)}$. We have proved that for all $r > 0$, there exists $\delta(r) > 0$ s.t. $B_F\left(\delta(r)\right) \subseteq \overline{TB_E(r)}$, and we may assume that $\delta(r) \leq r$. *Third step.* Let $r > 0$ and $y \in B_F\left(\delta\left(\frac{r}{2}\right)\right)$. Our aim is to find a $x \in B_E(r)$ s.t. $Tx = y$. We construct an approximate solution of the equation. As $y \in \overline{TB_E\left(\frac{r}{2}\right)}$, there exists $x_1 \in B_E\left(\frac{r}{2}\right)$ s.t. $y - Tx_1 \in B_F\left(\delta\left(\frac{r}{4}\right)\right) \subseteq \overline{TB_E\left(\frac{r}{4}\right)}$. Proceeding by induction, we construct a sequence $(x_n)_{n\in\mathbb{N}^*} \in E^{\mathbb{N}^*}$ s.t. $x_n \in B_E\left(2^{-n}r\right)$ and $y - T\left(x_1 + \cdots + x_n\right) \in B_F\left(\delta\left(2^{-(n+1)}r\right)\right)$ for all $n \in \mathbb{N}^*$. Write $z_n = x_1 + \cdots + x_n$ for $n \in \mathbb{N}^*$. Then the sequence $(z_n)_{n\in\mathbb{N}^*}$ is Cauchy so it converges to $z \in E$. We easily check that $y = Tz$ and $z \in B_E(r)$. This proves that $TB_E(r) \supseteq B_F\left(\delta\left(\frac{r}{2}\right)\right)$. □

Theorem 1.2.9 (Closed Graph Theorem)**.** *Let* E, F *be two complete metric vector spaces and* $T : E \to F$ *be a linear map. Then* T *is continuous* iff *its graph* $\mathcal{G}(T) = \{(x, Tx)\,,\ x \in E\}$ *is closed in* $E \times F$.

Proof. ($\Rightarrow$) Clear. ($\Leftarrow$) By assumption, $\mathcal{G}(T)$ is closed so it is a complete metric vector space. Let $\pi : E \times F \to E$ be the first projection. Then the restriction $\pi_{|\mathcal{G}(T)}$ is linear continuous and bijective. By Theorem 1.2.8, it is a homeomorphism, i.e. the inverse map $x \mapsto (x, Tx)$ is continuous. In particular, T is continuous. □

2 Convexity

Definition 2.0.1 (Dual space)**.** *If* E *is a topological vector space, its* dual space *is* $E^* = \mathcal{L}(E, \mathbb{K})$.

Remark 2.0.2.

(i) *If H is a Hilbert space, then H^* is isometric to H.*

(ii) *If X is a measured space, $p \in [1, +\infty[$ and $q \in]1, +\infty]$ is the conjugate exponent of p (i.e. $1 = \frac{1}{p} + \frac{1}{q}$), then $L^p(X)^*$ is isometric to $L^q(X)$.*

(iii) *However, in general, E^* may be very small.*

2.1 Locally convex topological vector spaces

Definition 2.1.1 (Local convexity)**.** *A topological vector space E is said to be* locally convex *if it admits a basis of convex neighbourhoods of* 0.

Example 2.1.2. *Normed spaces are locally convex.*

Proposition 2.1.3. *Let E be a topological vector space.*

(i) *Every neighbourhood of* 0 *contains a* balanced *neighbourhood, i.e. a neighbourhood V s.t. $\forall x \in V$, $\forall \lambda \in \mathbb{K}$, $|\lambda| \leq 1 \Longrightarrow \lambda x \in V$.*

(ii) *If E is locally convex, then every neighbourhood of* 0 *contains a balanced convex neighbourhood of* 0.

Proof. (i) Note that $\phi : (\lambda, x) \in \mathbb{K} \times E \longmapsto \lambda x \in E$ is continuous and $\phi(0,0) = 0 \in W$, so $\phi^{-1}(W)$ is a neighbourhood of $(0,0)$. Hence, there exists a neighbourhood U_1 of 0 in $\mathbb{K}$ and a neighbourhood V_1 of 0 in E s.t. $\phi(U_1 \times V_1) \subseteq W$, i.e. $U_1 V_1 \subseteq W$. We may assume that U_1 is balanced in $\mathbb{K}$; thus, $V = U_1 V_1$ is balanced in E. (ii) Let W be a neighbourhood of 0_E. As E is locally convex, we may assume that W is convex. Using point (i), W contains a balanced neighbourhood V_1 of 0. Now, we easily check that the convex hull V of V_1 is a balanced convex neighbourhood of 0, contained in W. □

Definition 2.1.4 (Semi-norm)**.** *If E is a vector space, a* semi-norm *on E is a map $p : E \to \mathbb{R}_+$ that is homogeneous and satisfies the triangle inequality, but not necessarily the separation property of norms.*

Remark 2.1.5. *If p is a semi-norm on a vector space E, then balls $B_p(r) = \{x \in E,\ p(x) < r\}$ are balanced and convex.*

Definition 2.1.6 (Topology defined by a separating family of semi-norms)**.** *Consider a vector space E equipped with a family of semi-norms $(p_\alpha)_{\alpha \in A}$ that is* separating, *i.e. s.t.*

$$\forall x \in E \backslash \{0\},\ \exists \alpha \in A,\ p_\alpha(x) \neq 0.$$

Then the family $(p_\alpha)_{\alpha \in A}$ defines a translation-invariant topology on E: this is the coarsest topology s.t. p_α is continuous (equivalently, continuous at 0) for every $\alpha \in A$. A basis of neighbourhoods of 0 for this topology is the collection of all sets of the form $\bigcap_{\alpha \in J} B_{p_\alpha}\left(\frac{1}{n}\right)$, where J is a finite subset of A and $n \in \mathbb{N}^*$.

Proposition 2.1.7 (Minkowski's Gauge)**.** *Let W be a balanced convex subset of a vector space E. Assume that W is absorbing and define:*

$$j_W : x \in E \longmapsto \inf \left\{ t > 0,\ \frac{1}{t} x \in W \right\}.$$

Then j_W is a semi-norm. In addition, $B = \{x \in E,\ j_W(x) < 1\}$ and $B' = \{x \in E,\ j_W(x) \leq 1\}$ satisfy:

$$B \subseteq W \subseteq B'.$$

Proof. Note that W is absorbing, so the set $\left\{t > 0,\ \frac{1}{t}x \in W\right\}$ is nonempty for all $x \in E$. Therefore, $j_W : E \to \mathbb{R}_+$ is well-defined. It is clear from the definition that j_W is positively homogeneous (i.e. $\forall \lambda \in \mathbb{R}_+,\ \forall x \in E,\ j_W(\lambda x) = \lambda j_W(x)$). Moreover, if $\mu \in \mathbb{K}$ is s.t. $|\mu| = 1$, then $\mu W = W$ as W is balanced, so $j_W(\mu x) = j_W(x)$. Therefore, j_W is homogeneous. For the triangle inequality, choose $x, y \in E$. Let $a > j_W(x)$ and $b > j_W(y)$. By convexity of W, $\mathbb{R}_+^* x \cap W$ is convex, so it is of the form Ix, where I is an interval of $\mathbb{R}_+^*$. Actually, $\mathbb{R}_+^* x \cap W = \left]0, \frac{1}{j_W(x)}\right) x$. Therefore, $\frac{1}{a}x \in W$; likewise, $\frac{1}{b}y \in W$. By convexity:

$$\frac{1}{a+b}(x+y) = \frac{a}{a+b} \cdot \frac{1}{a}x + \left(1 - \frac{a}{a+b}\right) \cdot \frac{1}{b}y \in W.$$

Therefore, $j_W(x+y) \leq a+b$. Taking infimums over a and b, we obtain $j_W(x+y) \leq j_W(x) + j_W(y)$. The inclusions $B \subseteq W \subseteq B'$ are easy to prove. □

Theorem 2.1.8. *If E is a locally convex topological vector space, then there exists a separating family of semi-norms inducing the topology of E.*

Proof. Let $\mathcal{B}$ be the set of balanced convex neighbourhoods of 0. According to Proposition 2.1.3, $\mathcal{B}$ is a basis of neighbourhoods of 0. For all $W \in \mathcal{B}$, Minkowski's Gauge j_W is a semi-norm (c.f. Proposition 2.1.7). Hence, we have a family $(j_W)_{W \in \mathcal{B}}$ of semi-norms; it is separating because of the fact that $\forall x \in E \backslash \{0\},\ \exists W \in \mathcal{B},\ x \notin W$ (because E is Hausdorff). Hence, $(j_W)_{W \in \mathcal{B}}$ defines a locally convex vector space topology on E; let $\mathcal{B}'$ be the set of balanced convexed neighbourhoods of 0 for this topology. Using Proposition 2.1.3 again, $\mathcal{B}'$ is a basis of neighbourhoods of 0 for the new topology on E. Therefore, it is enough to prove that $\mathcal{B} = \mathcal{B}'$. If $W \in \mathcal{B}$, then $W \supseteq B_{j_W}(1)$, so W is a neighbourhood of 0 in the new topology, and W is still balanced and convex; therefore $W \in \mathcal{B}'$. Conversely, if $W' \in \mathcal{B}'$, then W' contains a finite intersection of sets of the form $B_{j_W}(\varepsilon)$, with $\varepsilon > 0$ and $W \in \mathcal{B}$. Therefore, it is enough to prove that $B_{j_W}(\varepsilon) \in \mathcal{B}$ for all $\varepsilon > 0$ and $W \in \mathcal{B}$. We may actually assume that $\varepsilon = 1$. But according to the last part of Proposition 2.1.7:

$$B_{j_W}(1) \supseteq \left\{x \in E,\ j_W(x) \leq \frac{1}{2}\right\} = \frac{1}{2}\left\{x \in E,\ j_W(x) \leq 1\right\} \supseteq \frac{1}{2}W.$$

And $B_{j_W}(1)$ is balanced and convex, so $B_{j_W}(1) \in \mathcal{B}$. Hence $\mathcal{B} = \mathcal{B}'$. □

Proposition 2.1.9. *Let E and F be locally convex topological vector spaces, equipped with separating families of semi-norms $(p_\alpha)_{\alpha \in A}$ and $(q_\beta)_{\beta \in B}$ respectively.*

(i) *A sequence $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ converges towards $x \in E$* iff

$$\forall \alpha \in A,\ p_\alpha(x_n - x) \to 0.$$

(ii) *Let $T : E \to F$ be a linear map. Then T is continuous* iff *for every $\beta \in B$, there exists $C_\beta \in \mathbb{R}_+$ and a finite subset $J_\beta \subseteq A$ s.t.*

$$q_\beta \circ T \leq C_\beta \max_{j \in J_\beta} p_{\alpha_j}.$$

2.2 Fréchet spaces

Proposition 2.2.1. *If E is a locally convex topological vector space whose topology is defined by a countable family $(p_n)_{n \in \mathbb{N}}$ of semi-norms, then E is metrisable, with the distance d defined by:*

$$d(x,y) = \sum_{j \in \mathbb{N}} 2^{-j} \frac{p_n(y-x)}{1 + p_n(y-x)}.$$

Definition 2.2.2 (Fréchet space)**.** *A* Fréchet space *is a locally convex topological vector space E s.t.*

(i) *The topology of E is defined by a countable family of semi-norms (hence E is a metric vector space).*

(ii) *E is complete.*

Corollary 2.2.3. *Fréchet spaces satisfy the Uniform Boundedness Principle (Corollary 1.2.6), the Open Mapping Theorem (Theorem 1.2.8) and the Closed Graph Theorem (Theorem 1.2.9).*

Example 2.2.4.

(i) *If Ω is an open subset of $\mathbb{R}^n$, then the space $\mathcal{C}^0(\Omega)$ of continuous functions $\Omega \to \mathbb{K}$, equipped with the topology of uniform convergence on every compact set, is a Fréchet space.*

(ii) *If Ω is an open subset of $\mathbb{R}^n$, then the space $\mathcal{C}^\infty(\Omega)$ of smooth functions $\Omega \to \mathbb{K}$, equipped with the topology of uniform convergence of every partial derivative on every compact set, is a Fréchet space.*

(iii) *If Ω is an open subset of $\mathbb{C}$, then the space $\mathcal{H}(\Omega)$ of holomorphic functions $\Omega \to \mathbb{C}$, equipped with the topology of uniform convergence on every compact set, is a Fréchet space.*

(iv) *If K is a compact subset of $\mathbb{R}^n$, then the space $\mathcal{C}^\infty(K)$ consisting of restrictions to K of functions of $\mathcal{C}^\infty(\mathbb{R}^n)$ is a Fréchet space equipped with the family $(p_m)_{m\in\mathbb{N}}$ of semi-norms defined by:*

$$p_m(g) = \inf\left\{\sup\left\{\left\|\frac{\partial^m f}{\partial^{m_1}x_1\cdots\partial^{m_n}x_n}\right\|_{\mathbb{R}^n}^{\infty},\ m_1+\cdots+m_n=m\right\},\ f\in\mathcal{C}_c^\infty(\mathbb{R}^n),\ f_{|K}=g\right\}.$$

2.3 Hahn-Banach Theorem

Definition 2.3.1 (Inductive set)**.** *Let S be an ordered set. A* chain *of S is a subset $S' \subseteq S$ that is totally ordered. The set S is said to be* inductive *if every chain S' admits an upper-bound in S.*

Theorem 2.3.2 (Zorn's Lemma)**.** *If S is a nonempty inductive set, then S has a maximal element.*

Theorem 2.3.3 (Hahn-Banach Theorem)**.** *Let E be a real vector space, equipped with a function $p : E \to \mathbb{R}$ that is subadditive (i.e. $\forall (x,y)\in E^2$, $p(x+y)\le p(x)+p(y)$) and positively homogeneous (i.e. $\forall\lambda\in\mathbb{R}_+$, $\forall x\in E$, $p(\lambda x)=\lambda p(x)$). Let F be a subspace of E and $f : F\to\mathbb{R}$ be a linear form. Assume that $f\le p$ over F. Then there exists a linear form $\varphi : E\to\mathbb{R}$ s.t. $\varphi_{|F}=f$ and $\varphi\le p$ over E.*

Proof. Consider the set S of pairs (G,g), where G is a subspace of E containing F, and $g : G\to\mathbb{R}$ is a linear form s.t. $g_{|F}=f$ and $g\le p$ over G. S is ordered by $(G,g)\le(H,h)$ iff $G\subseteq H$ and $g=h_{|G}$. We affirm that S is inductive; according to Zorn's Lemma, it has a maximal element (M,φ). It remains to prove that $M=E$. Suppose for contradiction that $M\subsetneq E$ and choose $x\in E\backslash M$. Put $M'=M\oplus\mathbb{R}x$ and construct a linear form $\varphi' : M'\to\mathbb{R}$ defined by $\varphi'_{|M}=\varphi$ and $\varphi(x)=\lambda$, where λ is to be chosen. We want to have $\varphi'\le p$, i.e.

$$\forall (y,t)\in M\times\mathbb{R},\ \varphi'(y+tx)=\varphi(y)+t\lambda\le p(y+tx).$$

Because of positive homogeneity, we may restrict to $t\in\{\pm1\}$. This leads to the following inequalities:

$$\sup_{y\in M}\left(\varphi(y)-p(y-x)\right)\le\lambda\le\inf_{z\in M}\left(p(z+x)-\varphi(z)\right).$$

The choice of such a λ is possible because $\sup_{y\in M}(\varphi(y)-p(y-x))\le\inf_{z\in M}(p(z+x)-\varphi(z))$, since $\forall(y,z)\in M^2$, $\varphi(y)-p(y-x)\le p(z+x)-\varphi(z)$. Hence, we have constructed $(M',\varphi')\in S$, with $(M,\varphi)<(M',\varphi')$. This contradicts the maximality of (M,φ); therefore, $M=E$. □

Corollary 2.3.4. *The dual space E^* of a real locally convex topological vector space E separates the points of E : if $x,y\in E$ with $x\ne y$, then there exists $f\in E^*$ s.t. $f(x)\ne f(y)$.*

Corollary 2.3.5. *Let E be a real normed space. If $x\in E$, there exists $\varphi\in E^*$ s.t. $\varphi(x)=\|x\|$ and $\|\varphi\|=1$.*

2.4 Geometrical form of the Hahn-Banach Theorem

Lemma 2.4.1. *Let E be a real locally convex topological vector space and C be a nonempty convex open subset of E and $x \in E \backslash C$. Then there exists $\varphi \in E^* \backslash \{0\}$ s.t. $\sup_C \varphi \leq \varphi(x)$. In other words, C is contained in a half-space delimited by the closed affine hyperplane $x + \operatorname{Ker} \varphi$.*

Proof. As the lemma is translation-invariant, we may assume that $0 \in C$. We consider $j : y \in E \longmapsto \inf \left\{t > 0,\ \frac{1}{t} y \in C\right\}$. As C is absorbing (because it is a neighbourhood of 0), $j(y) < +\infty$ for all $y \in E$. Moreover, C is convex, so j is convex. Finally, j is positively homogeneous (but j might not be a semi-norm because C might not be balanced). Consider $F = \mathbb{R}x$ and define a linear form $f : F \to \mathbb{R}$ by $f(x) = j(x)$. We have $f \leq j$ on F. By the Hahn-Banach Theorem, there exists a linear form $\varphi : E \to \mathbb{R}$ s.t. $\varphi(x) = j(x)$ and $\varphi \leq j$ on E. In particular, for $y \in C$, $\varphi(y) \leq j(y) \leq 1$, so $\varphi^{-1}(]-2, +2[) \supseteq C$; by linearity, φ is continuous. Lastly, $\sup_C \varphi \leq 1 \leq \varphi(x)$. □

Theorem 2.4.2. *Let E be a real locally convex topological vector space. Consider two nonempty convex disjoint subsets A, B of E.*

(i) *If A is open and B is closed, then $\exists \varphi \in E^* \backslash \{0\}$, $\sup_A \varphi \leq \inf_B \varphi$.*

(ii) *If A is compact and B is closed, then $\exists \varphi \in E^* \backslash \{0\}$, $\sup_A \varphi < \inf_B \varphi$.*

Proof. (i) Define $C = A - B = \{a - b,\ (a, b) \in A \times B\}$. The set C is convex and open, and does not contain 0. According to Lemma 2.4.1, there exists $\varphi \in E^* \backslash \{0\}$ s.t. $\sup_C \varphi \leq \varphi(0) = 0$. As $\sup_C \varphi = \sup_A \varphi - \inf_B \varphi$, this gives the desired result. (ii) For $x \in A$, $E \backslash B$ is an open neighbourhood of x, so there exists a convex open neighbourhood V_x of 0 s.t. $x + V_x + V_x \subseteq E \backslash B$. Now $A \subseteq \bigcup_{x \in A} (x + V_x)$. Since A is compact, there are points $x_1, \ldots, x_N \in A$ s.t. $A \subseteq \bigcup_{j=1}^N \left(x_j + V_{x_j}\right)$. Define $V = \bigcap_{j=1}^N V_{x_j}$. V is an open convex neighbourhood of 0, and we have $A + V \subseteq E \backslash B$. Hence, $(A + V)$ is open, convex and nonempty, and $(A + V) \cap B = \varnothing$. By (i), there exists $\varphi \in E^* \backslash \{0\}$ s.t. $\sup_{A+V} \varphi \leq \inf_B \varphi$. But $\sup_{A+V} \varphi = \sup_A \varphi + \sup_V \varphi$. Since φ is linear and V is absorbing, $\sup_V \varphi > 0$, i.e. $\sup_A \varphi < \inf_B \varphi$. □

Corollary 2.4.3. *Let E be a real locally convex topological vector space, and let $F \subseteq E$ be a subspace. Then:*

(i) $\overline{F} = \left\{x \in E,\ \forall \varphi \in E^*,\ \left(\varphi_{|F} = 0 \Longrightarrow \varphi(x) = 0\right)\right\}$.

(ii) *F is dense in E* iff $\forall \varphi \in E^*,\ \left(\varphi_{|F} = 0 \Longrightarrow \varphi = 0\right)$.

Proof. Note that (ii) is a direct consequence of (i). For (i), apply Theorem 2.4.2 to the closed set $\overline{F}$ and the compact set $\{x\}$, for $x \in E \backslash \overline{F}$. □

2.5 Krein-Milman Theorem

Definition 2.5.1 (Extremal points)**.** *Let C be a nonempty convex subset of a vector space E. A point $x \in C$ is said to be an* extremal point *of C if:*

$$\forall (y, z) \in C^2,\ \forall \lambda \in]0, 1[,\ (x = (1 - \lambda) y + \lambda z) \Longrightarrow y = z = x.$$

The set of extremal points of C is denoted by $\operatorname{Extr}(C)$*.*

Notation 2.5.2. *If $S \subseteq E$ is a subset of a vector space E, then the convex hull of S is denoted by* $\operatorname{Conv}(S)$*.*

Theorem 2.5.3 (Krein-Milman Theorem)**.** *Let K be a compact convex subset of a real locally convex topological vector space E. Then:*

$$K = \overline{\operatorname{Conv}(\operatorname{Extr}(K))}.$$

In particular, $K \neq \varnothing \Longrightarrow \operatorname{Extr}(K) \neq \varnothing$.

Proof. We assume that $K \neq \varnothing$ (otherwise the statement is trivial). We say that a subset $S \subseteq K$ is extremal if:

$$\forall (x,y) \in K^2,\ \forall \lambda \in]0,1[,\ ((1-\lambda)x + \lambda y \in S) \Longrightarrow \{x,y\} \subseteq S.$$

In particular, note that $\{x\}$ is extremal iff $x \in \operatorname{Extr}(K)$. *First step*: $\operatorname{Extr}(K) \neq \varnothing$. Consider the set X of all nonempty closed convex extremal subsets of K, ordered by reverse inclusion. Since $K \in X$, $X \neq \varnothing$. If C is a chain in X, then $\bigcap_{S \in C} S \in X$, so X is inductive. By Zorn's Lemma, X has a maximal element S. Let us prove that S is a singleton. Suppose for contradiction that there exist $x \neq y$ in S. According to Corollary 2.3.4, there exists $f \in E^*$ s.t. $f(x) \neq f(y)$. Let $m = \sup_S f$; m is attained because S is compact and f is continuous. Hence, define $S' = S \cap f^{-1}(\{m\})$; this is a nonempty compact convex subset of K, and $S' \subsetneq S$ because f is not constant on S. It remains to prove that S' is extremal in K: let $(x,y) \in K^2$ and $\lambda \in]0,1[$ s.t. $(1-\lambda)\,x + \lambda y \in S'$. As $(1-\lambda)\,x + \lambda y \in S$, we have $\{x,y\} \subseteq S$; therefore:

$$m = f\left((1-\lambda)x + \lambda y\right) = (1-\lambda)\underbrace{f(x)}_{\leq m} + \lambda \underbrace{f(y)}_{\leq m} \leq m.$$

Hence, equality holds throughout and $f(x) = f(y) = m$, so $\{x,y\} \subseteq S'$. This proves that S' is extremal, i.e. $S' \in X$. Since $S' \subsetneq S$, this contradicts the maximality of S (for reverse inclusion), so S was a singleton, and $\operatorname{Extr}(K) \neq \varnothing$. *Second step*: $K = \overline{\operatorname{Conv}(\operatorname{Extr}(K))}$. The inclusion ($\supseteq$) is clear, so it is enough to prove ($\subseteq$). Define $K' = \overline{\operatorname{Conv}(\operatorname{Extr}(K))}$. We have $\varnothing \subsetneq K' \subseteq K$, and K' is compact and convex. Suppose for contradiction that $K' \subsetneq K$, i.e. there exists $x \in K \backslash K'$. By Theorem 2.4.2, there exists $\varphi \in E^*$ s.t.

$$\sup_{K'} \varphi < \varphi(x).$$

Define $M = \sup_K \varphi$. As above, define $K_1 = K \cap \varphi^{-1}(\{M\})$; this is a nonempty compact convex extremal subset of K. By the first step, K_1 has an extremal point $z \in \operatorname{Extr}(K_1) \subseteq \operatorname{Extr}(K) \subseteq K'$. But $\varphi(z) = M \geq \varphi(x) > \sup_{K'} \varphi$, so $z \notin K'$. This is a contradiction, hence $K = K'$. □

3 Duality

3.1 Weak-$*$ topology and weak topology

Remark 3.1.1. *If E is a normed space, E^* may be equipped with the dual norm. It makes E^* a Banach space (even if E is not Banach).*

Definition 3.1.2 (Weak-$*$ topology)**.** *Let E be a locally convex topological vector space. The* weak-$*$ topology *of E^* is the vector space topology defined by the separating family $(q_x)_{x \in E}$ of semi-norms, where:*

$$\forall x \in E,\ \forall f \in E^*,\ q_x(f) = |f(x)|\,.$$

The weak-$*$ topology is denoted by $\sigma(E^*, E)$, it is the topology of simple convergence *and makes E^* a locally convex topological vector space.*

Definition 3.1.3 (Weak topology)**.** *Let E be a locally convex topological vector space and write $\mathcal{T}$ for the topology of E. The* weak topology *of E is the vector space topology defined by the separating family $(p_f)_{f \in E^*}$ of semi-norms, where:*

$$\forall f \in E^*,\ \forall x \in E,\ p_f(x) = |f(x)|\,.$$

The weak topology is denoted by $\sigma(E, E^*)$, it is a new topology making E a locally convex topological vector space. It is the coarsest topology on E s.t. every $f \in E^*$ is continuous; therefore, $\sigma(E, E^*)$ is coarser than $\mathcal{T}$. We use the word "weak" *to refer to the topology $\sigma(E, E^*)$ and* "strong" *to refer to $\mathcal{T}$.*

Notation 3.1.4. *Let E be a locally convex topological vector space.*

(i) *If a sequence $(f_n)_{n\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$ converges to $f \in E^*$ for the topology $\sigma(E^*, E)$, we write $f_n \overset{*}{\rightharpoonup} f$; this is equivalent to $\forall x \in E,\ f_n(x) \to f(x)$.*

(ii) *If a sequence $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ converges to $x \in E$ for the topology $\sigma(E, E^*)$, we write $x_n \rightharpoonup x$; this is equivalent to $\forall f \in E^*,\ f(x_n) \to f(x)$.*

Proposition 3.1.5. *Let E be a locally convex topological vector space. Then:*

$$(E, \sigma(E, E^*))^* = E^*.$$

In other words, a linear form $f : E \to \mathbb{R}$ is weakly continuous iff *it is strongly continuous.*

Proposition 3.1.6. *Let E be a real locally convex topological vector space. A convex subset $C \subseteq E$ is weakly closed* iff *it is strongly closed.*

Proof. ($\Rightarrow$) Since the weak topology is coarser than the strong topology, any weakly closed (not necessarily convex) subset is also strongly closed. ($\Leftarrow$) Let C be a strongly closed convex subset of E. Let us show that C is weakly closed, i.e. $E\backslash C$ is weakly open. Let $x \in E\backslash C$. The sets $\{x\}$ and C are nonempty disjoint convex subsets of E, $\{x\}$ is strongly compact and C is strongly closed. According to Theorem 2.4.2, there exists a linear form $\varphi \in E^*$ s.t.

$$\varphi(x) < \inf_C \varphi.$$

Now choose α s.t. $\varphi(x) < \alpha < \inf_C \varphi$. The set $H = \{y \in E,\ \varphi(y) < \alpha\}$ is open for both topologies, and $x \in H \subseteq E\backslash C$, so $E\backslash C$ is a weak neighbourhood of x. Hence, $E\backslash C$ is weakly open. □

Proposition 3.1.7. *Let E be a locally convex topological vector space. Then any weak neighbourhood of 0 in E contains a linear subspace of E of finite codimension. Likewise, any weak-$*$ neighbourhood of 0 in E^* contains a linear subspace of E^* of finite codimension.*

3.2 Bidual

Proposition 3.2.1. *Let E be a locally convex topological vector space. Then the map:*

$$\delta : \left| \begin{array}{l} E \longrightarrow (E^*, \sigma(E^*, E))^* \\ x \longmapsto \delta_x : \left| \begin{array}{l} E^* \longrightarrow \mathbb{K} \\ f \longmapsto f(x) \end{array} \right. \end{array} \right.$$

is a linear isomorphism.

Proof. δ is a well-defined, injective, linear map. Let us prove the surjectivity of δ. Let $\varphi \in (E^*, \sigma(E^*, E))^*$. Since φ is weakly-$*$ continuous, there exist $x_1, \ldots, x_N \in E$ and $C \in \mathbb{R}_+$ s.t.

$$\forall f \in E,\ |\varphi(f)| \leq C \max_{1\leq j\leq N} q_{x_j}(f).$$

In particular $\bigcap_{j=1}^N \operatorname{Ker} \delta_{x_j} \subseteq \operatorname{Ker} \varphi$, which implies that $\varphi \in \operatorname{Vect}(\delta_{x_1}, \ldots, \delta_{x_N}) \subseteq \operatorname{Im} \delta$. □

Remark 3.2.2. *If E is a normed space, its* bidual *is defined as $E^{**} = (E^*, \|\cdot\|_*)^*$; it is different from $(E^*, \sigma(E^*, E))^*$.*

Proposition 3.2.3. *If E is a normed space, the map $\delta : E \to E^{**}$ defined as in Proposition 3.2.1 is a linear isometric embedding (but δ may not be surjective), i.e. $\forall x \in E,\ \|\delta(x)\|_{**} = \|x\|$.*

3.3 Weak or weak-$*$ convergence of sequences

Proposition 3.3.1. *Let E be a normed space. Let $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$, $x \in E$, $(f_n)_{n\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$, $f \in E^*$.*

(i) *If $x_n \rightharpoonup x$, then $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded and:*

$$\|x\| \leq \liminf_{n\to+\infty} \|x_n\| .$$

(ii) *If $f_n \overset{*}{\rightharpoonup} f$, then $(\|f_n\|_*)_{n\in\mathbb{N}}$ is bounded and:*

$$\|f\|_* \leq \liminf_{n\to+\infty} \|f_n\|_* .$$

Proof. (ii) For every $x \in E$, $(f_n(x))_{n\in\mathbb{N}}$ is bounded. By the Uniform Boundedness Principle (Corollary 1.2.6), $(\|f_n\|_*)_{n\in\mathbb{N}}$ is bounded (because $(E^*, \|\cdot\|_*)$ is a Banach space). The inequality can be obtained by taking the $\liminf$ in $\forall x \in E$, $\forall n \in \mathbb{N}$, $|f_n(x)| \leq \|x\| \, \|f_n\|_*$. (i) Apply (ii) to the space $F = (E^*, \|\cdot\|_*)$ (hence $F^* = E^{**}$) and to the sequence $(\delta_{x_n})_{n\in\mathbb{N}} \in (E^{**})^{\mathbb{N}}$. We have $\forall f \in E^*$, $\delta_{x_n}(f) = f(x_n) \to f(x) = \delta_x(f)$, so $\delta_{x_n} \overset{*}{\rightharpoonup} \delta_x$. Therefore, $(\|\delta_{x_n}\|_{**})_{n\in\mathbb{N}}$ is bounded and $\|\delta_x\|_{**} \leq \liminf_{n\to+\infty} \|\delta_{x_n}\|_{**}$. This provides the desired result since $x \mapsto \delta_x$ is an isometric embedding. □

Proposition 3.3.2. *Let E be a normed space. Let $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$, $x \in E$, $(f_n)_{n\in\mathbb{N}} \in (E^*)^{\mathbb{N}}$, $f \in E^*$.*

(i) *If $x_n \to x$ and $f_n \overset{*}{\rightharpoonup} f$, then $f_n(x_n) \to f(x)$.*

(ii) *If $x_n \rightharpoonup x$ and $f_n \to f$, then $f_n(x_n) \to f(x)$.*

Example 3.3.3. *Consider the Hilbert space $H = \ell^2(\mathbb{N})$. For $n \in \mathbb{N}$, define $e_n = (\delta_{np})_{p\in\mathbb{N}} \in H$ and $f_n = \langle e_n, \cdot \rangle$. Then $f_n \overset{*}{\rightharpoonup} 0$, $e_n \rightharpoonup 0$ but $f_n(e_n) = 1 \not\to 0$.*

3.4 Weak-$*$ compactness

Theorem 3.4.1 (Banach-Alaoglu Theorem)**.** *Let E be a normed space. Then the unit ball of $(E^*, \|\cdot\|_*)$ is weakly-$*$ compact.*

Proof. View E^* as a subspace of $\mathbb{K}^E$, endowed with the product topology, which is locally convex. It induces the weak-$*$ topology on E^*. Write $B_* = \{f \in E^*, \ \|f\|_* \leq 1\}$. It is enough to prove that B_* is compact in $\mathbb{K}^E$. If $\mathrm{Lin}_{\mathbb{K}}(E, \mathbb{K})$ denotes the set of linear forms $E \to \mathbb{K}$, we have:

$$B_* = \mathrm{Lin}_{\mathbb{K}}(E, \mathbb{K}) \cap \underbrace{\bigcap_{x\in E} \left\{\varphi \in \mathbb{K}^E, \ |\varphi(x)| \leq \|x\|\right\}}_{K} .$$

Since $K = \prod_{x\in E} \{y \in \mathbb{K}, \ |y| \leq \|x\|\}$, K is compact according to Tychonoff's Theorem. And the space $\mathrm{Lin}_{\mathbb{K}}(E, \mathbb{K})$ is closed in $\mathbb{K}^E$, so B_* is compact in $\mathbb{K}^E$, i.e. weakly-$*$ compact. □

Remark 3.4.2. *If E has infinite dimension, then Riesz's Theorem states that the unit ball of $(E^*, \|\cdot\|_*)$ is never compact for the normed topology of E^*.*

Remark 3.4.3. *In order for the Banach-Alaoglu Theorem to be useful, we want to be able to extract convergent sequences. For this to be possible, we need $(B_*, \sigma(E^*, E))$ to be metrisable.*

Theorem 3.4.4. *Let E be a Banach space. If $B_* = \{f \in E^*, \ \|f\|_* \leq 1\}$, then $(B_*, \sigma(E^*, E))$ is metrisable* iff *E is separable.*

Proof. ($\Leftarrow$) Assume that E is separable and consider a dense sequence $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$. For $n \in \mathbb{N}$, define:

$$x'_n = \begin{cases} x_n & \text{if } \|x_n\| \leq 1 \\ \frac{x_n}{\|x_n\|} & \text{otherwise} \end{cases}.$$

Then $(x'_n)_{n\in\mathbb{N}} \in B^{\mathbb{N}}$, and $(x'_n)_{n\in\mathbb{N}}$ is dense in B, where $B = \{x \in E,\ \|x\| \leq 1\}$. Now define a distance d on E^* by:

$$\forall (\varphi, \psi) \in (E^*)^2,\ d(\varphi,\psi) = \sum_{n\in\mathbb{N}} 2^{-n} |\varphi(x_n) - \psi(x_n)| \leq 2\|\varphi - \psi\|_*.$$

The topology $\mathcal{T}_d$ defined by d on E^* is the coarsest topology s.t. $\delta_{x_n} : \varphi \in E^* \longmapsto \varphi(x_n) \in \mathbb{R}$ is continuous for every $n \in \mathbb{N}$. In particular, $\mathcal{T}_d \subseteq \sigma(E^*, E)$ (because $\sigma(E^*, E)$ makes δ_x continuous for all $x \in E$). Now consider the topology induced by $\mathcal{T}_d$ on B_*. It is coarser than $\sigma(E^*, E)$. To show that it is finer than $\sigma(E^*, E)$, it is enough to prove that $\mathcal{T}_d$ makes $\delta_{x|B_*}$ continuous for all $x \in E$. Let $x \in B$. For $\varepsilon > 0$, there exists $n \in \mathbb{N}$ s.t. $\|x'_n - x\| < \varepsilon$. Hence, for every $(\varphi, \psi) \in (B_*)^2$ s.t. $d(\varphi, \psi) \leq 2^{-n}\varepsilon$, we have:

$$|\varphi(x) - \psi(x)| \leq \|\varphi\|_* \|x - x'_n\| + \|\psi\|_* \|x - x'_n\| + 2^n d(\varphi, \psi) \leq 3\varepsilon.$$

This proves that $\delta_{x|B_*}$ is continuous (for all $x \in B$, hence for all $x \in E$) when B_* is equipped with d.
($\Rightarrow$) Suppose that $(B_*, \sigma(E^*, E))$ is metrisable; in particular, 0 admits a countable basis of weak-$*$ neighbourhoods $(\mathcal{V}_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, $\mathcal{V}_n$ contains a finite intersection of kernels of continuous linear forms on $(E^*, \sigma(E^*, E))$. According to Proposition 3.2.1, these linear forms can be written as δ_x for $x \in E$; hence there exists a finite set $A_n \subseteq E$ s.t.

$$\mathcal{V}_n \supseteq \bigcap_{x\in A_n} \{f \in B^*,\ f(x) = 0\}.$$

Let $A = \bigcup_{n\in\mathbb{N}} A_n$; A is a countable subset of E. Using Corollary 2.4.3, let us show that $\mathrm{Vect}(A)$ is dense in E. Let $\varphi \in E^*$ (one may assume that $\varphi \in B_*$) s.t. $\varphi_{|A} = 0$; then:

$$\varphi \in \bigcap_{n\in\mathbb{N}} \bigcap_{x\in A_n} \{f \in B^*,\ f(x) = 0\} \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{V}_n = \{0\}.$$

Therefore, $\mathrm{Vect}(A)$ is dense in E, so $\mathrm{Vect}_{\mathbb{Q}}(A)$ is countable and dense in E. □

Remark 3.4.5. *Even if E is a separable Banach space, $(E^*, \sigma(E^*, E))$ may not be metrisable.*

Corollary 3.4.6. *Let E be a separable Banach space. If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $(E^*, \|\cdot\|_*)$, then it admits a weakly-$*$ converging subsequence.*

Example 3.4.7.

(i) *If Ω is an open subset of $\mathbb{R}^d$ and $p \in [1, +\infty[$, then $L^p(\mathbb{R})$ is a separable Banach space.*

(ii) *If $p \in [1, +\infty[$, then $\ell^p(\mathbb{N})$ is a separable Banach space.*

(iii) *The space $c_0 = \left\{a \in \mathbb{R}^{\mathbb{N}},\ \lim_{+\infty} a = 0\right\}$ is a separable Banach space.*

3.5 Reflexivity

Definition 3.5.1 (Reflexive space)**.** *Let E be a Banach space. The space E^* has two topologies: the weak-$*$ topology and the normed topology. According to Proposition 3.2.1, we have an isomorphism $(E^*, \sigma(E^*, E))^* \simeq E$. Recall that $E^{**} = (E^*, \|\cdot\|_*)^*$ by definition. In general, the map:*

$$\delta : \left| \begin{array}{l} E \longrightarrow E^{**} \\ x \longmapsto \delta_x : \left| \begin{array}{l} E^* \longrightarrow \mathbb{K} \\ f \longmapsto f(x) \end{array} \right. \end{array} \right.$$

is a linear isometric embedding, called the canonical injection. *The space E is said to be* reflexive *if δ is an isomorphism.*

Example 3.5.2.

(i) *If Ω is an open subset of $\mathbb{R}^d$ and $p \in]1, +\infty[$, then $L^p(\mathbb{R})$ is a reflexive space.*

(ii) *If $p \in]1, +\infty[$, then $\ell^p(\mathbb{N})$ is a reflexive.*

(iii) *For any nonempty open set $\Omega \subseteq \mathbb{R}^d$, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive. Likewise, $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are not reflexive.*

Lemma 3.5.3. *Let E be a real locally convex topological vector space. Let C be a convex subset of E.*

(i) *C is closed* iff *C is an (arbitrary) intersection of closed half-spaces.*

(ii) *$\overline{C}$ is the intersection of all closed half-spaces containing C.*

Lemma 3.5.4 (Goldstine's Lemma)**.** *Let E be a real Banach space. Then the $\sigma(E^{**}, E^*)$-closure of $\delta(B_E)$, where $B_E = \{x \in E,\ \|x\| \leq 1\}$, is $B_{E^{**}} = \{y \in E^{**},\ \|y\|_{**} \leq 1\}$.*

Proof. Apply Lemma 3.5.3 to $\delta(B_E)$ for $\sigma(E^{**}, E^*)$. For $f \in E^*$ and $\alpha \in \mathbb{R}$, set $H_{f,\alpha} = \{\varphi \in E^{**},\ \varphi(f) \leq \alpha\}$. Note that $\delta(B_E) \subseteq H_{f,\alpha}$ iff $\|f\|_* \leq \alpha$. Hence:

$$\overline{\delta(B_E)}^{w*} = \bigcap_{\substack{(f,\alpha) \in E^* \times \mathbb{R} \\ \delta(B_E) \subseteq H_{f,\alpha}}} H_{f,\alpha} = \bigcap_{f \in B_{E^*}} H_{f,1} = B_{E^{**}}.$$

□

Remark 3.5.5. *Let E be a Banach space. Then $\delta(B_E)$ is $\|\cdot\|_{**}$-closed.*

Theorem 3.5.6. *A real Banach space E is reflexive* iff *$B_E = \{x \in E,\ \|x\| \leq 1\}$ is weakly compact.*

Proof. ($\Rightarrow$) If E is reflexive, then E is isometric to $(E^*, \|\cdot\|_*)^*$, so accoding to the Banach-Alaoglu Theorem (Theorem 3.4.1), $\delta(B_E)$ is $\sigma(E^{**}, E^*)$-compact. But $\sigma(E^{**}, E^*) = \delta(\sigma(E, E^*))$, so B_E is $\sigma(E, E^*)$-compact. ($\Leftarrow$) Suppose that B_E is weakly compact. Since the topology induced by $\sigma(E^{**}, E^*)$ on $\delta(E)$ is $\delta(\sigma(E, E^*))$, $\delta(B_E)$ is weakly-$*$ compact, in particular weakly-$*$ closed. By Goldstine's Lemma (Lemma 3.5.4), $\delta(B_E) = B_{E^{**}}$, so $\delta(E) = E^{**}$ by linearity. □

3.6 Uniform convexity

Definition 3.6.1 (Uniform convexity)**.** *A normed space E is said to be* uniformly convex iff*:*

$$\forall \varepsilon > 0, \sup_{\substack{x,y \in B_E \\ \|x-y\| \geq \varepsilon}} \left\| \frac{x+y}{2} \right\| < 1.$$

Example 3.6.2.

(i) *Hilbert spaces are uniformly convex because of the Parallelogram Identity.*

(ii) *If Ω is an open subset of $\mathbb{R}^d$ and $p \in]1, +\infty[$, then $L^p(\Omega)$ is uniformly convex.*

(iii) *If $p \in]1, +\infty[$, then $\ell^p(\mathbb{N})$ is uniformly convex.*

(iv) *For any nonempty open set $\Omega \subseteq \mathbb{R}^d$, $L^1(\Omega)$ and $L^\infty(\Omega)$ are not uniformly convex. Likewise, $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are not uniformly convex.*

Theorem 3.6.3 (Milman–Pettis Theorem)**.** *If E is a uniformly convex real Banach space, then E is reflexive.*

Proof. Note that $\delta(E)$ is closed in $(E^{**}, \|\cdot\|_{**})$ because E is complete and δ is an isometric embedding. Hence, we have to prove that $\delta(E)$ is $\|\cdot\|_{**}$-dense in E^{**}. By linearity, it suffices to prove that $\overline{\delta(B_E)}^{\|\cdot\|_{**}}$ contains the unit sphere of E^{**}. So let $\xi \in E^{**}$ with $\|\xi\|_{**} = 1$. Let $\varepsilon > 0$. Set $1 - \alpha = \sup_{\substack{x,y \in B_E \\ \|x-y\| \geq \varepsilon}} \left\| \frac{x+y}{2} \right\|$, with $\alpha > 0$ (because E is uniformly convex). By definition of $\|\cdot\|_{**}$, there exists $\eta \in E^*$ s.t.

$$1 - \alpha < \xi(\eta) \leq 1 \qquad \text{and} \qquad \|\eta\|_* = 1.$$

Define $V = \{\varphi \in E^{**}, \varphi(\eta) > 1 - \alpha\}$; V is a $\sigma(E^{**}, E^*)$-open half-space of E^{**} containing ξ. In particular, V is a weak-$*$ neighbourhood of ξ. By Goldstine's Lemma (Lemma 3.5.4), V meets $\delta(B_E)$: there exists $x \in B_E$ s.t. $\delta_x \in V \cap \delta(B_E)$. Now, note that if $y \in B_E$ is s.t. $\delta_y \in V \cap \delta(B_E)$, then $\eta(x) > 1 - \alpha$ and $\eta(y) > 1 - \alpha$, so:

$$1 - \alpha < \eta\left(\frac{x+y}{2}\right) \leq \|\eta\|_* \left\|\frac{x+y}{2}\right\| = \left\|\frac{x+y}{2}\right\|.$$

By definition of α, we infer that $\|y - x\| \leq \varepsilon$. In other words, $V \cap \delta(B_E) \subseteq \delta\left(x + \varepsilon \overline{B}_E\right)$. But $\delta\left(x + \varepsilon \overline{B}_E\right)$ is convex, $\|\cdot\|_{**}$-closed, so it is $\sigma(E^{**}, E^*)$-closed according to Proposition 3.1.6. Therefore, $\xi \in \overline{V \cap \delta(B_E)}^{w*} \subseteq \delta\left(x + \varepsilon \overline{B}_E\right)$, so $\|\xi - \delta_x\|_{**} \leq \varepsilon$. Hence, $\overline{\delta(B_E)}^{\|\cdot\|_{**}}$ contains the unit sphere of E^{**}. □

3.7 Adjoint operators

Definition 3.7.1 (Adjoint operator). *Let E and F be two locally convex topological vector spaces. If $T \in \mathcal{L}(E, F)$, define:*

$$T^* : \left| \begin{array}{rcl} F^* & \longrightarrow & E^* \\ \ell & \longmapsto & \ell \circ T \end{array} \right. .$$

We have $T^* \in \mathcal{L}(F^*, E^*)$.

Proposition 3.7.2. *Let E and F be two normed spaces. For any $T \in \mathcal{L}(E, F)$, the linear map $T^* : F^* \to E^*$ is continuous when F^* and E^* are equipped with their normed topologies (we already know that it is continuous when F^* and E^* are equipped with their weak-$*$ topologies). Moreover, $\|T^*\|_* = \|T\|$.*

Proposition 3.7.3. *Let E and F be two locally convex topological vector spaces. Let $T \in \mathcal{L}(E, F)$. Consider $T^{**} \in \mathcal{L}(E^{**}, F^{**})$, where $E^{**} = (E^*, \sigma(E^*, E))^*$ and $F^{**} = (F^*, \sigma(F^*, F))^*$. Then the following diagram is commutative:*

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\ E^{**} & \xrightarrow{\quad T^{**} \quad} & F^{**} \end{array}$$

In other words, for all $x \in E$, $T^{**}\delta_x = \delta_{Tx}$.

4 Theory of distributions

Notation 4.0.1. *In what follows, Ω is a nonempty open subset of $\mathbb{R}^d$.*

Notation 4.0.2. *If K is a compact subset of Ω, we write $K \Subset \Omega$.*

4.1 Test functions

Definition 4.1.1 (Support of a function). *Let $f : \Omega \to \mathbb{K}$ be a function. We define the* support *of f by:*

$$\operatorname{Supp} f = \Omega \backslash \bigcup_{\substack{\mathcal{O} \text{ open in } \Omega \\ f_{|\mathcal{O}}=0}} \mathcal{O}.$$

$\operatorname{Supp} f$ *is a closed subset of* Ω.

Definition 4.1.2 (Compactly supported function). *A function $f : \Omega \to \mathbb{K}$ is said to be* compactly supported *if* $\operatorname{Supp} f$ *is compact.*

Definition 4.1.3 (Test functions).

(i) *If $K \Subset \Omega$, we define $\mathcal{D}_K(\Omega) = \{f \in \mathcal{C}^\infty(\Omega), \operatorname{Supp} f \Subset K\}$. We equip $\mathcal{D}_K(\Omega)$ with the (countable) family $\left(\|\cdot\|_{N,K}\right)_{N\in\mathbb{N}}$ of semi-norms defined by:*

$$\|f\|_{N,K} = \max_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|=N}} \|\partial^\alpha f\|_{L^\infty}.$$

$\mathcal{D}_K(\Omega)$ is a Fréchet space.

(ii) *We define the space of* test functions $\mathcal{D}(\Omega) = \{f \in \mathcal{C}^\infty(\Omega),\ \operatorname{Supp} f \Subset \Omega\} = \bigcup_{K\Subset\Omega} \mathcal{D}_K(\Omega)$. *We equip $\mathcal{D}(\Omega)$ with the finest topology s.t. for every $K \Subset \Omega$, the inclusion $\mathcal{D}_K(\Omega) \subseteq \mathcal{D}(\Omega)$ is continuous. Hence, $\mathcal{D}(\Omega)$ is a locally convex topological vector space (but not a Fréchet space).*

Proposition 4.1.4. *Let E be a locally convex topological vector space. If $f : \mathcal{D}(\Omega) \to E$ is a linear map, then the following assertions are equivalent:*

(i) *$f : \mathcal{D}(\Omega) \to E$ is continuous.*

(ii) *For every $K \Subset \Omega$, $f_{|\mathcal{D}_K(\Omega)} : \mathcal{D}_K(\Omega) \to E$ is continuous.*

Proposition 4.1.5. *For every $\omega \in \Omega$ and $0 < r < d(z, \partial\Omega)$, there exists a function $u \in \mathcal{D}(\Omega)$ s.t. $u \geq 0$ and $u_{|B(z,r)} = 1$. In particular, $\mathcal{D}(\Omega)$ is nontrivial.*

Proof. Use the function $\varphi : t \in \mathbb{R} \longmapsto \begin{cases} \exp\left(-\frac{1}{t(1-t)}\right) & \text{if } t \in]0,1[\\ 0 & \text{otherwise} \end{cases}$, which is $\mathcal{C}^\infty$. □

Proposition 4.1.6 (Partitions of unity). *Let $\Gamma \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ be a collection of open subsets of $\mathbb{R}^d$. Set $\Omega = \bigcup_{\mathcal{O}\in\Gamma} \mathcal{O} \subseteq \mathbb{R}^d$. Then there exists a sequence $(\Psi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\Omega)^{\mathbb{N}}$ s.t.*

(i) $\forall n \in \mathbb{N},\ \Psi_n \geq 0$,

(ii) $\forall n \in \mathbb{N},\ \exists \mathcal{O}_n \in \Gamma,\ \operatorname{Supp} \Psi_n \Subset \mathcal{O}_n$,

(iii) $\sum_{n\in\mathbb{N}} \Psi_n = 1$ *on Ω and the sum is locally finite.*

We say that $(\Psi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\Omega)^{\mathbb{N}}$ is a partition of unity *subordinated to* Γ.

Proof. *First step.* For $m \in \mathbb{N}^*$, let $K_m = \left\{x \in \Omega,\ d(x, \partial\Omega) \geq \frac{1}{m} \text{ and } \|x\| \leq m\right\}$. Hence $K_m \Subset K_{m+1} \Subset \Omega$ and $\Omega = \bigcup_{m\in\mathbb{N}^*} K_m$. Given $m \in \mathbb{N}^*$, for all $x \in K_m$, there exists $\omega_x \in \Gamma$ s.t. $x \in \omega_x$; choose $r_x > 0$ s.t. $x \in B(x, 2r_x) \subseteq \omega_x$ and set $V_x = B(x, r_x)$: thus $x \in \overline{V}_x \Subset \omega_x$. Hence, the compact set K_m is covered by $(V_x)_{x\in K_m}$, so there exists a finite subset $F_m \subseteq K_m$ s.t. $(V_x)_{x\in F_m}$ covers K_m. Now set $F = \bigcup_{m\in\mathbb{N}^*} F_m$; F is countable so we may write $F = \{x_j,\ j \in \mathbb{N}\}$. Thus $\Omega = \bigcup_{j\in\mathbb{N}} V_{x_j}$. Now for any $j \in \mathbb{N}$, using Proposition 4.1.5, there exists $\varphi_j \in \mathcal{D}(\Omega)$ s.t. $\operatorname{Supp} \varphi_j \Subset B\left(x_j, \frac{3}{2} r_{x_j}\right) \subseteq \omega_{x_j}$, $0 \leq \varphi_j \leq 1$ and $\varphi_{j|V_{x_j}} = 1$. *Second step.* For $j \in \mathbb{N}$, define $\Psi_j = \varphi_j \prod_{k=0}^{j-1} (1 - \varphi_k)$. We have $0 \leq \Psi_j \leq 1$, $\operatorname{Supp} \Psi_j \Subset \omega_{x_j}$ and $\sum_{j\in\mathbb{N}} \Psi_j = 1$ (with the sum locally finite). □

4.2 Distributions

Definition 4.2.1 (Distributions). *We denote by* $\mathcal{D}'(\Omega)$ *the dual space of* $\mathcal{D}(\Omega)$*, equipped with the weak-*$*$ *topology.* $\mathcal{D}'(\Omega)$ *is called the* space of distributions *on* Ω.

Remark 4.2.2. *Let* $\Lambda : \mathcal{D}(\Omega) \to \mathbb{K}$ *be a linear form. Then* Λ *is continuous* iff

$$\forall K \Subset \Omega,\ \exists N_K \in \mathbb{N},\ \exists C_K < +\infty,\ \forall \varphi \in \mathcal{D}(\Omega),\ \operatorname{Supp}\varphi \subseteq K \Longrightarrow |\langle \Lambda, \varphi \rangle| \leq C_K \max_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq N_K}} \|\partial^\alpha \varphi\|_\infty .$$

If N_K *can be chosen independent of* K*, we say that* Λ *is of* order *less than or equal to* N.

Proposition 4.2.3. *Let* $(\Lambda_n)_{n\in\mathbb{N}} \in \mathcal{D}'(\Omega)^{\mathbb{N}}$*; let* $\Lambda : \mathcal{D}(\Omega) \to \mathbb{K}$ *be a linear form s.t.*

$$\forall \varphi \in \mathcal{D}(\Omega),\ \langle \Lambda_n, \varphi \rangle \to \langle \Lambda, \varphi \rangle .$$

Then $\Lambda \in \mathcal{D}'(\Omega)$ *(i.e.* Λ *is continuous) and* $\Lambda_n \overset{*}{\rightharpoonup} \Lambda$.

Proof. Use the Uniform Boundedness Principle (Corollary 1.2.6). □

Remark 4.2.4. *Distributions of order* 0 *correspond to continuous linear forms on the space of continuous functions with compact support, i.e. to locally finite measures on* Ω.

Example 4.2.5.

(i) *If* μ *is a locally finite measure on* Ω*, then* $\Lambda_\mu : \varphi \in \mathcal{D}(\Omega) \longmapsto \int_\Omega \varphi \, \mathrm{d}\mu$ *is a distribution.*

(ii) *In particular, if* $a \in \Omega$*, then the Dirac mass* $\delta_a : \varphi \in \mathcal{D}(\Omega) \longmapsto \varphi(a)$ *is a distribution.*

(iii) *If* $f \in L^1_{\mathrm{loc}}(\Omega)$*, then* $\Lambda_f : \varphi \in \mathcal{D}(\Omega) \longmapsto \int_\Omega f\varphi$ *is a distribution, sometimes simply denoted by* f.

4.3 Operations on distributions

Remark 4.3.1. *Given an operator* $T \in \mathcal{L}(\mathcal{D}(\Omega))$*, we have its adjoint* $T^* \in \mathcal{L}(\mathcal{D}'(\Omega))$.

Definition 4.3.2 (Multiplication by a function). *If* $\theta \in \mathcal{C}^\infty(\Omega)$*, we consider:*

$$M_\theta : \varphi \in \mathcal{D}(\Omega) \longmapsto \theta\varphi \in \mathcal{D}(\Omega) .$$

We have: $\forall f \in L^1_{\mathrm{loc}}(\Omega),\ M_\theta^* \Lambda_f = \Lambda_{M_\theta f}$*. Hence,* M_θ^* *will be called* multiplication by θ*, and we will write* $\theta\Lambda$ *instead of* $M_\theta^*\Lambda$.

Definition 4.3.3 (Differentiation). *If* $j \in \{1, \ldots, d\}$*, we consider:*

$$\partial_j : \varphi \in \mathcal{D}(\Omega) \longmapsto \frac{\partial \varphi}{\partial x_j} \in \mathcal{D}(\Omega) .$$

We have: $\forall f \in \mathcal{C}^1(\Omega),\ \partial_j^* \Lambda_f = -\Lambda_{\partial_j f}$*. Hence, we will write* $-\partial_j\Lambda$ *instead of* $\partial_j^*\Lambda$*. More generally, if* $\alpha \in \mathbb{N}^d$ *is a multi-index, we write* $\partial^\alpha \Lambda = (-1)^{|\alpha|} (\partial^\alpha)^* \Lambda$.

Proposition 4.3.4 (Leibniz's Formula). *Let* $\Lambda \in \mathcal{D}'(\Omega)$ *and* $\theta \in \mathcal{C}^\infty(\Omega)$*. For any multi-index* $\alpha \in \mathbb{N}^d$*, we have:*

$$\partial^\alpha (\theta\Lambda) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \left(\partial^\beta \theta\right) \left(\partial^{\alpha-\beta} \Lambda\right) .$$

4.4 Support of a distribution

Definition 4.4.1 (Extension operator)**.** *If ω is an open subset of Ω, we consider:*

$$\mathrm{Ext}_\omega : \theta \in \mathcal{D}(\omega) \longmapsto \theta \mathbb{1}_\omega \in \mathcal{D}(\Omega).$$

We have: $\forall f \in L^1_{\mathrm{loc}}(\Omega)$, $\mathrm{Ext}^*_\omega \Lambda_f = \Lambda_{f_{|\omega}}$. Hence, Ext^*_ω will be called restriction to ω *and we will write $\Lambda_{|\omega}$ instead of $\mathrm{Ext}^*_\omega \Lambda$.*

Vocabulary 4.4.2. *A distribution $\Lambda \in \mathcal{D}(\Omega)$ is said to* vanish *over an open subset $\omega \subseteq \Omega$ if $\Lambda_{|\omega} = 0$, i.e.*

$$\forall \varphi \in \mathcal{D}(\Omega), \ \mathrm{Supp}\, \varphi \Subset \omega \Longrightarrow \langle \Lambda, \varphi \rangle = 0.$$

Lemma 4.4.3. *Let Γ be a collection of open subsets of Ω; consider $U = \bigcup_{\omega \in \Gamma} \omega$. Let $\Lambda \in \mathcal{D}'(\Omega)$ s.t. $\forall \omega \in \Gamma$, $\Lambda_{|\omega} = 0$. Then $\Lambda_{|U} = 0$.*

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ s.t. $\mathrm{Supp}\, \varphi \Subset U$. Since $\mathrm{Supp}\, \varphi$ is compact, there exists a finite subset $J \subseteq \Gamma$ s.t. $\mathrm{Supp}\, \varphi \Subset \bigcup_{\omega \in J} \omega$. Now, consider a partition of unity $(\theta_n)_{n\in\mathbb{N}}$ subordinated to J (c.f. Proposition 4.1.6). For $n \in \mathbb{N}$, there exists $\omega_n \in J$ s.t. $\mathrm{Supp}\, \theta_n \Subset \omega_n$. Therefore:

$$\langle \Lambda, \varphi \rangle = \left\langle \Lambda, \sum_{n\in\mathbb{N}} \theta_n \varphi \right\rangle = \sum_{n\in\mathbb{N}} \langle \Lambda, \theta_n \varphi \rangle = \sum_{n\in\mathbb{N}} \left\langle \Lambda_{|\omega_n}, \theta_n \varphi \right\rangle = 0.$$

□

Definition 4.4.4 (Support of a distribution)**.** *Let $\Lambda \in \mathcal{D}'(\Omega)$. We define the* support *of Λ by:*

$$\mathrm{Supp}\, \Lambda = \Omega \backslash \bigcup_{\substack{\omega \text{ open in } \Omega \\ f_{|\omega} = 0}} \omega.$$

$\mathrm{Supp}\, \Lambda$ *is a closed subset of Ω. Moreover, by Lemma 4.4.3, $\Lambda_{|\Omega \backslash \mathrm{Supp}\, \Lambda} = 0$.*

Definition 4.4.5 (Compactly supported distribution)**.** *A distribution $\Lambda \in \mathcal{D}'(\Omega)$ is said to be* compactly supported *if $\mathrm{Supp}\, \Lambda$ is compact. We write $\mathcal{E}'(\Omega)$ for the space of compactly supported distributions over Ω.*

Theorem 4.4.6. *If $\Lambda \in \mathcal{E}'(\Omega)$ is a compactly supported distribution, then:*

$$\exists K \Subset \Omega, \ \exists N \in \mathbb{N}, \ \exists C \in \mathbb{R}_+, \ \forall \varphi \in \mathcal{D}(\Omega), \ |\langle \Lambda, \varphi \rangle| \leq C \max_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq N}} \|\partial^\alpha \varphi\|_{\infty}^{K}.$$

In particular, Λ has finite order (because $\|\cdot\|_\infty^K \leq \|\cdot\|_\infty^\Omega$).

Proof. Choose $\varepsilon > 0$ s.t. $\mathrm{Supp}\, \Lambda + \overline{B}(0, \varepsilon) \Subset \Omega$. There exists $\Psi \in \mathcal{D}(\Omega)$ s.t. $0 \leq \Psi \leq 1$ and $\Psi_{|\mathrm{Supp}\, \Lambda + \overline{B}(0,\varepsilon)} = 1$. Let $K = \mathrm{Supp}\, \Psi \Subset \Omega$. Since $\Lambda_{|\mathcal{D}_K(\Omega)}$ is continuous, there exist $C \in \mathbb{R}_+$ and $N \in \mathbb{N}$ s.t.

$$\forall \theta \in \mathcal{D}(\Omega), \ \mathrm{Supp}\, \theta \subseteq K \Longrightarrow |\langle \Lambda, \theta \rangle| \leq C \max_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq N}} \|\partial^\alpha \theta\|_{L^\infty}.$$

Now, if $\varphi \in \mathcal{D}(\Omega)$, write $\varphi = \Psi \varphi + (1 - \Psi) \varphi$. Note that $\mathrm{Supp}\, ((1 - \Psi) \varphi) \subseteq \mathrm{Supp}\, (1 - \Psi) \subseteq \Omega \backslash \mathrm{Supp}\, \Lambda$ so $\langle \Lambda, (1 - \Psi) \varphi \rangle = 0$. Thus:

$$|\langle \Lambda, \varphi \rangle| = |\langle \Lambda, \Psi \varphi \rangle| \leq C \max_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq N}} \|\partial^\alpha (\Psi \varphi)\|_{L^\infty} \leq C \max_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq N}} \|\partial^\alpha \varphi\|_{\infty}^{K}.$$

□

Corollary 4.4.7. *A compactly supported distribution $\Lambda \in \mathcal{E}'(\Omega)$ induces a unique continuous linear form over $\mathcal{C}^\infty(\Omega)$ (where the topology of $\mathcal{C}^\infty(\Omega)$ is given by the family $\left(\|\cdot\|_{N,K}\right)_{\substack{N\in\mathbb{N}\\ K\Subset\Omega}}$ of semi-norms defined by $\|\varphi\|_{N,K} = \max_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le N}} \|\partial^\alpha\varphi\|_{\substack{\infty\\ K}}$).*

Proof. Note that $\mathcal{D}(\Omega)$ is dense in $\mathcal{C}^\infty(\Omega)$, and that elements of $\mathcal{E}'(\Omega)$ are $\mathcal{C}^\infty(\Omega)$-continuous over the dense subspace $\mathcal{D}(\Omega)$. ☐

Remark 4.4.8. *Conversely, if $\Lambda \in \mathcal{C}^\infty(\Omega)^*$, then $\Lambda_{|\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$.*

Notation 4.4.9. *We shall write $\mathcal{E}(\Omega) = \mathcal{C}^\infty(\Omega)$. This notation is coherent with the fact that $\mathcal{E}'(\Omega) = \mathcal{E}(\Omega)^*$.*

Proposition 4.4.10. *Fix $a \in \Omega$ and write $\delta_a \in \mathcal{E}'(\Omega)$ for the Dirac mass at a. If $\Lambda \in \mathcal{D}'(\Omega)$ is s.t. $\operatorname{Supp}\Lambda \subseteq \{a\}$, then $\Lambda \in \operatorname{Vect}\left(\partial^\alpha\delta_a,\ \alpha\in\mathbb{N}^d\right)$.*

Proof. By a standard algebraic argument, it is enough to prove the existence of $N \in \mathbb{N}$ s.t.

$$\operatorname{Ker}\Lambda \supseteq \bigcap_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le N}} \operatorname{Ker}\partial^\alpha\delta_a.$$

Let $\Psi \in \mathcal{D}\left(\mathbb{R}^d\right)$ s.t. $0 \le \Psi \le 1$ and $\psi_{|B(0,1)} = 1$. Define $\Psi_n : x \in \mathbb{R}^d \longmapsto \Psi(n(x-a))$. Now consider a closed ball $\overline{B} \Subset \Omega$ centred at a. We have $\operatorname{Supp}\Psi_n = a + \frac{1}{n}\operatorname{Supp}\Psi \subseteq \overline{B}$ for n larger than or equal to some $n_0 \in \mathbb{N}^*$. By continuity of Λ, there exist $C \in \mathbb{R}_+$, $N \in \mathbb{N}$ s.t.

$$\forall\theta \in \mathcal{D}(\Omega),\ \operatorname{Supp}\theta \subseteq \overline{B} \Longrightarrow |\langle\Lambda,\theta\rangle| \le C \max_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le N}} \|\partial^\alpha\theta\|_{L^\infty}.$$

If $\varphi \in \mathcal{D}(\Omega)$, then $\operatorname{Supp}(\Psi_n\varphi) \subseteq \operatorname{Supp}\Psi_n \subseteq \overline{B}$ for $n \ge n_0$. Therefore:

$$\forall\varphi \in \mathcal{D}(\Omega),\ \forall n \ge n_0,\ |\langle\Lambda,\Psi_n\varphi\rangle| \le C \max_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le N}} \|\partial^\alpha(\Psi_n\varphi)\|_{L^\infty}.$$

Now, let $\varphi \in \bigcap_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le N}} \operatorname{Ker}\partial^\alpha\delta_a$, i.e. $|\alpha| \le N \Longrightarrow \partial^\alpha\varphi(a) = 0$. By Taylor's formula, $\partial^\alpha\varphi(x) = \mathcal{O}_a\left(|x-a|^{N+1-\alpha}\right)$ if $|\alpha| \le N$. By Leibniz's formula, we obtain $|\partial^\alpha(\Psi_n\varphi)(x)| \le C'n^{|\alpha|-N-1}$ for some $C' \in \mathbb{R}_+$. Therefore, there is a constant $C'' \in \mathbb{R}_+$ s.t. $|\langle\Lambda,\Psi_n\varphi\rangle| \le \frac{C''}{n}$ for all $n \ge n_0$. Now, for $n \ge n_0$, $\operatorname{Supp}(\varphi - \Psi_n\varphi) \cap \operatorname{Supp}\Lambda = \varnothing$, so $|\langle\Lambda,\varphi\rangle| = |\langle\Lambda,\Psi_n\varphi\rangle| \le \frac{C''}{n}$. By making $n \to +\infty$, we obtain $\langle\Lambda,\varphi\rangle = 0$, i.e. $\varphi \in \operatorname{Ker}\Lambda$ as wanted. ☐

4.5 Assembling distributions

Proposition 4.5.1. *Let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^d$. Let $\Lambda_1 \in \mathcal{D}'(\Omega_1)$, $\Lambda_2 \in \mathcal{D}'(\Omega_2)$ and assume that:*

$$\Lambda_{1|\Omega_1\cap\Omega_2} = \Lambda_{2|\Omega_1\cap\Omega_2}.$$

Then there exists a unique distribution $\Lambda \in \mathcal{D}'(\Omega_1\cup\Omega_2)$ s.t. $\Lambda_{|\Omega_1} = \Lambda_1$ and $\Lambda_{|\Omega_2} = \Lambda_2$.

Proof. *Uniqueness.* Assume that Λ exists. Let $\varphi \in \mathcal{D}(\Omega_1\cup\Omega_2)$. Note that there exist $\Psi_1, \Psi_2 \in \mathcal{D}(\Omega_1\cup\Omega_2)$ s.t. $\operatorname{Supp}\Psi_1 \Subset \Omega_1$, $\operatorname{Supp}\Psi_2 \Subset \Omega_2$ and $\Psi_1 + \Psi_2 = 1$ on $\operatorname{Supp}\varphi$. Therefore:

$$\langle\Lambda,\varphi\rangle = \langle\Lambda,(\Psi_1+\Psi_2)\varphi\rangle = \langle\Lambda_1,\Psi_1\varphi\rangle + \langle\Lambda_2,\Psi_2\varphi\rangle. \tag{$*$}$$

This proves the uniqueness. *Existence.* Let us prove that the right-hand side of $(*)$ does not depend on the choice of (Ψ_1,Ψ_2). Let (Ψ_1',Ψ_2') be another pair satisfying the same conditions. Then:

$$(\langle\Lambda_1,\Psi_1'\varphi\rangle + \langle\Lambda_2,\Psi_2'\varphi\rangle) - (\langle\Lambda_1,\Psi_1\varphi\rangle + \langle\Lambda_2,\Psi_2\varphi\rangle) = \langle\Lambda_1,(\Psi_1'-\Psi_1)\varphi\rangle - \langle\Lambda_2,(\Psi_2-\Psi_2')\varphi\rangle.$$

Now, consider $\theta = (\Psi_1' - \Psi_1)\varphi = (\Psi_2 - \Psi_2')\varphi$. We have $\operatorname{Supp}\theta \subseteq \Omega_1 \cap \Omega_2$. Since $\Lambda_{1|\Omega_1\cap\Omega_2} = \Lambda_{2|\Omega_1\cap\Omega_2}$, this gives $\langle \Lambda_1, \theta\rangle = \langle \Lambda_2, \theta\rangle$, therefore $\langle \Lambda_1, \Psi_1'\varphi\rangle + \langle \Lambda_2, \Psi_2'\varphi\rangle = \langle \Lambda_1, \Psi_1\varphi\rangle + \langle \Lambda_2, \Psi_2\varphi\rangle$ as wanted. Hence, we can define a linear form Λ using $(*)$ as wanted. Let us check that Λ is continuous. Let $K \Subset \Omega_1 \cup \Omega_2$. There exist $\Psi_1, \Psi_2 \in \mathcal{D}(\Omega_1 \cup \Omega_2)$ s.t. $\operatorname{Supp}\Psi_1 \Subset \Omega_1$, $\operatorname{Supp}\Psi_2 \Subset \Omega_2$ and $\Psi_1 + \Psi_2 = 1$ on K. For any $\varphi \in \mathcal{D}(\Omega_1 \cup \Omega_2)$ with $\operatorname{Supp}\varphi \subseteq K$, we have $\langle \Lambda, \varphi\rangle = \langle \Lambda_1, \Psi_1\varphi\rangle + \langle \Lambda_2, \Psi_2\varphi\rangle$. Hence, we easily obtain the continuity of Λ from that of Λ_1 and Λ_2. Now, let us check that $\Lambda_{|\Omega_1} = \Lambda_1$. Let $\varphi \in \mathcal{D}(\Omega_1 \cup \Omega_2)$ with $\operatorname{Supp}\varphi \subseteq \Omega_1$. If Ψ_1, Ψ_2 are chosen as in the construction of Λ, we have $\operatorname{Supp}(\Psi_2\varphi) \subseteq \Omega_1 \cap \Omega_2$, so $\langle \Lambda, \varphi\rangle = \langle \Lambda_1, \Psi_1\varphi\rangle + \langle \Lambda_2, \Psi_2\varphi\rangle = \langle \Lambda_1, \varphi\rangle$, which proves that $\Lambda_{|\Omega_1} = \Lambda_1$. Likewise, $\Lambda_{|\Omega_2} = \Lambda_2$. □

5 Convolution of distributions

5.1 Generalities

Lemma 5.1.1. *If $\Gamma \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$, then $\psi : y \in \mathbb{R}^d \longmapsto \langle \Gamma, \varphi(\cdot + y)\rangle$ is an element of $\mathcal{D}\left(\mathbb{R}^d\right)$.*

Proof. We easily prove that $\operatorname{Supp}\psi \subseteq \operatorname{Supp}\varphi - \operatorname{Supp}\Gamma$, so ψ is compactly supported. For the continuity of ψ, we prove that, for all $y \in \mathbb{R}^d$, $|\psi(y+h) - \psi(y)| = \mathcal{O}_0(h)$, so ψ is continuous. Likewise, for $j \in \{1, \ldots, d\}$, we have $\left|\psi(y+h) - \psi(y) - \left\langle \Gamma, \frac{\partial\varphi}{\partial x_j}(\cdot + y)\right\rangle\right| = \mathcal{O}_0\left(h^2\right)$. By induction, ψ is $\mathcal{C}^\infty$, and:

$$\forall \alpha \in \mathbb{N}^d,\ \forall y \in \mathbb{R}^d,\ \partial^\alpha\psi(y) = \langle \Gamma, \partial^\alpha\varphi(\cdot + y)\rangle .$$

□

Remark 5.1.2. *With the notations above, one can also show that if Γ is a (not necessarily compactly supported) distribution, then ψ is an element of $\mathcal{C}^\infty\left(\mathbb{R}^d\right)$.*

Definition 5.1.3 (Convolution)**.** *Let $\Lambda, \Gamma \in \mathcal{D}'\left(\mathbb{R}^d\right)$. Assume that Λ or Γ is compactly supported. Then we can define a linear map $\Lambda * \Gamma : \mathcal{D}\left(\mathbb{R}^d\right) \to \mathbb{K}$ as follows. For any test function $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$, set $\psi : y \in \mathbb{R}^d \longmapsto \langle \Gamma, \varphi(\cdot + y)\rangle$ and define:*

$$\langle \Lambda * \Gamma, \varphi\rangle = \langle \Lambda, \psi\rangle .$$

Then $\Lambda * \Gamma$ is a distribution.

Proposition 5.1.4. *If $f, g \in L^1\left(\mathbb{R}^d\right)$ s.t. f or g is compactly supported, then $\Lambda_f * \Lambda_g = \Lambda_{f*g}$.*

Proposition 5.1.5. *Let $\delta_0 \in \mathcal{E}'\left(\mathbb{R}^d\right)$ be the Dirac mass at 0. Then:*

$$\forall \Lambda \in \mathcal{D}\left(\mathbb{R}^d\right),\ \delta_0 * \Lambda = \Lambda = \Lambda * \delta_0.$$

δ_0 is the unit of the convolution product.

Remark 5.1.6. *If $\Lambda, \Gamma \in \mathcal{E}'\left(\mathbb{R}^d\right)$, then $\Lambda * \Gamma \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $\operatorname{Supp}(\Lambda * \Gamma) \subseteq \operatorname{Supp}\Lambda + \operatorname{Supp}\Gamma$. Therefore, $\mathcal{E}'\left(\mathbb{R}^d\right)$ is an algebra for $*$, and $\mathcal{D}'\left(\mathbb{R}^d\right)$ is an $\mathcal{E}'\left(\mathbb{R}^d\right)$-module.*

Proposition 5.1.7. *If $\Lambda, \Gamma \in \mathcal{D}'\left(\mathbb{R}^d\right)$ s.t. Λ or Γ is compactly supported, then:*

$$\forall \alpha \in \mathbb{N}^d,\ \partial^\alpha(\Lambda * \Gamma) = (\partial^\alpha\Lambda) * \Gamma = \Lambda * (\partial^\alpha\Gamma) .$$

In particular, the maps $\partial^\alpha : \mathcal{D}'\left(\mathbb{R}^d\right) \to \mathcal{D}'\left(\mathbb{R}^d\right)$ are $\mathcal{E}'\left(\mathbb{R}^d\right)$-linear.

Proposition 5.1.8. *If $\Gamma_1, \Gamma_2 \in \mathcal{E}'\left(\mathbb{R}^d\right)$ and $\Lambda \in \mathcal{D}'\left(\mathbb{R}^d\right)$, then $(\Lambda * \Gamma_1) * \Gamma_2 = \Lambda * (\Gamma_1 * \Gamma_2)$.*

5.2 Applications to partial differential equations

Vocabulary 5.2.1 (Linear PDE with constant coefficient)**.** *A* linear partial differential equation *(PDE)* with constant coefficients *is an equation of the form:*

$$Lu = \Gamma,$$

where $\Gamma \in \mathcal{D}'(\Omega)$ is a given distribution and L is of the form $L = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha$, with $N \in \mathbb{N}$ and $(c_\alpha)_{\alpha \in \mathbb{N}^d} \in \mathbb{R}^{\mathbb{N}^d}$.

Definition 5.2.2 (Fundamental solution)**.** *A distribution $v \in \mathcal{D}'\left(\mathbb{R}^d\right)$ is said to be a* fundamental solution *for L if:*

$$Lv = \delta_0,$$

where δ_0 is the Dirac mass at 0*.*

Proposition 5.2.3. *If $v \in \mathcal{D}'\left(\mathbb{R}^d\right)$ is a fundamental solution for L, then for any $\Gamma \in \mathcal{E}'\left(\mathbb{R}^d\right)$, the distribution $(v * \Gamma)$ satisfies $L(v * \Gamma) = \Gamma$.*

Example 5.2.4.

(i) *If $d = 1$ and $L = \frac{\mathrm{d}}{\mathrm{d}x}$, then Heaviside's function $\mathbb{1}_{\mathbb{R}_+}$ is a fundamental solution for L.*

(ii) *If $d \geq 2$ and $L = \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2}$, we have a fundamental solution $E \in L^1_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ for L given by:*

$$E(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & \text{if } d = 2 \\ \frac{1}{d(d-2)V_d} |x|^{2-d} & \text{if } d > 2 \end{cases},$$

where V_d is the volume of the unit ball of $\mathbb{R}^d$. Therefore, if f is a compactly supported $\mathcal{C}^2$ function, then $(E * f)$ is also $\mathcal{C}^2$, so $(E * f)$ is a solution of $\Delta u = f$ in the ordinary sense.

5.3 The Schwartz class

Definition 5.3.1 (Schwartz class)**.** *A function $f \in \mathcal{C}^\infty\left(\mathbb{R}^d, \mathbb{K}\right)$ is said to have* rapid decay *if one the three following equivalent conditions is satisfied:*

(i) $\forall (\alpha, \beta) \in \left(\mathbb{N}^d\right)^2$, $\sup_{x \in \mathbb{R}^d} \left|x^\alpha \partial^\beta f(x)\right| < +\infty$.

(ii) $\forall (\alpha, \beta) \in \left(\mathbb{N}^d\right)^2$, $\lim_{|x| \to +\infty} x^\alpha \partial^\beta f(x) = 0$.

(iii) $\forall (\alpha, \beta) \in \left(\mathbb{N}^d\right)^2$, $\int_{\mathbb{R}^d} \left|x^\alpha \partial^\beta f(x)\right| \mathrm{d}x < +\infty$.

The Schwartz class *if the space $\mathcal{S}\left(\mathbb{R}^d\right)$ of $\mathcal{C}^\infty$ fucntions with rapid decay. $\mathcal{S}\left(\mathbb{R}^d\right)$ is equipped with the countable family $(\|\cdot\|_N)_{N \in \mathbb{N}}$ of semi-norms defined by:*

$$\|f\|_N = \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\partial^\alpha f(x)|.$$

Proposition 5.3.2. *$\mathcal{S}\left(\mathbb{R}^d\right)$ is a Fréchet space.*

Proof. It suffices to prove that $\mathcal{S}\left(\mathbb{R}^d\right)$ is complete, which comes from the fact that the space of continuous functions which converge to 0 at ∞ is complete, equipped with $\|\cdot\|_{L^\infty}$, and from the fact that if a sequence of functions is such that the derivatives of the functions all converge, then one can compute the derivatives of the limit of the sequence. ☐

Vocabulary 5.3.3 (Slow growth)**.** *A function $f \in \mathcal{C}^\infty\left(\mathbb{R}^d, \mathbb{K}\right)$ is said to have* slow growth *if every derivative of f grows at most polynomially.*

Proposition 5.3.4. *Let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

(i) *If $\alpha \in \mathbb{N}^d$, then $\partial^\alpha f \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

(ii) *If $g \in \mathcal{C}^\infty\left(\mathbb{R}^d, \mathbb{K}\right)$ has slow growth, then $gf \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

Moreover, these operators $f \mapsto \partial^\alpha f$ and $f \mapsto gf$ are linear continuous.

Proposition 5.3.5. *We have the (continuous) inclusions:*

$$\mathcal{D}\left(\mathbb{R}^d\right) \subseteq \mathcal{S}\left(\mathbb{R}^d\right) \subseteq \mathcal{E}\left(\mathbb{R}^d\right).$$

Moreover, $\mathcal{D}\left(\mathbb{R}^d\right)$ is dense in $\mathcal{S}\left(\mathbb{R}^d\right)$.

Proof. Choose a function $\psi \in \mathcal{D}\left(\mathbb{R}^d\right)$ s.t. $0 \leq \psi \leq 1$ and $\psi = 1$ on $B_1 = \left\{x \in \mathbb{R}^d,\ |x| \leq 1\right\}$. For $n \in \mathbb{N}^*$, define $\psi_n(x) = \psi\left(\frac{x}{n}\right)$; hence $\psi_n = 1$ on B_n. If $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, show that $\psi_n \varphi \to \varphi$ in $\mathcal{S}\left(\mathbb{R}^d\right)$, and $\psi_n \varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$. □

5.4 The Fourier transform

Definition 5.4.1 (Fourier transform in L^1)**.** *If $f \in L^1\left(\mathbb{R}^d\right)$, then the* Fourier transform *of f is defined by:*

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x)\ \mathrm{d}x.$$

$\mathcal{F}$ is a continuous linear operator from $L^1\left(\mathbb{R}^d\right)$ to the space of continuous functions on $\mathbb{R}^d$ which converge to 0 at ∞.

Proposition 5.4.2.

(i) *Let $f \in \mathcal{C}^N\left(\mathbb{R}^d\right)$ s.t. $\forall |\alpha| \leq N$, $\partial^\alpha f \in L^1\left(\mathbb{R}^d\right)$. Then, for $|\alpha| \leq N$:*

$$\mathcal{F}\left(\partial^\alpha f\right)(\xi) = i^{|\alpha|} \xi^\alpha \mathcal{F}f(\xi).$$

(ii) *Let $f \in L^1\left(\mathbb{R}^d\right)$ s.t. $\forall |\alpha| \leq N$, $x^\alpha f \in L^1\left(\mathbb{R}^d\right)$. Then $\mathcal{F}f \in \mathcal{C}^N\left(\mathbb{R}^d\right)$ and, for $|\alpha| \leq N$:*

$$\partial^\alpha\left(\mathcal{F}f\right)(\xi) = (-1)^{|\alpha|} \mathcal{F}\left(x^\alpha f\right)(\xi).$$

(iii) *If $f, g \in L^1\left(\mathbb{R}^d\right)$, then $(f * g) \in L^1\left(\mathbb{R}^d\right)$ and:*

$$\mathcal{F}(f * g) = (2\pi)^{d/2} \left(\mathcal{F}f\right)\left(\mathcal{F}g\right).$$

(iv) *Let $f \in L^1\left(\mathbb{R}^d\right)$ s.t. $\mathcal{F}f \in L^1\left(\mathbb{R}^d\right)$. Then:*

$$f = \overline{\mathcal{F}}\mathcal{F}f,$$

where $\overline{\mathcal{F}}$ is defined by $\overline{\mathcal{F}}g(x) = \overline{\mathcal{F}\overline{g}}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\xi} g(\xi)\ \mathrm{d}\xi$.

(v) *$L^1\left(\mathbb{R}^d\right) \cap L^2\left(\mathbb{R}^d\right)$ is dense in $L^2\left(\mathbb{R}^d\right)$, and $\mathcal{F}_{|L^1\left(\mathbb{R}^d\right) \cap L^2\left(\mathbb{R}^d\right)}$ is a linear isometry, so $\mathcal{F}$ can be extended uniquely to a linear isometry $\mathcal{F} : L^2\left(\mathbb{R}^d\right) \to L^2\left(\mathbb{R}^d\right)$, which satisfies $\mathcal{F}^{-1} = \overline{\mathcal{F}}$.*

Proposition 5.4.3. *The Schwartz class $\mathcal{S}\left(\mathbb{R}^d\right)$ is stable under the Fourier transform, and the operator $\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$ is an isomorphism.*

5.5 Tempered distributions

Definition 5.5.1 (Tempered distributions)**.** *The dual space $\mathcal{S}'\left(\mathbb{R}^d\right) = \mathcal{S}\left(\mathbb{R}^d\right)^*$ is called the space of* tempered distributions*.*

Proposition 5.5.2. *Since $\mathcal{D}\left(\mathbb{R}^d\right) \subseteq \mathcal{S}\left(\mathbb{R}^d\right) \subseteq \mathcal{E}\left(\mathbb{R}^d\right)$, we have:*

$$\mathcal{E}'\left(\mathbb{R}^d\right) \subseteq \mathcal{S}'\left(\mathbb{R}^d\right) \subseteq \mathcal{D}'\left(\mathbb{R}^d\right).$$

Proposition 5.5.3. *Let $\Lambda \in \mathcal{D}'\left(\mathbb{R}^d\right)$. Then Λ is tempered (i.e. Λ can be extended to a continous linear form $\mathcal{S}\left(\mathbb{R}^d\right) \to \mathbb{K}$)* iff*:*

$$\exists N \in \mathbb{N},\ \exists C \in \mathbb{R}_+,\ \forall \varphi \in \mathcal{D}\left(\mathbb{R}^d\right),\ \left|\langle \Lambda, \varphi \rangle\right| \leq C \left\|\varphi\right\|_N,$$

where $\|\cdot\|_N$ was defined in Definition 5.3.1.

Definition 5.5.4 (Differentiation and multiplication by a function with slow growth)**.** *Let $\Lambda \in \mathcal{S}'\left(\mathbb{R}^d\right)$ and $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$.*

(i) *If $\alpha \in \mathbb{N}^d$, then $\partial^\alpha \Lambda$ is defined by:*

$$\langle \partial^\alpha \Lambda, \varphi \rangle = (-1)^{|\alpha|} \langle \Lambda, \partial^\alpha \varphi \rangle.$$

(ii) *If $g \in \mathcal{C}^\infty\left(\mathbb{R}^d, \mathbb{K}\right)$ has slow growth, then $g\Lambda$ is defined by:*

$$\langle g\Lambda, \varphi \rangle = \langle \Lambda, g\varphi \rangle.$$

Hence, we define operators $\mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{S}'\left(\mathbb{R}^d\right)$.

5.6 Fourier transform of tempered distributions

Definition 5.6.1 (Fourier transform of a tempered distribution)**.** *If $\Lambda \in \mathcal{S}'\left(\mathbb{R}^d\right)$, then $\mathcal{F}\Lambda$ is the tempered distribution defined by:*

$$\langle \mathcal{F}\Lambda, \varphi \rangle = \langle \Lambda, \mathcal{F}\varphi \rangle,$$

for $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. In other words, $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{S}'\left(\mathbb{R}^d\right)$ is the adjoint operator of the isomorphism $\mathcal{F} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{S}\left(\mathbb{R}^d\right)$; it is also an isomorphism and its inverse is $\overline{\mathcal{F}} : \mathcal{S}'\left(\mathbb{R}^d\right) \to \mathcal{S}'\left(\mathbb{R}^d\right)$.

Proposition 5.6.2. *Let $f \in L^1\left(\mathbb{R}^d\right)$. Then $\Lambda_f \in \mathcal{S}'\left(\mathbb{R}^d\right)$, and:*

$$\mathcal{F}\Lambda_f = \Lambda_{\mathcal{F}f}.$$

Proof. This comes from the fact that if $f, g \in L^1\left(\mathbb{R}^d\right)$, then:

$$\int_{\mathbb{R}^d} \left(\mathcal{F}f\right)(\xi) \cdot g(\xi)\ \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \cdot \left(\mathcal{F}g\right)(x)\ \mathrm{d}x.$$

□

Example 5.6.3. *Let $\omega \in \mathbb{R}^d$ and consider $f_\omega : x \in \mathbb{R}^{\text{'}} \longrightarrow e^{i\omega \cdot x}$. Since f_ω is $\mathcal{C}^\infty$ and bounded, it defines a tempered distribution (even though f is neither L^1 nor L^2). And we have:*

$$\mathcal{F}f_\omega = (2\pi)^{d/2}\, \delta_\omega.$$

Proposition 5.6.4. *Let $\alpha, \beta \in \mathbb{N}^d$. For any $\Lambda \in \mathcal{S}'\left(\mathbb{R}^d\right)$, we have:*

$$\mathcal{F}\left(x^\beta \partial^\alpha \Lambda\right) = i^{|\alpha|+|\beta|} \partial^\beta \left(x^\alpha \mathcal{F}\Lambda\right).$$

5.7 Fourier transform of compactly supported distributions

Theorem 5.7.1. *Let $\Lambda \in \mathcal{S}'\left(\mathbb{R}^d\right)$ and $M \in \mathcal{E}'\left(\mathbb{R}^d\right) \subseteq \mathcal{S}'\left(\mathbb{R}^d\right)$.*

(i) *$\mathcal{F}M$ is a $\mathcal{C}^\infty$ function with slow growth.*

(ii) *$\Lambda * M \in \mathcal{S}'\left(\mathbb{R}^d\right)$.*

(iii) *$\mathcal{F}\left(\Lambda * M\right) = (2\pi)^{d/2}\mathcal{F}M \cdot \mathcal{F}\Lambda$.*

Proof. (i) For $x \in \mathbb{R}^d$, define $z_x : \xi \in \mathbb{R}^d \longmapsto (2\pi)^{-d/2}\exp\left(-ix \cdot \xi\right)$, and set:

$$f : x \in \mathbb{R}^d \longmapsto \langle M, z_x \rangle ,$$

which is meaningful because $z_x \in \mathcal{E}\left(\mathbb{R}^d\right)$ and $M \in \mathcal{E}'\left(\mathbb{R}^d\right)$. Show that f is $\mathcal{C}^\infty$ with slow growth. Now, for $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$, write:

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\varphi(x)\ \mathrm{d}x = \frac{1}{(2\pi)^{d/2}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^d} \sum_{x\in\varepsilon\mathbb{Z}^d} e^{-ix\cdot\xi}\varphi(x),$$

and use this to prove that $\langle \mathcal{F}M, \varphi \rangle = \langle f, \varphi \rangle$. Therefore, $\mathcal{F}M = f$. (ii) Use Proposition 5.5.3, as well as Theorem 4.4.6. (iii) Let $\varphi \in \mathcal{S}\left(\mathbb{R}^d\right)$. We have:

$$\langle \mathcal{F}\left(\Lambda * M\right), \varphi \rangle = \langle \Lambda, \psi \rangle ,$$

where $\psi(y) = \langle M, \mathcal{F}\varphi\left(\cdot + y\right)\rangle$. Now, $\mathcal{F}\varphi(x + y) = \mathcal{F}\theta_y(x)$, where $\theta_y(\xi) = e^{-iy\cdot\xi}\varphi(\xi)$. From this, we show that:

$$\psi(y) = (2\pi)^{d/2}\mathcal{F}\left(f\varphi\right)(y),$$

with $f = \mathcal{F}M$. As a consequence, $\langle \mathcal{F}\left(\Lambda * M\right), \varphi \rangle = (2\pi)^{d/2}\langle f\mathcal{F}\Lambda, \varphi \rangle$. □

Corollary 5.7.2. *If $M_1, M_2 \in \mathcal{E}'\left(\mathbb{R}^d\right)$, then $M_1 * M_2 = M_2 * M_1$.*

6 Sobolev spaces

6.1 Sobolev spaces of integral order

Remark 6.1.1. *Let $p \in [1, +\infty]$. If $f \in L^p(\Omega)$, then $f \in L^1_{\mathrm{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$.*

Vocabulary 6.1.2. *Let $p \in [1, +\infty]$. A distribution $\Lambda \in \mathcal{D}'(\Omega)$ is said to be in $L^p(\Omega)$ if there exists a $f \in L^p(\Omega)$ s.t. $\Lambda = \Lambda_f$.*

Proposition 6.1.3. *Assume that $p \in \left]1, +\infty\right]$. Then a distribution $\Lambda \in \mathcal{D}'(\Omega)$ is in $L^p(\Omega)$ iff:*

$$\exists C_\Lambda \in \mathbb{R}_+,\ \forall \varphi \in \mathcal{D}(\Omega),\ |\langle \Lambda, \varphi \rangle| \le C_\Lambda \|\varphi\|_{L^q},$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 6.1.4 ($W^{k,p}$)**.** *Let $k \in \mathbb{N}$, $p \in [1, +\infty]$. We define:*

$$W^{k,p}(\Omega) = \left\{u \in \mathcal{D}'(\Omega),\ \forall \alpha \in \mathbb{N}^d,\ \left(|\alpha| \le k \Longrightarrow \partial^\alpha u \in L^p(\Omega)\right)\right\} \subseteq L^p(\Omega) \subseteq \mathcal{D}'(\Omega).$$

$W^{k,p}(\Omega)$ is a vector space which we equip with the norm $\|\cdot\|_{W^{k,p}}$ defined by:

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p}^p\right)^{1/p}.$$

Corollary 6.1.5. *Assume that $p \in]1,+\infty]$. Then a distribution $\Lambda \in \mathcal{D}'(\Omega)$ is in $W^{k,p}(\Omega)$ iff:*

$$\exists C_\Lambda \in \mathbb{R}_+,\ \forall (\varphi_\alpha)_{\alpha\in\mathbb{N}^d} \in \mathcal{D}(\Omega)^{\mathbb{N}^d},\quad \left|\left\langle \Lambda, \sum_{|\alpha|\le k} \partial^\alpha \varphi_\alpha \right\rangle\right| \le C_\Lambda \sum_{|\alpha|\le k} \|\varphi_\alpha\|_{L^q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proposition 6.1.6. *Let $k \in \mathbb{N}$, $p \in [1,+\infty]$.*

(i) *$W^{k,p}(\Omega)$ is a Banach space.*

(ii) *If $p < +\infty$, then $W^{k,p}(\Omega)$ is separable.*

(iii) *If $1 < p < +\infty$, then $W^{k,p}(\Omega)$ is reflexive.*

(iv) *If $p = 2$, then $W^{k,p}(\Omega)$ is a Hilbert space.*

Proof. Define $I_k = \left\{\alpha \in \mathbb{N}^d,\ |\alpha| \le k\right\}$ and consider:

$$\mathcal{J} : \left| \begin{array}{rcl} W^{k,p}(\Omega) & \longrightarrow & L^p(I_k \times \Omega) \\ u & \longmapsto & (\partial^\alpha u)_{\alpha\in I_k} \end{array} \right. .$$

$\mathcal{J}$ is a linear isometric embedding, and $L^p(I_k\times\Omega)$ is a Banach space. Therefore, $W^{k,p}(\Omega)$ is isometric to $\operatorname{Im}\mathcal{J}$. Hence, for (i), (ii) and (iii), it suffices to show that $\operatorname{Im}\mathcal{J}$ is closed in $L^p(I_k\times\Omega)$. To prove it, consider $(u_n)_{n\in\mathbb{N}} \in W^{k,p}(\Omega)^{\mathbb{N}}$ s.t. $\mathcal{J}u_n \to g = (g_\alpha)_{\alpha\in I_k} \in L^p(I_k\times\Omega)$. Set $u = g_0$. We have $u_n \overset{L^p}{\to} u$, so $u_n \overset{\mathcal{D}'}{\rightharpoonup} u$. By continuity of ∂^α, we obtain $\partial^\alpha u_n \overset{\mathcal{D}'}{\rightharpoonup} \partial^\alpha u$ for all $\alpha \in I_k$. But since $\partial^\alpha u_n \overset{L^p}{\to} g_\alpha$, we also have $\partial^\alpha u_n \overset{\mathcal{D}'}{\rightharpoonup} g_\alpha$, which yields $g_\alpha = \partial^\alpha u$, and $g = \mathcal{J}u \in \operatorname{Im}\mathcal{J}$. For (iv), simply notice that $\|u\|_{W^{2,p}} = (u,u)_k$, where:

$$(u,v)_k = \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha u(x) \partial^\alpha \overline{v}(x)\ \mathrm{d}x.$$

□

Proposition 6.1.7. *Assume that $u \in W^{k,p}(\Omega)$ is compactly supported in Ω. Define:*

$$\tilde{u} : x \in \mathbb{R}^d \longmapsto \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

Then $\tilde{u} \in W^{k,p}\left(\mathbb{R}^d\right)$ and $\|\tilde{u}\|_{W^{k,p}} = \|u\|_{W^{k,p}}$.

Remark 6.1.8. *In Proposition 6.1.7, it is crucial to assume that u has compact support. For instance, take $u = 1$ on $\Omega =]0,1[\subseteq \mathbb{R}$. Then $u \in W^{k,p}(\Omega)$ for all k,p. However, $\tilde{u} = \mathbb{1}_{]0,1[}$, so $\frac{\mathrm{d}}{\mathrm{d}x}\tilde{u} = \delta_0 - \delta_1 \notin L^p(\mathbb{R})$ for all p.*

6.2 Approximation by smooth functions

Lemma 6.2.1. *Assume that $p \in [1,+\infty[$. Let $\rho \in \mathcal{D}\left(\mathbb{R}^d\right)$ s.t. $\int_{\mathbb{R}^d}\rho = 1$ and $\rho \ge 0$. Set $\rho_n(x) = n^d\rho(nx)$. Then, for every element $u \in W^{k,p}\left(\mathbb{R}^d\right)$, we have:*

(i) $\forall n \in \mathbb{N}$, $\rho_n * u \in \mathcal{C}^\infty\left(\mathbb{R}^d\right) \cap W^{k,p}\left(\mathbb{R}^d\right)$.

(ii) $\forall n \in \mathbb{N}$, $\operatorname{Supp}(\rho_n * u) \subseteq \operatorname{Supp} u + \frac{1}{n}\operatorname{Supp}\rho$.

(iii) $\|\rho_n * u - u\|_{W^{k,p}} \to 0$.

In particular, $\mathcal{C}^\infty\left(\mathbb{R}^d\right)\cap W^{k,p}\left(\mathbb{R}^d\right)$ *is dense in* $W^{k,p}\left(\mathbb{R}^d\right)$*.*

Proof. Note that $\mathcal{D}\left(\mathbb{R}^d\right)*L^p\left(\mathbb{R}^d\right)\subseteq\mathcal{C}^\infty\left(\mathbb{R}^d\right)\cap L^p\left(\mathbb{R}^d\right)$. Using the fact that $\partial^\alpha\left(\rho_n*u\right)=\rho_n*(\partial^\alpha u)$ for $|\alpha|\leq k$, we obtain $\rho_n*u\in\mathcal{C}^\infty\left(\mathbb{R}^d\right)*W^{k,p}\left(\mathbb{R}^d\right)$ and $\|\partial^\alpha\left(\rho_n*u\right)\|_{W^{k,p}}\leq\|\partial^\alpha u\|_{W^{k,p}}$; therefore $\|\rho_n*u\|_{W^{k,p}}\leq\|u\|_{W^{k,p}}$. Moreover, it is clear that $\operatorname{Supp}\left(\rho_n*u\right)\subseteq\operatorname{Supp}u+\frac{1}{n}\operatorname{Supp}\rho$. Finally, write:

$$\partial^\alpha\left(\rho_n*u\right)(x)-\partial^\alpha u(x)=\int_{\mathbb{R}^d}\rho_n(y)\left(\partial^\alpha u(x-y)-\partial^\alpha u(x)\right)\mathrm{d}y,$$

and use this to show that $\|\partial^\alpha\left(\rho_n*u\right)-\partial^\alpha u\|_{L^p}\to 0$. □

Theorem 6.2.2. *Assume that* $p\in[1,+\infty[$*. Then* $\mathcal{C}^\infty(\Omega)\cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$*.*

Proof. Choose a locally finite covering of $\Omega:\ \Omega=\bigcup_{j\in\mathbb{N}}\omega_j$, with $\overline{\omega}_j\Subset\Omega$. Now, choose a partition of unity $\left(\Psi_j\right)_{j\in\mathbb{N}}\in\mathcal{D}(\Omega)^{\mathbb{N}}$ s.t. $\operatorname{Supp}\Psi_j\subseteq\omega_j$, $\Psi_j\geq 0$ and $\sum_{j\in\mathbb{N}}\Psi_j=1$. For $u\in W^{k,p}(\Omega)$, set $u_j=\Psi_j u$ for all $j\in\mathbb{N}$ and extend u_j by 0 to a function $\widetilde{u}_j\in W^{k,p}\left(\mathbb{R}^d\right)$, as in Proposition 6.1.7. Use Lemma 6.2.1 to find $v_j\in\mathcal{C}^\infty\left(\mathbb{R}^d\right)\cap W^{k,p}\left(\mathbb{R}^d\right)$ with:

$$\|v_j-\widetilde{u}_j\|_{W^{k,p}}\leq 2^{-j}\varepsilon,$$

and $\operatorname{Supp}v_j\subseteq\omega_j$. Now, set $v=\sum_{j\in\mathbb{N}}v_{j|\Omega}\in\mathcal{C}^\infty(\Omega)$; check that $v\in W^{k,p}(\Omega)$ and $\|v-u\|_{W^{k,p}}\leq 2\varepsilon$. □

Remark 6.2.3. *Using Theorem 6.2.2, in the case where* $p\in[1,+\infty[$*, we may define* $W^{k,p}(\Omega)$ *as the completion of the space* $X^{k,p}(\Omega)=\{u\in\mathcal{C}^\infty(\Omega),\ \|u\|_{W^{k,p}}<+\infty\}$ *for* $\|\cdot\|_{W^{k,p}}$*.*

Proposition 6.2.4. *If* $p\in[1,+\infty[$*, then* $\mathcal{D}\left(\mathbb{R}^d\right)$ *is dense in* $W^{k,p}\left(\mathbb{R}^d\right)$*.*

Proof. Prove that $W^{k,p}\left(\mathbb{R}^d\right)\cap\mathcal{E}'\left(\mathbb{R}^d\right)$ is dense in $W^{k,p}\left(\mathbb{R}^d\right)$ (by using a function $\psi\in\mathcal{D}\left(\mathbb{R}^d\right)$ s.t. $\psi=1$ on $B_1=\left\{x\in\mathbb{R}^d,\ \|x\|\leq 1\right\}$ and by considering $\psi_n(x)=\psi\left(\frac{x}{n}\right)$) and apply Lemma 6.2.1. □

Definition 6.2.5 ($W_0^{k,p}$)**.** *For* $k\in\mathbb{N}$ *and* $p\in[1,+\infty]$*, define* $W_0^{k,p}(\Omega)$ *to be the closure of* $\mathcal{D}(\Omega)$ *in* $W^{k,p}(\Omega)$*.*

Corollary 6.2.6. *If* $p\in[1,+\infty[$ *and* $\Omega=\mathbb{R}^d$*, then* $W_0^{k,p}\left(\mathbb{R}^d\right)=W^{k,p}\left(\mathbb{R}^d\right)$*.*

6.3 Extension by zero

Notation 6.3.1. *If* u *is a function defined (a.e.) on* Ω*, and* $\Omega_1\supseteq\Omega$*, we set:*

$$\widetilde{u}:x\in\Omega_1\longmapsto\begin{cases}u(x) & \text{if } x\in\Omega\\ 0 & \text{otherwise}\end{cases}.$$

Proposition 6.3.2. *Let* $\Omega\subseteq\Omega_1$ *be open subsets of* $\mathbb{R}^d$*. If* $u\in W_0^{k,p}(\Omega)$*, then* $\widetilde{u}\in W_0^{k,p}(\Omega_1)$*.*

Proof. Note that there exists a sequence $\left(\varphi_n\right)_{n\in\mathbb{N}}\in\mathcal{D}(\Omega)^{\mathbb{N}}$ s.t. $\|\varphi_n-u\|_{W^{k,p}}\to 0$. Now, $\widetilde{\varphi}_n\in\mathcal{D}(\Omega_1)$ and since $\|\widetilde{\varphi}_m-\widetilde{\varphi}_n\|_{W^{k,p}}=\|\varphi_m-\varphi_n\|_{W^{k,p}}$, $\left(\widetilde{\varphi}_n\right)_{n\in\mathbb{N}}$ is Cauchy, so it converges to a limit $v\in W^{k,p}(\Omega)$. Show that $v=\widetilde{u}$ in $\mathcal{D}'(\Omega_1)$ by computing $\langle v,\theta\rangle$ for $\theta\in\mathcal{D}(\Omega_1)$; hence $\widetilde{u}\in W_0^{k,p}(\Omega_1)$. □

Notation 6.3.3. *Write* $\mathbb{R}_+^d=\mathbb{R}^{d-1}\times\,]0,+\infty[$ *and* $\mathbb{R}_-^d=\mathbb{R}^{d-1}\times\,]-\infty,0[$*.*

Proposition 6.3.4. *Assume that* $p\in[1,+\infty[$*. Let* $u\in W^{k,p}\left(\mathbb{R}_+^d\right)$*. Then:*

$$\widetilde{u}\in W^{k,p}\left(\mathbb{R}^d\right)\Longleftrightarrow u\in W_0^{k,p}\left(\mathbb{R}_+^d\right).$$

Proof. It suffices to prove ($\Rightarrow$). Therefore, suppose that $\tilde{u} \in W^{k,p}\left(\mathbb{R}^d\right)$. For $\varepsilon > 0$, define $u_\varepsilon(x) = \tilde{u}\left(x - \varepsilon e_d\right)$, where e_d is the d-th vector in the canonical basis of $\mathbb{R}^d$. We have $\operatorname{Supp} u_\varepsilon \subseteq \mathbb{R}^{d-1} \times [\varepsilon, +\infty[$ and $\|u_\varepsilon - \tilde{u}\|_{W^{k,p}} \to 0$. Because the subspace $W_0^{k,p}\left(\mathbb{R}_+^d\right)$ is closed, it suffices to prove that $u_{\varepsilon|\mathbb{R}_+^d} \in W_0^{k,p}\left(\mathbb{R}_+^d\right)$. From now on, ε is fixed. Approximate u_ε by functions in $\varphi_n \in \mathcal{D}\left(\mathbb{R}^d\right)$, choose a function $\theta \in \mathcal{C}^\infty(\mathbb{R})$ s.t. $\theta_{|\mathbb{R}_-} = 0$ and $\theta_{|[\varepsilon,+\infty[} = 1$ and consider $\psi_n(x_1, \ldots, x_d) = \theta(x_d)\, \varphi_n(x_1, \ldots, x_d)$; show that $\left\|(\psi_n - u_\varepsilon)_{|\mathbb{R}_+^d}\right\|_{W^{k,p}} \to 0$. ☐

6.4 Existence of a right inverse of the restriction operator

Remark 6.4.1. *A natural question is to find an operator $P : W^{k,p}(\Omega) \to W^{k,p}\left(\mathbb{R}^d\right)$ that is linear and continuous and s.t. $R \circ P = \mathrm{id}_{W^{k,p}(\Omega)}$, where $R : u \in W^{k,p}\left(\mathbb{R}^d\right) \longmapsto u_{|\Omega} \in W^{k,p}(\Omega)$. If $k = 0$, it suffices to take the extension by 0.*

Theorem 6.4.2. *Assume that $p \in [1, +\infty[$ and $\Omega = \mathbb{R}_+^d$. Then there exists an extension operator $P : W^{k,p}\left(\mathbb{R}_+^d\right) \to W^{k,p}\left(\mathbb{R}^d\right)$ that is a right inverse of the restriction operator.*

Proof. For $u \in W^{k,p}\left(\mathbb{R}_d^+\right)$, define:

$$Pu(x_1, \ldots, x_d) = \begin{cases} u(x_1, \ldots, x_d) & \text{if } x_d > 0 \\ \sum_{j=1}^{k+1} a_j u(x_1, \ldots, x_{d-1}, -jx_d) & \text{if } x_d \leq 0 \end{cases},$$

where $a_1, \ldots, a_{k+1}$ are determined by the following Vandermonde linear system:

$$\forall m \in \{0, \ldots, k\}, \sum_{j=1}^{k+1} (-j)^m a_j = 1.$$

It is clear that P is a linear map satisfying $R \circ P = \mathrm{id}_{W^{k,p}\left(\mathbb{R}_+^d\right)}$; it remains to show that $\operatorname{Im} P \subseteq W^{k,p}\left(\mathbb{R}_+^d\right)$ and that P is continuous. ☐

Theorem 6.4.3. *Assume that $p \in [1, +\infty[$ and let Ω be a bounded domain with a $\mathcal{C}^k$ boundary. Then there exists an extension operator $P : W^{k,p}(\Omega) \to W^{k,p}\left(\mathbb{R}^d\right)$ that is a right inverse of the restriction operator.*

6.5 Embeddings of distribution spaces

Definition 6.5.1 (Distribution space)**.** *A* distribution space *is a Banach space E that is included in $\mathcal{D}'(\Omega)$ s.t. for all $\varphi \in \mathcal{D}(\Omega)$, the map $u \in E \longmapsto \langle u, \varphi \rangle \in \mathbb{K}$ is continuous.*

Remark 6.5.2. *If F is a distribution space and E is a closed subspace of F s.t. the inclusion $E \subseteq F$ is continuous, then E is also a distribution space.*

Example 6.5.3.

(i) *$L^p(\Omega)$ is a distribution space for $p \in [1, +\infty]$.*

(ii) *$W^{k,p}(\Omega)$ is a distribution space for $p \in [1, +\infty]$.*

(iii) *$\mathcal{C}^0\left(\overline{\Omega}\right) \cap L^\infty\left(\overline{\Omega}\right)$ is a distribution space, equipped with $\|\cdot\|_{L^\infty}$.*

(iv) *$\mathcal{C}^\alpha\left(\overline{\Omega}\right) = \left\{u \in \mathcal{C}^0\left(\overline{\Omega}\right) \cap L^\infty\left(\overline{\Omega}\right), \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty\right\}$ is a distribution space for $\alpha \in]0, 1[$, equipped with $\|\cdot\|_{\mathcal{C}^\alpha}$ defined by:*

$$\|f\|_{\mathcal{C}^\alpha} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Lemma 6.5.4. *Let E and F be two distribution spaces over Ω and assume that $E \subseteq F$. Then the inclusion $E \subseteq F$ is continuous.*

Proof. Consider $X = \{(u,u),\ u \in E\} \subseteq E \times F$; X is the graph of the inclusion map $E \subseteq F$. By the Closed Graph Theorem (Theorem 1.2.9), it suffices to prove that X is a closed subspace of $E \times F$. Hence, let $(u_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ s.t. $(u_n, u_n) \to (u, v)$ in $E \times F$. Then $u_n \to u$ in E, so $u_n \overset{*}{\rightharpoonup} u$ in $\mathcal{D}'$. Likewise, $u_n \overset{*}{\rightharpoonup} v$ in $\mathcal{D}'$, so $u = v$ and $(u, v) \in X$. □

Lemma 6.5.5. *Let E and F be two distribution spaces over Ω and let D be a dense subspace of E s.t. $D \subseteq F$. Assume that there exists $C \in \mathbb{R}_+$ s.t.*

$$\forall u \in D,\ \|u\|_F \leq C \|u\|_E .$$

Then $E \subseteq F$, with a continuous inclusion.

Proof. Let $u \in E$. Then there exists $(u_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ s.t. $u_n \to u$ in E. The sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy in E, and therefore in F because $\|u_p - u_q\|_F \leq C \|u_p - u_q\|_E$ for all $p, q \in \mathbb{N}$. Since F is a Banach space, there exists $v \in F$ s.t. $u_n \to v$ in F. Now, $u_n \overset{*}{\rightharpoonup} u$ in $\mathcal{D}'$ and $u_n \overset{*}{\rightharpoonup} v$ in $\mathcal{D}'$ so $u = v \in F$. Moreover, $\|u\|_F = \lim_{n\to+\infty} \|u_n\|_F \leq C \lim_{n\to+\infty} \|u_n\|_E = \|u\|_E$. □

6.6 Sobolev embeddings

Theorem 6.6.1 (Morrey's Theorem)**.** *Let Ω be either $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a bounded domain with a $\mathcal{C}^1$ boundary. Assume that $d < p < +\infty$. Then:*

$$W^{1,p}(\Omega) \subseteq \mathcal{C}^{\alpha}\left(\overline{\Omega}\right),$$

with $\alpha = 1 - \frac{d}{p} \in]0,1[$.

Proof. We only prove the case where $\Omega = \mathbb{R}^d$ (for the other cases, use the extension operators given by Theorems 6.4.2 and 6.4.3). Let $E = W^{1,p}\left(\mathbb{R}^d\right)$, $F = \mathcal{C}^{\alpha}\left(\mathbb{R}^d\right)$ and $D = \mathcal{D}\left(\mathbb{R}^d\right) \subseteq E \cap F$. According to Proposition 6.2.4, D is dense in E. Therefore, by Lemma 6.5.5, it suffices to prove the existence of a constant $C \in \mathbb{R}_+$ s.t. $\forall u \in D$, $\|u\|_{\mathcal{C}^\alpha} \leq C \|u\|_{W^{1,p}}$. To do this, show firstly that if B_r is any (closed) ball of radius r containing a point $x \in \mathbb{R}^d$, then:

$$\left| u(x) - \frac{1}{\lambda(B_r)} \int_{B_r} u(y)\ \mathrm{d}y \right| \leq \underbrace{\frac{2}{\lambda(B_1)^{1/p}} \left(\int_0^1 t^{-d/p}\ \mathrm{d}t \right)}_{C_1} \|\nabla u\|_{L^p} r^{\alpha}.$$

Hence, if $x, y \in \mathbb{R}^d$ and if $r = \frac{1}{2}|x - z|$, by choosing B_r to be the ball with centre $\frac{x+y}{2}$ and with radius r, we obtain:

$$|u(x) - u(z)| \leq 2^{1-\alpha} C_1 \|\nabla u\|_{L^p} |x - z|^{\alpha}.$$

Next, we need to show that u is bounded. To do this, note that, if B_1 is any (closed) ball of radius 1, then:

$$\left| \frac{1}{\lambda(B_1)} \int_{B_1} u(y)\ \mathrm{d}y \right| \leq \frac{1}{\lambda(B_1)^{1/p}} \|u\|_{L^p}.$$

Using this and the previous inequalities, we obtain an upper bound for $\|u\|_{L^\infty}$, and then for $\|u\|_{\mathcal{C}^\alpha}$. □

Theorem 6.6.2 (Gagliardo–Nirenberg Theorem)**.** *Let Ω be either $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a bounded domain with a $\mathcal{C}^1$ boundary. Assume that $1 \leq p < d$. Then:*

$$W^{1,p}(\Omega) \subseteq L^{p^*}(\Omega),$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

Proof. The strategy is exactly the same as for Morrey's Theorem (Theorem 6.6.1): we assume that $\Omega = \mathbb{R}^d$ and we work with functions in $\mathcal{D}\left(\mathbb{R}^d\right)$. Firstly, assume that $p = 1$. In this case, prove that, for any compactly supported $\mathcal{C}^1$ function u, we have $\|u\|_{L^{d/(d-1)}} \leq \|u\|_{W^{1,1}}$. For the general case, fix $s = \frac{p(d-1)}{d-p} > 1$, so that $p^* = \frac{sd}{d-1}$. Note that $t \mapsto |t|^s$ is a $\mathcal{C}^1$ function. Thus, $|u|^s$ is a compactly supported $\mathcal{C}^1$ function. Apply the previous case and obtain the desired inequality. □

Corollary 6.6.3. *Let Ω be either $\mathbb{R}^d$, $\mathbb{R}^d_+$ or a bounded domain with a $\mathcal{C}^1$ boundary. Assume that $1 \leq p < d$. Then:*

$$\forall r \in [p, p^*]\,,\ W^{1,p}(\Omega) \subseteq L^r(\Omega)\,,$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

6.7 Compact embeddings

Definition 6.7.1 (Compact embedding)**.** *Let E and F be two distribution spaces s.t. $E \subseteq F$. We say the the embedding $E \subseteq F$ is* compact *if the unit ball B_E of E is relatively compact in F. Equivalently, from every sequence $(u_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ that is bounded in E, we can extract a subsequence which converges in F.*

Remark 6.7.2. *If E is of infinite dimension, then the embedding $E \subseteq E$ is never compact.*

Theorem 6.7.3. *Let Ω be a bounded domain in $\mathbb{R}^d$ with a $\mathcal{C}^1$ boundary.*

(i) *If $p > d$ and $0 \leq \beta < 1 - \frac{d}{p}$, then the embedding $W^{1,p}(\Omega) \subseteq \mathcal{C}^\beta\left(\overline{\Omega}\right)$ (given by Theorem 6.6.1) is compact.*

(ii) *If $p < d$ and $1 \leq r < p^*$, then the embedding $W^{1,p}(\Omega) \subseteq L^r(\Omega)$ (given by Corollary 6.6.3) is compact.*

6.8 Sobolev spaces of fractional order

Lemma 6.8.1. *Let $k \in \mathbb{N}$. Then:*

$$W^{k,2}\left(\mathbb{R}^d\right) = \left\{u \in \mathcal{S}'\left(\mathbb{R}^d\right),\ \left(\left(1+|\xi|^2\right)^{k/2}\mathcal{F}u\right) \in L^2\left(\mathbb{R}^d\right)\right\}.$$

In addition, $\|\cdot\|_{W^{k,2}}$ is equivalent to the norm $\|\cdot\|$ defined by:

$$\|u\| = \left\|\left(1+|\xi|^2\right)^{k/2}\mathcal{F}u\right\|_{L^2}.$$

Definition 6.8.2 ($H^s\left(\mathbb{R}^d\right)$)**.** *For $s \in \mathbb{R}$, define:*

$$H^s\left(\mathbb{R}^d\right) = \left\{u \in \mathcal{S}'\left(\mathbb{R}^d\right),\ \left(\left(1+|\xi|^2\right)^{s/2}\mathcal{F}u\right) \in L^2\left(\mathbb{R}^d\right)\right\}.$$

$H^s\left(\mathbb{R}^d\right)$ can also be denoted by $W^{s,2}\left(\mathbb{R}^d\right)$. We equip it with the scalar product $((\cdot,\cdot))_{H^s}$ defined by:

$$((u,v))_{H^s} = \int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^s\overline{\mathcal{F}u(\xi)}\mathcal{F}v(\xi)\ \mathrm{d}\xi,$$

(in the case where $\mathbb{K} = \mathbb{R}$, we have to take the real part because the Fourier transform is not necessarily real-valued). Thus, $H^s\left(\mathbb{R}^d\right)$ is a Hilbert space.

Proposition 6.8.3.

(i) *For $k \in \mathbb{N}$, the new definition of $H^k\left(\mathbb{R}^d\right) = W^{k,2}\left(\mathbb{R}^d\right)$ agrees with the original one.*

(ii) *If $s \leq \sigma$, then $H^s\left(\mathbb{R}^d\right) \supseteq H^\sigma\left(\mathbb{R}^d\right)$.*

(iii) *$H^0\left(\mathbb{R}^d\right) = L^2\left(\mathbb{R}^d\right)$.*

(iv) *If $s \geq 0$, $H^s\left(\mathbb{R}^d\right) \subseteq L^2\left(\mathbb{R}^d\right)$. In particular, the elements of $H^s\left(\mathbb{R}^d\right)$ are functions.*

(v) *For $s \in \mathbb{R}$, $\mathcal{D}\left(\mathbb{R}^d\right)$ is dense in $H^s\left(\mathbb{R}^d\right)$.*

Definition 6.8.4 ($H^s(\Omega)$)**.** *If Ω is a domain of $\mathbb{R}^d$ with a $\mathcal{C}^1$ boundary, we define:*

$$H^s(\Omega) = \left\{u_{|\Omega},\, u \in H^s\left(\mathbb{R}^d\right)\right\},$$

and we equip this space with the norm $\|\cdot\|_{H^s}$ defined by:

$$\|v\|_{H^s} = \inf_{\substack{u \in H^s\left(\mathbb{R}^d\right) \\ u_{|\Omega} = v}} \|u\|_{H^s}.$$

Hence, $H^s(\Omega)$ is a Hilbert space.

6.9 Trace theorems

Theorem 6.9.1. *Let $s \in \left]\frac{1}{2}, +\infty\right[$. Then the linear map $u \in \mathcal{D}\left(\mathbb{R}^d\right) \longmapsto u_{|\{x_d=0\}} \in \mathcal{D}\left(\mathbb{R}^{d-1}\right)$ extends uniquely to a continuous linear operator $\gamma : H^s\left(\mathbb{R}^d\right) \to H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)$. In addition, there exists a continuous linear operator $R : H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right) \to H^s\left(\mathbb{R}^d\right)$ s.t. $\gamma \circ R = \mathrm{id}$. In particular, γ is surjective (and open).*

Proof. For the existence and uniqueness of γ, by density of $\mathcal{D}\left(\mathbb{R}^d\right)$ in $H^s\left(\mathbb{R}^d\right)$, it suffices to prove the existence of $C \in \mathbb{R}_+$ s.t.

$$\forall u \in \mathcal{D}\left(\mathbb{R}^d\right),\ \left\|u_{|\{x_d=0\}}\right\|_{H^{s-\frac{1}{2}}} \leq C \|u\|_{H^s}.$$

For the existence of R, choose $\theta \in \mathcal{D}(\mathbb{R})$ s.t. $\int_{\mathbb{R}} \theta = 1$. Now, for $g \in H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)$, define:

$$h : \xi \in \mathbb{R}^d \longmapsto \sqrt{2\pi} \cdot \theta\left(\frac{\xi_d}{\sqrt{1+|\xi'|^2}}\right) \cdot \frac{1}{\sqrt{1+|\xi'|^2}} \cdot \mathcal{F}g(\xi'),$$

with $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and let $Rg = \mathcal{F}^{-1}h \in \mathcal{S}'\left(\mathbb{R}^d\right)$. Check that $Rg \in H^s\left(\mathbb{R}^d\right)$ and that the linear map $R : H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right) \to H^s\left(\mathbb{R}^d\right)$ thus defined is continuous, then show that $\gamma \circ R = \mathrm{id}$. □

Remark 6.9.2. *In Theorem 6.9.1, the lifting R is not unique.*

Corollary 6.9.3. *Let $s \in \left]\frac{1}{2}, +\infty\right[$. Then there exists a continuous linear operator $\gamma_0 : H^s\left(\mathbb{R}^d_+\right) \to H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right)$ s.t. the following diagram commutes:*

$$\begin{array}{ccc} H^s\left(\mathbb{R}^d\right) & \xrightarrow{\ \gamma\ } & H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right) \\ \big\downarrow & \nearrow_{\gamma_0} & \\ H^s\left(\mathbb{R}^d_+\right) & & \end{array}$$

where γ is as in Theorem 6.9.1 and $H^s\left(\mathbb{R}^d\right) \to H^s\left(\mathbb{R}^d_+\right)$ is the restriction. Moreover, there exists a continuous linear operator $R : H^{s-\frac{1}{2}}\left(\mathbb{R}^{d-1}\right) \to H^s\left(\mathbb{R}^d_+\right)$ s.t. $\gamma_0 \circ R = \mathrm{id}$. In particular, γ_0 is surjective (and open).

References

[1] H. Brezis. *Analyse fonctionnelle*.