Advanced Geometry

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1.1 Generalities

Definition 1.1.1 (Chart, transition map, atlas)**.** *Let X be a topological space; let $n \in \mathbb{N}$.*

(i) *A n-*dimensional chart *on X is a homeomorphism $\varphi : U \to V$, where U is an open set in X and V is an open set in $\mathbb{R}^n$. We say that φ is* centred *at $x \in U$ if $\varphi(U) = 0$.*

(ii) *If $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are two n-dimensional charts on X, the associated* transition maps *are $\psi_{1,2} = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ and its inverse map $\psi_{2,1} = \varphi_1 \circ \varphi_2^{-1}$.*

(iii) *A n-*dimensional atlas *on X is a collection $(\varphi_i)_{i\in I} \in \prod_{i\in I} V_i^{U_i}$ of n-dimensional charts s.t. the $(U_i)_{i\in I}$ cover X.*

Definition 1.1.2 (Locally Euclidean space)**.** *A topological space X admitting a n-dimensional atlas is said to be* locally Euclidean of dimension *n (it is true, but by no means obvious, that the dimension of X is well-defined).*

Definition 1.1.3 (Compatibility)**.** *Let X be a topological space.*

(i) *Two charts on X are said to be* (smoothly) compatible *if the associated transition maps are smooth (i.e. $\mathcal{C}^\infty$).*

(ii) *A n-dimensional atlas on X is said to be* smooth *if all its transition maps are smooth.*

(iii) *Two smooth atlases are said to be* (smoothly) compatible *if all the transition maps between the two atlases are smooth. This is an equivalence relation of atlases.*

Definition 1.1.4 (Differentiable manifold)**.** *A* n-dimensional differentiable manifold *is a topological space M equipped with a maximal n-dimensional smooth atlas (or with an equivalence class of smooth atlases of dimension n, which amounts to the same), with the following two properties:*

(i) *M is Hausdorff.*

(ii) *M admits a smoothly compatible countable atlas.*

Remark 1.1.5.

(i) *The fact that the dimension of a differentiable manifold is well-defined is a consequence of the fact that if there exists a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^m$, then $n = m$.*

(ii) *In the definition, one can replace "smooth" by $\mathcal{C}^k$ ($k \in \mathbb{N}^*$) or "real-analytic" to obtain a notion of $\mathcal{C}^k$-manifolds or real-analytic manifolds.*

(iii) *If $n = 2m$, $m \in \mathbb{N}^*$, one can identify $\mathbb{R}^n \simeq \mathbb{C}^m$ and require the transition maps to be holomorphic functions. One obtains a notion of complex (analytic) manifolds, with a completely different theory.*

Definition 1.1.6 (Isomorphism of differentiable manifolds)**.** *Two differentiable manifolds M_1 and M_2 are said to be* $\mathcal{C}^\infty$-diffeomorphic *or* isomorphic *if there exists a bijection $\varphi : M_1 \to M_2$ taking the maximal atlas of M_1 (or a smooth atlas of M_1) to the maximal atlas of M_2 (or a smooth atlas of M_2). In this case, φ is called a* $\mathcal{C}^\infty$-diffeomorphism*.*

Example 1.1.7. *The following spaces are n-dimensional differentiable manifolds:*

(i) *Open subsets of $\mathbb{R}^n$,*

(ii) *The n-sphere $\mathbb{S}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1},\ \sum_{i=0}^n x_i^2 = 1\}$.*

(iii) *The n-torus $\mathbb{T}^n = (\mathbb{S}^1)^n$.*

(iv) *The real projective space of dimension n: $\mathbb{PR}^n = (\mathbb{R}^{n+1} \backslash \{0\}) / \mathbb{R}^*$.*

(v) *The abstract Möbius band $\mathbb{M} = (\mathbb{R} \times]-1, 1[) / \mathbb{Z}$, where the action $\mathbb{Z} \curvearrowright \mathbb{R} \times]-1, 1[$ is given by $n \cdot (x, y) = (x + n, (-1)^n y)$.*

1.2 Submanifolds of $\mathbb{R}^N$

Definition 1.2.1 (Submanifold of $\mathbb{R}^N$)**.** *Recall that a n-dimensional* submanifold of $\mathbb{R}^N$ *is a subset $M \subseteq \mathbb{R}^N$ s.t. every $x \in M$ admits a* straightening chart*, i.e. a diffeomorphism $\Phi : \hat{U} \subseteq \mathbb{R}^n \longrightarrow \hat{V} \subseteq \mathbb{R}^n$ s.t. $\Phi\left(M \cap \hat{U}\right) = (\mathbb{R}^n \times \{0\}) \cap \hat{V}$. To such a straightening chart, we associate the map $\varphi = \mathrm{pr}_{\mathbb{R}^n} \circ \Phi_{|M \cap \hat{U}}$. This is a homeomorphism, and therefore a chart on M.*

Proposition 1.2.2. *Every n-dimensional submanifold of $\mathbb{R}^N$ is a n-dimensional differentiable manifold.*

Definition 1.2.3. *Differentiable manifolds which are isomorphic to submanifolds of $\mathbb{R}^N$ are said to* embed *in $\mathbb{R}^N$.*

Remark 1.2.4. *Whitney proved that every n-dimensional manifold embeds in $\mathbb{R}^{2n}$.*

2 Differentiable maps and tangent bundle

2.1 Differentiable maps

Definition 2.1.1 (Differentiable map)**.** *Let $f : M \to N$ be a map between two manifolds and let $x \in M$.*

(i) *The map f is said to be* differentiable (resp. $\mathcal{C}^k$, smooth) at x *if there exist charts φ for M at x and ψ for N at $f(x)$ s.t. $\psi \circ f \circ \varphi^{-1}$ is differentiable (resp. $\mathcal{C}^k$, smooth) at $\varphi(x)$. Equivalently, this condition is true for all such charts φ and ψ.*

(ii) *The map f is said to be* differentiable (resp. $\mathcal{C}^k$, smooth) on M *if it is differentiable (resp. $\mathcal{C}^k$, smooth) at every point of M. Equivalently, M and N admit respective atlases $((U_i, \varphi_i))_{i\in I}$ and $((V_i, \psi_i))_{i\in I}$ s.t. $f(U_i) \subseteq V_i$ and $\psi_i \circ f \circ \varphi_i^{-1}$ is differentiable (resp. $\mathcal{C}^k$, smooth) on $\varphi(U_i)$ for all $i \in I$.*

Notation 2.1.2. *If M and N are two manifolds and $k \in \mathbb{N} \cup \{\infty\}$, we write $\mathcal{C}^k(M, N)$ for the set of $\mathcal{C}^k$ maps from M to N.*

Proposition 2.1.3. *Let $f : M \to N$ be a map between two manifolds.*

(i) *If $N = \mathbb{R}$, f is smooth* iff *$f \circ \varphi^{-1}$ is smooth for all φ in some atlas on M.*

(ii) *If $N = \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ is smooth* iff *$f_1, \ldots, f_n$ are smooth.*

(iii) *If M is an open subset of $\mathbb{R}^n$, f is smooth* iff *$\psi \circ f$ is smooth for all ψ in some atlas on N.*

(iv) *If $f \in \mathcal{C}^k(M, N)$ and U is an open subset of M, then $f_{|U} \in \mathcal{C}^k(U, N)$.*

(v) *If $(U_i)_{i\in I}$ is an open cover of M and $\forall i \in I$, $f_{|U_i} \in \mathcal{C}^k(U_i, N)$, then $f \in \mathcal{C}^k(M, N)$.*

(vi) *If $f \in \mathcal{C}^k(M, N)$ and $g \in \mathcal{C}^k(N, P)$, then $g \circ f \in \mathcal{C}^k(M, P)$.*

(vii) *The set $\mathcal{C}^k(M, \mathbb{R})$ is a unital subalgebra of $\mathcal{C}^0(M, \mathbb{R})$.*

2.2 Diffeomorphisms between manifolds

Definition 2.2.1 (Diffeomorphism)**.** *A map $f : M \to N$ between manifolds is said to be a* diffeomorphism *if it is bijective and f and f^{-1} are smooth. This is equivalent to f being an isomorphism of manifolds (c.f. Definition 1.1.6).*

Definition 2.2.2 (Group of diffeomorphisms)**.** *If M is a manifold, then the set of diffeomorphisms of M is a group for composition, denoted by* $\mathrm{Diff}(M)$.

Remark 2.2.3. *Let M and N be two manifolds. Filipkiewicz proved that if $\Phi : \mathrm{Diff}(M) \to \mathrm{Diff}(N)$ is a group isomorphism, then there exists a unique diffeomorphism $\varphi : M \to N$ s.t. $\forall f \in \mathrm{Diff}(M)$, $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$. In particular, $\mathrm{Diff}(M)$ (as a group) determines M (as a manifold).*

Proposition 2.2.4. *Let M be a connected manifold.*

(i) $\mathrm{Diff}(M)$ *is transitive.*

(ii) *If in addition $\dim M \geq 2$, then $\mathrm{Diff}(M)$ is k-transitive for all $k \in \mathbb{N}^*$, i.e. if $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are two k-uples of distinct points in M, then there exists $f \in \mathrm{Diff}(M)$ s.t. $\forall i \in \{1, \ldots, k\}$, $y_i = f(x_i)$.*

2.3 Tangent bundle

Definition 2.3.1 (Tangent space at a point). *Let M be a n-dimensional manifold and let $x \in M$. We denote by S the set of maps $\gamma :]-\varepsilon, +\varepsilon[\to M$ for $\varepsilon > 0$ s.t. $\gamma(0) = x$ and γ is differentiable at 0 (here, we could suppose that γ is smooth and this would lead to the same notion of tangent space). We endow S with the equivalence relation $\mathcal{R}$ defined by $\gamma_1 \mathcal{R} \gamma_2$* iff *$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for some (or for all) chart φ at x. A* tangent vector *of M at x is an equivalence class of S for $\mathcal{R}$, i.e. an element of $S/\mathcal{R}$. The* tangent space *T_xM of M at x is the set of all tangent vectors; it is equipped with a linear structure defined as follows. We choose a chart φ at x and we note that the map $[\gamma] \in S/\mathcal{R} \longmapsto (\varphi \circ \gamma)'(0) \in \mathbb{R}^n$ is a bijection, denoted by $\mathrm{d}\varphi_x$; we use this bijection to transport the linear structure from $\mathbb{R}^n$ to T_xM by setting:*

$$\forall (\lambda_1, \lambda_2) \in \mathbb{R}^2,\ \forall (v_1, v_2) \in T_xM^2,\ \lambda_1 v_1 + \lambda_2 v_2 = \mathrm{d}\varphi_x^{-1} \left(\lambda_1 \mathrm{d}\varphi_x (v_1) + \lambda_2 \mathrm{d}\varphi_x (v_2)\right).$$

This linear structure is well-defined and independent of the choice of φ. Hence, T_xM is a real vector space and $\dim T_xM = \dim M$.

Remark 2.3.2. *There are two other common (equivalent) definitions of T_xM:*

(i) *T_xM is the set of equivalence classes of elements of the form $[\varphi, v]$, where φ is a centered chart at x, $v \in \mathbb{R}^n$ and $(\varphi_1, v_1) \sim (\varphi_2, v_2) \iff v_2 = \mathrm{d}(\psi_{1,2})_0 \cdot v_1$.*

(ii) *Consider the $\mathbb{R}$-algebra $A = \mathcal{C}^\infty(M, \mathbb{R})$. A* derivation *of A is a linear map $D : A \to A$ s.t. $\forall (f, g) \in A^2, D(fg) = fDg + gDf$. Let $\mathcal{D}(M)$ be the vector space of all derivations of $\mathcal{C}^\infty(M, \mathbb{R})$ and $\mathcal{D}_x(M)$ be the subspace of all $D \in \mathcal{D}(M)$ s.t. $\forall f \in \mathcal{C}^\infty(M, \mathbb{R}),\ (Df)(x) = 0$. Then we define $T_xM = \mathcal{D}(M)/\mathcal{D}_x(M)$.*

Remark 2.3.3. *The definition of the tangent space of a manifold at a point agrees with that of the tangent space of a submanifold of $\mathbb{R}^N$ at a point.*

Definition 2.3.4 (Tangent bundle). *Let M be a n-dimensional manifold. The* tangent bundle *of M is defined by:*

$$TM = \bigsqcup_{x \in M} T_xM = \bigcup_{x \in M} (\{x\} \times T_xM).$$

It is equipped with a natural projection $\pi : TM \to M$. It has a fibered atlas *defined in the following way. To each chart $\varphi : U \subseteq M \to V \subseteq \mathbb{R}^n$, we associate the* fibered chart *$T\varphi : \pi^{-1}(U) \to V \times \mathbb{R}^n$ defined by $T\varphi(x, v) = (\varphi(x), \mathrm{d}\varphi_x \cdot v)$. Hence, we obtain a smooth atlas on TM. Therefore, TM is a manifold.*

2.4 Tangent map

Definition 2.4.1 (Differential at a point). *Let $f : M \to N$ be a map between manifolds which is differentiable at a point $x \in M$. If φ and ψ are charts of M at x and of N at $f(x)$ respectively, the* differential *$\mathrm{d}f_x : T_xM \to T_{f(x)}N$ is defined by:*

$$\mathrm{d}f_x = \mathrm{d}\psi_{f(x)}^{-1} \circ \mathrm{d}\left(\psi \circ f \circ \varphi^{-1}\right)_{\varphi(x)} \circ \mathrm{d}\varphi_x.$$

The map $\mathrm{d}f_x$ is linear and independent of the choice of φ and ψ.

Definition 2.4.2 (Tangent map). *If $f : M \to N$ is a differentiable map between manifolds, we define its tangent map $Tf : TM \to TN$ by:*

$$\forall (x, v) \in TM,\ Tf(x, v) = (f(x), \mathrm{d}f_x \cdot v).$$

Proposition 2.4.3. *Let $f : M \to N$ and $g : N \to P$ be differentiable maps between manifolds.*

(i) *If f is $\mathcal{C}^k$, then Tf is $\mathcal{C}^{k-1}$.*

(ii) *If f is smooth, then Tf is smooth.*

(iii) *$T(f \circ g) = Tf \circ Tg$.*

(iv) *If f is bijective and f^{-1} is differentiable, then $T(f^{-1}) = (Tf)^{-1}$.*

2.5 Immersions and submersions

Theorem 2.5.1 (Classification of linear maps in finite dimension)**.** *Let E and F be two vector spaces of respective dimensions m and n over a field k. If $u : E \to F$ is a linear map with rank r, then there exist linear isomorphisms $P : E \to k^m$ and $Q : F \to k^n$ s.t. the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "P"'] & F \arrow[d, "Q"] \\ k^m \arrow[r, "\ell_{m,n,r}"] & k^n \end{tikzcd}$$

where $\ell_{m,n,r} : (x_1, \dots, x_m) \in k^m \longmapsto (x_1, \dots, x_r, 0, \dots, 0) \in k^n$. Moreover, if u is injective (i.e. $r = m$), then we can choose P arbitrarily; if u is surjective (i.e. $r = n$), then we can choose Q arbitrarily.

Definition 2.5.2 (Immersion, submersion, constant rank)**.** *Let $f : M \to N$ be a $\mathcal{C}^1$ map between manifolds and let $x \in M$.*

(i) *f is said to be an* immersion *at x if $\mathrm{d}f_x : T_xM \to T_{f(x)}N$ is injective (thus $\dim M \leq \dim N$).*

(ii) *f is said to be a* submersion *at x if $\mathrm{d}f_x : T_xM \to T_{f(x)}N$ is surjective (thus $\dim M \geq \dim N$).*

(iii) *f is said to have* constant rank *at x if the map $y \mapsto \mathrm{rk}\,(\mathrm{d}f_y)$ is constant in a neighbourhood of x.*

The map f is said to be an immersion (resp. a submersion) on M if it is an immersion (resp. a submersion) at every point of M.

Proposition 2.5.3. *If E and F are two finite dimensional real vector spaces, then the map $\mathrm{rk} : \mathcal{L}(E, F) \to \mathbb{R}_+$ is upper-semicontinuous, i.e. for all $m \in \mathbb{R}_+$, $\mathrm{rk}^{-1}\,(]m, +\infty[)$ is open in $\mathcal{L}(E, F)$.*

Proof. Use the fact that a linear map has rank at least r iff it has a nonzero minor of order r. $\square$

Corollary 2.5.4. *Let $f : M \to N$ be a $\mathcal{C}^1$ map. Then $\{x \in M,\ \mathrm{rk}\,(\mathrm{d}f_x) = \min\,(\dim M, \dim N)\}$ is an open set. In particular:*

(i) *If $\dim M \leq \dim N$, then $\{x \in M,\ f \text{ is an immersion at } x\}$ is open.*

(ii) *If $\dim M \geq \dim N$, then $\{x \in M,\ f \text{ is a submersion at } x\}$ is open.*

Theorem 2.5.5 (Constant Rank Theorem)**.** *Let $f : M \to N$ be a $\mathcal{C}^k$ map ($k \in \mathbb{N}^* \cup \{\infty\}$) between manifolds and let $p \in M$. If f has constant rank r at p, then there exists $\mathcal{C}^k$ charts (U, φ) centred at p and (V, ψ) centred at $f(p)$ s.t. the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \begin{array}{c}\varphi(U)\\ \subseteq \mathbb{R}^m\end{array} \arrow[r, "\ell_{m,n,r}"] & \begin{array}{c}\psi(V)\\ \subseteq \mathbb{R}^n\end{array} \end{tikzcd}$$

Moreover, if f is a immersion at p, then we can choose φ arbitrarily; if f is a submersion at p, then we can choose ψ arbitrarily.

2.6 Submanifolds and embeddings

Definition 2.6.1 (Submanifold)**.** *Let M be a manifold of dimension n. A subset $W \subseteq M$ is said to be a* submanifold *of M of dimension m (or of codimension $(n-m)$) if every $x \in W$ admits a smooth* straightening chart *between open sets $\Phi : \hat{U} \subseteq M \to \hat{V} \subseteq \mathbb{R}^n$ s.t. $\Phi\left(W \cap \hat{U}\right) = (\mathbb{R}^m \times \{0\}) \cap \hat{V}$.*

Remark 2.6.2. *A submanifold is naturally a manifold.*

Definition 2.6.3 (Topological embedding)**.** *A map $f : X \to Y$ between topological spaces is called a* topological embedding *if f is continuous, injective, and $f : X \to f(X)$ is a homeomorphism.*

Definition 2.6.4 (Embedding)**.** *A smooth map $f : W \to M$ between manifolds is called an* embedding *if it is an injective immersion and a topological embedding.*

Theorem 2.6.5. *Let $f : W \to M$ be a smooth injective immersion between manifolds. Then f is an embedding* iff *$f(W)$ is a submanifold of M of dimension* $\dim W$*.*

2.7 Regular and critical points, regular and critical values

Definition 2.7.1 (Regular and critical points)**.** *Let $f : M \to N$ be a $\mathcal{C}^1$ map between manifolds.*

(i) *A point $x \in M$ is said to be a* regular point *of f if f is a submersion at x (thus $\dim M \geq \dim N$). The set of regular points of f is denoted by $\mathrm{Reg}(f)$; it is an open subset of M.*

(ii) *A point $x \in M$ is said to be a* critical point *of f if it is not regular. The set of critical points of f is denoted by $\mathrm{Crit}(f)$; it is a closed subset of M.*

Definition 2.7.2 (Regular and critical values)**.** *Let $f : M \to N$ be a $\mathcal{C}^1$ map between manifolds.*

(i) *A point $y \in N$ is said to be a* regular value *of f if $f^{-1}(\{y\}) \subseteq \mathrm{Reg}(f)$.*

(ii) *A point $y \in N$ is said to be a* critical value *of f if $y \in f(\mathrm{Crit}(f))$.*

Remark 2.7.3. *Note that a regular value of f may not be attained, i.e. it may not be an element of $f(M)$.*

Theorem 2.7.4. *Let $f : M \to N$ be a smooth map between manifolds. Consider a regular value y which is attained (i.e. $f^{-1}(\{y\}) \neq \varnothing$). Then $f^{-1}(\{y\})$ is a submanifold of M of codimension* $\dim N$*.*

Remark 2.7.5. *Theorem 2.7.4 has a local converse: if W is a submanifold of M and $x \in W$, then there exists an open neighbourhood U of x in M and a submersion $f : U \to \mathbb{R}^{\operatorname{codim} W}$ s.t. $W \cap U = f^{-1}(\{0\})$.*

2.8 Partitions of unity

Definition 2.8.1 (Standard plateau function)**.** *Let M be a manifold of dimension n. A* standard plateau function *on M is a smooth map $\rho : M \to [0,1]$ s.t. there exists a chart $\varphi : U \to \mathbb{R}^n$ with $\rho = 1$ on $\varphi^{-1}(B^n(0,1))$, $\rho > 0$ on $\varphi^{-1}(B^n(0,2))$ and $\rho = 0$ on $M \backslash \varphi^{-1}\left(\overline{B}^n(0,2)\right)$.*

Remark 2.8.2. *If M is a manifold of dimension n, we may obtain an atlas of M of the form $(\varphi_i : U_i \to \mathbb{R}^n)_{i \in I}$ by restricting charts so that they are homeomorphisms $\hat{\varphi}_i : \hat{U}_i \to B^n(0,\varepsilon)$, and by using the fact that $B^n(0,\varepsilon)$ is diffeomorphic to $\mathbb{R}^n$.*

Remark 2.8.3. *By using the smooth map $f : x \in \mathbb{R} \longmapsto \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$, one may construct standard plateau functions.*

Proposition 2.8.4. *Any compact manifold can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.*

Proposition 2.8.5. *Any manifold M admits a* compact exhaustion*, i.e. a sequence $(K_n)_{n\in\mathbb{N}} \in \mathcal{P}(M)^{\mathbb{N}}$ of compact subsets of M s.t. $\forall n \in \mathbb{N}$, $K_n \subseteq \mathring{K}_{n+1}$ and $M = \bigcup_{n\in\mathbb{N}} K_n$.*

Definition 2.8.6 (Local finiteness)**.** *Let X be a topological space.*

(i) *A family $(A_i)_{i\in I} \in \mathcal{P}(X)^I$ is said to be* locally finite *if every $x \in X$ has a neighbourhood V in X s.t. $\{i \in I,\ V \cap A_i \neq \varnothing\}$ is finite.*

(ii) *A family $(f_i)_{i\in I} \in \mathcal{C}^0(X,\mathbb{R})^I$ is said to be locally finite if $(\operatorname{Supp} f_i)_{i\in I}$ is locally finite, where $\operatorname{Supp} f_i = \overline{\{x \in X,\ f_i(x) \neq 0\}}$ is the support of f_i. In this case, the function $\sum_{i\in I} f_i$ is well-defined and continuous. If in addition X is a manifold and $(f_i)_{i\in I} \in \mathcal{C}^\infty(X,\mathbb{R})^I$, then $\sum_{i\in I} f_i$ is smooth.*

Lemma 2.8.7. *Let X be a topological space. If $(A_i)_{i\in I} \in \mathcal{P}(X)^I$ is locally finite, then:*

$$\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A}_i.$$

Definition 2.8.8 (Partition of unity)**.** *Let X be a topological space (resp. a manifold). If $(f_i)_{i\in I}$ is a locally finite family of continuous (resp. smooth) functions $X \to [0,+\infty[$ s.t. $\sum_{i\in I} f_i = 1$, we say that $(f_i)_{i\in I}$ is a* partition of unity*. If in addition $(U_j)_{j\in J}$ is an open covering of X s.t. $\forall i \in I$, $\exists j \in J$, $\operatorname{Supp} f_i \subseteq U_j$, we say that the partition of unity $(f_i)_{i\in I}$ is* subordinated *to $(U_j)_{j\in J}$.*

Theorem 2.8.9. *If M is a manifold and $(U_j)_{j\in J}$ is an open covering of M, then there exists a partition of unity $(f_i)_{i\in I}$ subordinated to $(U_j)_{j\in J}$.*

3 Vector fields

3.1 Generalities

Definition 3.1.1 (Vector field)**.** *Let M be a manifold. A (smooth)* vector field *on M is a smooth section $X : M \to TM$ of the projection $TM \to M$, i.e. a smooth map s.t.*

$$\forall x \in M,\ X(x) \in T_xM.$$

We write $\mathcal{X}(M)$ for the set of all vector fields on M.

Proposition 3.1.2. *If M is a manifold, then $\mathcal{X}(M)$ is a $\mathcal{C}^\infty(M,\mathbb{R})$-module.*

Remark 3.1.3. *If $(X_i)_{i\in I}$ is a locally finite family of vector fields on a manifold M, then $\sum_{i\in I} X_i$ is a vector field.*

3.2 Vector fields in coordinate systems

Notation 3.2.1. *Let M be a manifold. Consider a local system of coordinates (i.e. a chart) $\varphi = (x_1,\dots,x_n) : U \to \mathbb{R}^n$ on M. Let $(e_1,\dots,e_n)$ be the canonical basis of $\mathbb{R}^n$. Then the vector field defined on U by $p \mapsto (p, \mathrm{d}\varphi_p{}^{-1} \cdot e_i)$ is denoted by $\frac{\partial}{\partial x_i}$.*

Proposition 3.2.2.

(i) *If $V \subseteq \mathbb{R}^n$ is an open set, then $\mathcal{X}(V)$ is isomorphic to $\mathcal{C}^\infty(V,\mathbb{R}^n)$.*

(ii) *If U is the domain of a chart on a manifold M of dimension n and $X \in \mathcal{X}(U)$ is the restriction to U of a vector field on M, then we can find a unique family $(f_1,\dots,f_n) \in \mathcal{C}^\infty(U,\mathbb{R})^n$ s.t.*

$$\forall p \in U,\ X(p) = \sum_{i=1}^n f_i(p) \left(\frac{\partial}{\partial x_i}\right)_p.$$

Corollary 3.2.3. *If U is the domain of a chart on a manifold M of dimension n, then $\mathcal{X}(U)$ is a free $\mathcal{C}^\infty(U,\mathbb{R})$-module with basis $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$.*

3.3 Vector fields viewed as derivations

Definition 3.3.1 (Derivation)**.** *Let A be a unital (not necessarily commutative) k-algebra, where k is a field. A* derivation *of A is a linear map $D : A \to A$ satisfying* Leibniz's rule*:*

$$\forall (x, y) \in A^2,\ D(xy) = (Dx)\, y + x\, (Dy)\, .$$

Definition 3.3.2 (Lie derivative along a vector field)**.** *Let X be a vector field on a manifold M. If $f \in \mathcal{C}^\infty(M, \mathbb{R})$, we define the* Lie derivative *of f along X by:*

$$\forall p \in M,\ (L_X f)(p) = \mathrm{d}f_p \cdot X(p).$$

$L_X f$ is also written as $\mathrm{d}f \cdot X$ or $X \cdot f$. The $\mathbb{R}$-linear operator L_X is a derivation on $\mathcal{C}^\infty(M, \mathbb{R})$.

Lemma 3.3.3 (Locality Lemma)**.** *Let M be a manifold and consider a derivation D on $\mathcal{C}^\infty(M, \mathbb{R})$. Consider an open subset $U \subseteq M$.*

(i) *If $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$ s.t. $f_{|U} = g_{|U}$, then $(Df)_{|U} = (Dg)_{|U}$.*

(ii) *There exists a unique derivation D_U on $\mathcal{C}^\infty(U, \mathbb{R})$ s.t. $D_U\left(f_{|U}\right) = (Df)_{|U}$ for all $f \in \mathcal{C}^\infty(M, \mathbb{R})$.*

Proof. (i) By linearity, we may assume that $g = 0$. Fix $x \in U$. There exists a function $\rho \in \mathcal{C}^\infty(M, \mathbb{R})$ s.t. $\operatorname{Supp} \rho \subseteq U$ and $\rho_{|W} = 1$ for a sufficiently small neighbourhood $W \subseteq U$ of x in M. Thus, $f = (1 - \rho) f$ on M, so $Df = fD(1 - \rho) + (1 - \rho) Df$, which gives $Df(x) = 0 = Dg(x)$. (ii) Let $f \in \mathcal{C}^\infty(U, \mathbb{R})$. If $x \in U$ and $\rho \in \mathcal{C}^\infty(M, \mathbb{R})$ are as above, we extend ρf to a function $g \in \mathcal{C}^\infty(M, \mathbb{R})$ (with $g_{|M \setminus U} = 0$). We now define $(D_U f)(x) = Dg(x)$. One checks that D_U is well-defined, and satisfies $D_U\left(f_{|U}\right) = (Df)_{|U}$ for all $f \in \mathcal{C}^\infty(M, \mathbb{R})$. The uniqueness follows from (i). ☐

Theorem 3.3.4. *If M is a manifold and D is a derivation on $\mathcal{C}^\infty(M, \mathbb{R})$, then there exists a unique vector field $X \in \mathcal{X}(M)$ s.t. $L_X = D$.*

Proof. *First step*: M is diffeomorphic to $\mathbb{R}^n$ with (global) coordinates $(x_1, \dots, x_n)$. Define $X = \sum_{i=1}^n Dx_i \frac{\partial}{\partial x_i}$ and check that $L_X = D$. *Second step*: general case. Cover M by open sets diffeomorphic to $\mathbb{R}^n$, use the first step and the Locality Lemma (Lemma 3.3.3). ☐

3.4 Autonomous ordinary differential equations on a manifold

Definition 3.4.1 (Autonomous ODE asociated to a vector field)**.** *Consider a manifold M and a vector field $X \in \mathcal{X}(M)$. We associate to X the* autonomous ordinary differential equation *(ODE) $x' = X(x)$, whose solutions are differentiable maps $\gamma : I \to M$, where I is an interval of $\mathbb{R}$, s.t.*

$$\forall t \in I,\ \gamma'(t) = X(\gamma(t))\, .$$

Since X is smooth, every solution of $x' = X(x)$ is also smooth.

Theorem 3.4.2 (Cauchy–Lipschitz Theorem for manifolds)**.** *Consider a manifold M and a vector field $X \in \mathcal{X}(M)$.*

(i) *If $x_0 \in M$, the equation $x' = X(x)$ has a unique solution γ s.t. $\gamma(0) = x_0$, defined on a maximal open interval $I(x_0) \ni 0$. This solution is denoted by $\gamma(t) = \varphi_X^t(x_0) = X^t(x_0)$.*

(ii) *The set $\mathcal{U} = \bigcup_{x_0 \in M} I(x_0) \times \{x_0\} \subseteq \mathbb{R} \times M$ is open and the global solution map $\Phi : (t, x) \in \mathcal{U} \longmapsto \varphi_X^t(x) \in M$ is smooth.*

Proposition 3.4.3. *Consider a manifold M and a vector field $X \in \mathcal{X}(M)$. The following equality holds for all $(t, s, x) \in \mathbb{R}^2 \times M$ s.t. the right-hand side is defined:*

$$X^{t+s}(x) = X^t\left(X^s(x)\right).$$

The family $\left(X^t\right)_{t \in \mathbb{R}}$ is called the flow *of X.*

3.5 Complete vector fields and one-parameter subgroups

Definition 3.5.1 (Orbit). *Consider a manifold M and a vector field $X \in \mathcal{X}(M)$. For $x \in M$, the map $t \in I(x) \longmapsto X^t(x)$ is called the* orbit *(or* trajectory*) of x.*

Definition 3.5.2 (Complete vector field). *Let M be a manifold. A vector field $X \in \mathcal{X}(M)$ is said to be* complete *if all its orbits are defined on $\mathbb{R}$, i.e. the global solution map is defined on $\mathbb{R} \times M$.*

Proposition 3.5.3. *All vector fields on a compact manifold are complete.*

Definition 3.5.4 (One-parameter subgroup). *Let M be a manifold. A* one-parameter subgroup *of* $\mathrm{Diff}(M)$ *is a group homomorphism $t \in (\mathbb{R}, +) \longmapsto \varphi_t \in (\mathrm{Diff}(M), \circ)$ s.t. the map $(t, x) \in \mathbb{R} \times M \longmapsto \varphi_t(x) \in M$ is smooth.*

Proposition 3.5.5. *If X is a complete vector field on a manifold M, then the flow $\left(X^t\right)_{t\in\mathbb{R}}$ is a one-parameter subgroup of* $\mathrm{Diff}(M)$.

Proposition 3.5.6. *Let M be a manifold. If $\left(\varphi_t\right)_{t\in\mathbb{R}}$ is a one-parameter subgroup of* $\mathrm{Diff}(M)$, *then there exists a unique $X \in \mathcal{X}(M)$ s.t. $\varphi_t = X^t$ for all $t \in \mathbb{R}$.*

Proof. Define $X(x) = \left(\frac{\mathrm{d}\varphi_t(x)}{\mathrm{d}t}\right)_{t=0}$. □

3.6 Nonautonomous ODEs on a manifold

Definition 3.6.1 (Nonautonomous vector field). *A* nonautonomous vector field *on a manifold M is a map $t \in I \longmapsto X_t \in \mathcal{X}(M)$, where I is an interval of $\mathbb{R}$, s.t. $(t, x) \in I \times M \mapsto X_t(x) \in T_xM$ is smooth. We associate to $\left(X_t\right)_{t\in I}$ the* nonautonomous ODE $x' = X_t(x)$, *whose solutions are the differntiable maps $\gamma : J \to M$, where $J \subseteq I$ is an interval of $\mathbb{R}$, s.t.*

$$\forall t \in J,\ \gamma'(t) = X_t\left(\gamma(t)\right).$$

Theorem 3.6.2 (Nonautonomous Cauchy–Lipschitz Theorem for manifolds). *Consider a manifold M and a nonautonomous vector field $\left(X_t\right)_{t\in I}$ on M.*

(i) *If $(t_0, x_0) \in I \times M$, the equation $x' = X_t(x)$ has a unique solution γ s.t. $\gamma\left(t_0\right) = x_0$, defined on a maximal open interval $I\left(t_0, x_0\right) \ni t_0$. This solution is denoted by $\gamma(t) = \varphi_X^{t_0,t}\left(x_0\right) = X^{t_0,t}\left(x_0\right)$.*

(ii) *The set $\mathcal{U} = \bigcup_{(t_0,x_0)\in I\times M} \{t_0\} \times I\left(t_0, x_0\right) \times \{x_0\} \subseteq I \times I \times M$ is open and the global solution map $\Phi : (t_0, t, x) \in \mathcal{U} \longmapsto \varphi_X^{t_0,t}(x) \in M$ is smooth.*

4 Lie bracket of vector fields

4.1 Generalities

Definition 4.1.1 (Commutant of two derivations). *If A is a unital k-algebra and D_1, D_2 are two derivations of A, then the* commutant *of D_1 and D_2 is defined by:*

$$[D_1, D_2] = D_1D_2 - D_2D_1.$$

$[D_1, D_2]$ is again a derivation of A.

Corollary 4.1.2. *Let M be a manifold. If $X, Y \in \mathcal{X}(M)$, then $[L_X, L_Y]$ is a derivation of $\mathcal{C}^\infty(M, \mathbb{R})$, so according to Theorem 3.3.4, there exists a unique vector field $[X, Y] \in \mathcal{X}(M)$ s.t.*

$$L_{[X,Y]} = \left[L_X, L_Y\right].$$

$[X, Y]$ is called the Lie bracket *of X and Y.*

Proposition 4.1.3. *Let M be a manifold.*

(i) $[\cdot,\cdot] : \mathcal{X}(M)^2 \to \mathcal{X}(M)$ *is a skew-symmetric $\mathbb{R}$-bilinear map.*

(ii) *If X, Y, Z are three vector fields, we have* Jacobi's Identity*:*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Hence, the $\mathbb{R}$-vector space $\mathcal{X}(M)$ endowed with the bilinear map $[\cdot,\cdot]$ is a Lie algebra*.*

Remark 4.1.4. *Let M be a manifold. The map $[\cdot,\cdot] : \mathcal{X}(M)^2 \to \mathcal{X}(M)$ is $\mathbb{R}$-bilinear but not $\mathcal{C}^\infty(M,\mathbb{R})$-bilinear. In fact, for $X, Y \in \mathcal{X}(M)$ and $f, g \in \mathcal{C}^\infty(M,\mathbb{R})$, we have:*

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Example 4.1.5. *Let M be a manifold. If $(x_1,\ldots,x_n)$ is a local system of coordinates for M, then $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ for all $i, j \in \{1,\ldots,n\}$.*

4.2 Flow-box Theorem

Theorem 4.2.1 (Flow-box Theorem)**.** *Let M be a manifold and $p_0 \in M$. If $X \in \mathcal{X}(M)$ is s.t. $X(p_0) \neq 0$, then there exist local coordinates $(x_1,\ldots,x_n)$ near p_0 s.t. $X = \frac{\partial}{\partial x_1}$. Equivalently, if $\varphi = (x_1,\ldots,x_n)$, we have $X^t(\varphi^{-1}(x_1,\ldots,x_n)) = \varphi^{-1}(x_1+t, x_2,\ldots,x_n)$ for $t, x_1,\ldots,x_n$ sufficiently small.*

4.3 Pushforwards and pullbacks of vector fields

Definition 4.3.1 (Pushforward and pullback)**.** *Let $\varphi : M \to N$ be an isomorphism between manifolds.*

(i) *φ induces an isomorphism $\varphi_* : \mathcal{X}(M) \to \mathcal{X}(N)$. For $X \in \mathcal{X}(M)$, $\varphi_* X$ is called the* pushforward *of X and is defined by:*

$$\forall y \in N,\ (\varphi_* X)(y) = \mathrm{d}\varphi_{\varphi^{-1}(y)} \cdot X\left(\varphi^{-1}(y)\right).$$

(ii) *φ induces an isomorphism $\varphi^* : \mathcal{X}(N) \to \mathcal{X}(M)$. For $Y \in \mathcal{X}(N)$, $\varphi^* Y$ is called the* pullback *of Y and is defined by:*

$$\forall x \in M,\ (\varphi^* Y)(x) = \mathrm{d}\varphi_x^{-1} \cdot Y(\varphi(x)).$$

We have $\varphi^* = (\varphi_*)^{-1}$.

Proposition 4.3.2. *Let $\varphi : M \to N$ be an isomorphism between manifolds.*

(i) *For $X_1, X_2 \in \mathcal{X}(M)$, we have $\varphi_*[X_1, X_2] = [\varphi_* X_1, \varphi_* X_2]$.*

(ii) *For $Y_1, Y_2 \in \mathcal{X}(N)$, we have $\varphi^*[Y_1, Y_2] = [\varphi^* Y_1, \varphi^* Y_2]$.*

(iii) *If $X \in \mathcal{X}(M)$, then $\varphi \circ X^t = (\varphi_* X)^t \circ \varphi$ whenever this is defined.*

(iv) *If $Y \in \mathcal{X}(N)$, then $Y^t \circ \varphi = \varphi \circ (\varphi^* Y)^t$ whenever this is defined.*

Proposition 4.3.3. *If M is a manifold, then any vector field $X \in \mathcal{X}(M)$ is invariant by the flow $(X^t)_t$, i.e.*

$$\left(X^t\right)_* X = X = \left(X^t\right)^* X.$$

4.4 Commuting vector fields

Definition 4.4.1 (Commuting flows)**.** *Let M be a manifold and $X, Y \in \mathcal{X}(M)$. We say that the (partially defined) flows $(X^t)_t$ and $(Y^u)_u$* commute *if $X^t \circ Y^u = Y^u \circ X^t$ for all t, u small enough.*

Proposition 4.4.2 (Second formula for the Lie bracket)**.** *Let M be a manifold and $X, Y \in \mathcal{X}(M)$. Then:*

$$[X, Y] = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(X^t\right)^* Y\right)_{t=0}.$$

Definition 4.4.3 (Commuting vector fields)**.** *Let M be a manifold and $X, Y \in \mathcal{X}(M)$. The following assertions are equivalent:*

(i) *The flows $(X^t)_t$ and $(Y^u)_u$ commute.*

(ii) *$(X^t)^* Y = Y$ whenever this is defined.*

(iii) *$[X, Y] = 0$.*

We then say that X and Y commute.

4.5 Normal form of an independent family of commuting vector fields

Theorem 4.5.1. *Let M be a manifold and $p_0 \in M$. Let $X_1, \ldots, X_k \in \mathcal{X}(M)$ be commuting vector fields (i.e. $[X_i, X_j] = 0$ for all $i, j \in \{1, \ldots, k\}$) s.t. $X_1(p_0), \ldots, X_k(p_0)$ are (linearly) independent. Then there exist local coordinates $(x_1, \ldots, x_n)$ near p_0 s.t. $X_i = \frac{\partial}{\partial x_i}$ for all $i \in \{1, \ldots, k\}$.*

4.6 Distributions of tangent spaces

Definition 4.6.1 (Distribution)**.** *Let M be a manifold and $0 < k < n = \dim M$. A* distribution *$\mathcal{D}$ of dimension k in M is a map $x \in M \mapsto \mathcal{D}_x$, where $\mathcal{D}_x$ is a subspace of T_xM of dimension k, which is smooth in the following sense: for every $x_0 \in M$, there exist local vector fields $X_1, \ldots, X_k$ s.t. $\mathcal{D}_x = \mathrm{Vect}\left(X_1(x), \ldots, X_k(x)\right)$ near x_0.*

Definition 4.6.2 (Vector field tangent to a distribution)**.** *Let M be a manifold. A vector field $X \in \mathcal{X}(M)$ is said to be* tangent *to a distribution $\mathcal{D}$ if $\forall x \in M$, $X(x) \in \mathcal{D}_x$. We denote by $\mathcal{X}(\mathcal{D})$ the space of vector fields that are tangent to $\mathcal{D}$; this is a $\mathcal{C}^\infty(M, \mathbb{R})$-submodule of $\mathcal{X}(M)$.*

Definition 4.6.3 (Integral submanifold of a distribution)**.** *Let M be a manifold. A submanifold $W \subseteq M$ is said to be an* integral submanifold *of a distribution $\mathcal{D}$ if $TW \subseteq \mathcal{D}$, i.e. $\forall x \in W$, $T_xW \subseteq \mathcal{D}_x$.*

Proposition 4.6.4. *Let M be a manifold. If $\mathcal{D}$ is a distribution, W is an integral submanifold of $\mathcal{D}$ and $X, Y \in \mathcal{X}(\mathcal{D})$ s.t. $X_{|W}, Y_{|W}, [X, Y]_{|W} \in \mathcal{X}(W)$, then:*

$$[X, Y]_{|W} = \left[X_{|W}, Y_{|W}\right].$$

Definition 4.6.5 (Integrable distribution)**.** *Let $\mathcal{D}$ be a distribution over a manifold M.*

(i) *$\mathcal{D}$ is said to be* integrable *if $\mathcal{X}(\mathcal{D})$ is a Lie subalgebra of $\mathcal{X}(M)$, i.e.*

$$X, Y \in \mathcal{X}(\mathcal{D}) \Longrightarrow [X, Y] \in \mathcal{X}(\mathcal{D}).$$

(ii) *$\mathcal{D}$ is said to be* maximally nonintegrable *if $\mathcal{X}(\mathcal{D})$ generates $\mathcal{X}(M)$ as a Lie subalgebra.*

Theorem 4.6.6 (Frobenius' Theorem)**.** *Let $\mathcal{D}$ be an integrable k-dimensional distribution over a manifold M. Then for each $p \in M$, there exist local coordinates $(x_1, \ldots, x_n)$ near p s.t. $\mathcal{D} = \mathrm{Vect}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right)$. Hence, maximal integral submanifolds are of the form $W_1 \times \{c\}$, where W_1 is an open subset of $\mathbb{R}^k$.*

5 Exterior algebra

Notation 5.0.1. *In this section, V will be a vector space over $\mathbb{R}$ (or over any field of characteristic $\neq 2$).*

5.1 Exterior forms

Definition 5.1.1 (Exterior p-forms)**.** *For $p \in \mathbb{N}$, we denote by $\otimes^p V^*$ the space of p-linear forms $u : V^p \to \mathbb{R}$. Moreover, we denote by $\Lambda^p V^*$ the subspace of $\otimes^p V^*$ consisting of alternating (or equivalently: antisymmetric) p-linear forms. The elements of $\Lambda^p V^*$ are called* exterior p-forms *on V.*

Example 5.1.2. $\otimes^0 V^* = \mathbb{R}$ *and* $\otimes^1 V^* = V^*$.

Remark 5.1.3. *We define a linear left action of $\mathfrak{S}_p$ on $\otimes^p V^*$ by:*

$$(\sigma \cdot u)(x) = \varepsilon(\sigma) u \left(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(p)} \right).$$

Thus, $\Lambda^p V^*$ is the space of $\mathfrak{S}_p$-invariant forms.

Notation 5.1.4. *For $p, \in \mathbb{N}$, we define:*

$$\mathcal{J}_{p,n} = \left\{ (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p ,\ i_1 < \cdots < i_p \right\}.$$

We have $|\mathcal{J}_{p,n}| = \binom{n}{p}$.

Proposition 5.1.5. *Assume that V is finite-dimensional and equipped with a basis $(e_1, \ldots, e_n)$. Then the map $\Phi : \Lambda^p V^* \to \mathbb{R}^{\mathcal{J}_{p,n}}$ defined by:*

$$\Phi(u) = \left(u \left(e_{i_1}, \ldots, e_{i_p} \right) \right)_{(i_1, \ldots, i_p) \in \mathcal{J}_{p,n}} \in \mathbb{R}^{\mathcal{J}_{p,n}}.$$

is a linear isomorphism. In particular:

$$\dim \Lambda^p V^* = \binom{n}{p}.$$

Remark 5.1.6. *If $(i_1, \ldots, i_p) \in \mathcal{J}_{p,n}$ and if $f_{(i_1, \ldots, i_p)}$ is the basis element of $\mathbb{R}^{\mathcal{J}_{p,n}}$ associated to $(i_1, \ldots, i_p)$, the element $\Phi^{-1}\left(f_{(i_1, \ldots, i_p)}\right)$ will later be denoted by $e^*_{i_1} \wedge \cdots \wedge e^*_{i_p} \in \Lambda^p V^*$.*

Remark 5.1.7. $\Lambda^0 V^* = \mathbb{R}$. *On the other hand, $\Lambda^n V^*$ is isomorphic to $\mathbb{R}$, but not canonically. A generating element $u \in \Lambda^n V^*$ will be called an* oriented volume element*.*

5.2 Exterior product

Definition 5.2.1 (Exterior product)**.** *We want to define a product $\wedge : \Lambda^p V^* \times \Lambda^q V^* \to \Lambda^{p+q} V^*$.*

- *We start by defining $\otimes : \otimes^p V^* \times \otimes^q V^* \to \otimes^{p+q} V^*$ by:*

  $$(u \otimes v)(x_1, \ldots, x_{p+q}) = u(x_1, \ldots, x_p)\, v(x_{p+1}, \ldots, x_{p+q}).$$

  $\otimes$ is "partially antisymmetric", i.e. if $\sigma \in H_{p,q} = \mathfrak{S}_p \times \mathfrak{S}_q \subseteq \mathfrak{S}_{p+q}$, then $\sigma \cdot (u \otimes v) = u \otimes v$ as soon as $u \in \Lambda^p V^*$ and $v \in \Lambda^q V^*$. Therefore, it makes sense to define $[\sigma] \cdot (u \otimes v) = \sigma \cdot (u \times v)$ for $[\sigma] \in \mathfrak{S}_{p+q} / H_{p,q}$.

- *We can now define* $\wedge : \Lambda^p V^* \times \Lambda^q V^* \to \Lambda^{p+q} V^*$ *by:*

$$u \wedge v = \sum_{[\sigma] \in \mathfrak{S}_{p+q}/H_{p,q}} [\sigma] \cdot (u \otimes v) \,.$$

  Explicitly, if $A_{p,q} = \{\sigma \in \mathfrak{S}_{p+q},\ \sigma$ *increases on* $\{1, \dots, p\}$ *and on* $\{p+1, \dots, p+q\}\}$*, then* $A_{p,q}$ *is a set of representatives of left cosets of* $H_{p,q}$*, and:*

$$u \wedge v = \sum_{\sigma \in A_{p,q}} \sigma \cdot (u \otimes v) \,.$$

Thus, we have defined a bilinear map $\wedge : \Lambda^p V^* \times \Lambda^q V^* \to \Lambda^{p+q} V^*$*; it is called the* exterior product.

Proposition 5.2.2. *The exterior product is associative, i.e. if* $u \in \Lambda^p V^*$*,* $v \in \Lambda^q V^*$*,* $w \in \Lambda^r V^*$*, then* $(u \wedge v) \wedge w = u \wedge (v \wedge w)$*.*

Proof. Note that:

$$\begin{aligned} (u \wedge v) \wedge w &= \sum_{\sigma \in A_{p+q,r}} \sigma \cdot ((u \wedge v) \otimes w) = \sum_{\sigma \in A_{p+q,r}} \sum_{\tau \in A_{p,q}} \sigma \cdot ((\tau \cdot (u \otimes v)) \otimes w) \\ &= \sum_{\sigma \in A_{p+q,r}} \sum_{\tau \in A_{p,q}} \sigma \circ (\tau \times \mathrm{id}) \cdot (u \otimes v \otimes w) \,. \end{aligned}$$

Now, the map $(\sigma, \tau) \in A_{p+q,r} \times A_{p,q} \longmapsto \sigma \circ (\tau \times \mathrm{id}) \in A_{p,q,r}$ is a bijection, where $A_{p,q,r}$ is the subset of $\mathfrak{S}_{p+q+r}$ consisting of permutations which are increasing on $\{1, \dots, p\}$, on $\{p+1, \dots, p+q\}$ and on $\{p+q+1, \dots, p+q+r\}$. Therefore:

$$(u \wedge v) \wedge w = \sum_{\theta \in A_{p,q,r}} \theta \cdot (u \otimes v \otimes w) \,.$$

By symmetry, we obtain $(u \wedge v) \wedge w = u \wedge (v \wedge w)$. $\square$

Remark 5.2.3. *If* $(e_1, \dots, e_n)$ *is a basis of* V*, and if* $(i_1, \dots, i_p) \in \mathcal{J}_{p,n}$*, then the element* $e_{i_1}^* \wedge \dots \wedge e_{i_p}^* \in \Lambda^p V^*$ *is well-defined and its definition coincides with that of Remark 5.1.6.*

5.3 Exterior algebra

Definition 5.3.1 (Exterior algebra)**.** *We define:*

$$\Lambda^* V^* = \bigoplus_{p \in \mathbb{N}} \Lambda^p V^*.$$

An element of $\Lambda^p V^* \subseteq \Lambda^* V^*$ *is called* pure of degree p. *In general, if:*

$$u = \sum_{i=0}^{d} \underbrace{u_i}_{\in \Lambda^i V^*} \,,$$

with $u_d \neq 0$*, we define* $\deg u = d$*. The exterior product* $\wedge : \Lambda^* V^* \times \Lambda^* V^* \to \Lambda^* V^*$ *is defined by extending the product of pure elements by bilinearity. We obtain an algebra over* $\mathbb{R}$*, which is:*

(i) *Associative,*

(ii) *Unital, because* $\mathbb{R} = \Lambda^0 V^* \subseteq \Lambda^* V^*$*,*

(iii) *Graded, because* $\deg (u \wedge v) = \deg u + \deg v$*,*

(iv) *Anticommutative (or graded-commutative), because* $u \wedge v = (-1)^{(\deg u)(\deg v)} v \wedge u$*.*

This algebra is called the exterior algebra*. To summarise these properties, we shall say that it is a* graded algebra over $\mathbb{R}$*.*

Remark 5.3.2.

(i) *If* $u \in \mathbb{R} = \Lambda^0 V^*$ *and* $v \in \Lambda^q V^*$*, then* $u \wedge v = uv$*.*

(ii) *If* $u \in V^* = \Lambda^1 V^*$ *and* $v \in \Lambda^q V^*$*, then:*

$$(u \wedge v)(x_0, \ldots, x_q) = \sum_{i=0}^{q} (-1)^i u(x_i)\, v(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_q).$$

5.4 Functoriality

Definition 5.4.1. *If* V, W *are two real vector spaces, any linear map* $f : V \to W$ *induces linear maps* $\Lambda^p f^* : \Lambda^p W^* \to \Lambda^p V^*$ *(sometimes simply denoted by* f^**), defined by:*

$$\Lambda^p f^* u = u \circ (f \times \cdots \times f).$$

By linear extension, f *induces a linear map* $\Lambda^* f^* : \Lambda^* W^* \to \Lambda^* V^*$ *(sometimes simply denoted by* f^**).*

Remark 5.4.2. $\Lambda^0 f^* = \mathrm{id}_{\mathbb{R}}$ *and* $\Lambda^1 f^*$ *is the usual dual map.*

Proposition 5.4.3. Λ^* *is a contravariant functor from the category of* $\mathbb{R}$*-vector spaces to the category of graded* $\mathbb{R}$*-algebras.*

5.5 Exterior forms of degree 2

Definition 5.5.1 (Rank of an exterior form of degree 2)**.** *If* $u \in \Lambda^2 V^*$ *and* $\dim V < +\infty$*, define* $\operatorname{rk} u = \operatorname{rk}(x \in V \mapsto u(x, \cdot) \in V^*) = \operatorname{rk}(y \in V \mapsto u(\cdot, y) \in V^*)$*.*

Proposition 5.5.2. *Assume that* V *is finite-dimensional. If* $u \in \Lambda^2 V^*$*, then* $\operatorname{rk} u$ *is even, and there exists a basis* $(e_1, \ldots, e_n)$ *of* V *s.t.*

$$u = e_1^* \wedge e_2^* + \cdots + e_{r-1}^* \wedge e_r^*,$$

where $r = \operatorname{rk} u$*.*

Corollary 5.5.3. *Assume that* V *is finite-dimensional and let* $u \in \Lambda^2 V^*$*.*

(i) *If* $\dim V$ *is odd, then* u *is always degenerate.*

(ii) *If* $\dim V = 2m$*, then* u *is nondegenerate* iff $u^m = u \wedge \cdots \wedge u \neq 0$.

5.6 Interior product

Definition 5.6.1 (Interior product)**.** *If* $x \in V$ *and* $u \in \Lambda^p V^*$*, we define the* interior product $\iota_x u \in \Lambda^{p-1} V^*$ *by:*

$$\iota_x u(x_1, \ldots, x_{p-1}) = u(x, x_1, \ldots, x_{p-1}).$$

We extend the map $\iota_x : \Lambda^p V^* \to \Lambda^{p-1} V^*$ *by linearity to a map* $\iota_x : \Lambda^* V^* \to \Lambda^* V^*$ *(with* $\Lambda^{-1} V^* = 0$*).*

Proposition 5.6.2. *If* $x \in V$*, then* ι_x *is a* derivation *of the graded algebra* $\Lambda^* V^*$*, i.e.*

$$\iota_x(u \wedge v) = (\iota_x u) \wedge v + (-1)^{\deg u} u \wedge (\iota_x v).$$

6 Differential forms

6.1 Cotangent bundle

Definition 6.1.1 (Cotangent bundle). *Let M be a n-dimensional manifold. The* cotangent bundle *of M is defined by:*

$$T^*M = \bigsqcup_{x \in M} T_x^*M,$$

where $T_x^*M = (T_xM)^*$ is the cotangent space *of M at x. T^*M is equipped with a natural projection $\pi : T^*M \to M$. It has a* fibered atlas *defined in the following way. To each chart $\varphi : U \subseteq M \to V \subseteq \mathbb{R}^n$, we associate the* fibered chart *$\Phi_\varphi : \pi^{-1}(U) \to V \times \mathbb{R}^n$ defined by $\Phi_\varphi(x, v) = (\varphi(x), v \circ \mathrm{d}\varphi_x^{-1})$. Hence, we obtain a smooth atlas on T^*M. Therefore, T^*M is a manifold.*

Remark 6.1.2. *If M is a manifold, then $\pi : T^*M \to M$ is a vector bundle of rank* $\dim M$*.*

Remark 6.1.3. *If M is a manifold, it turns out that T^*M is isomorphic to TM.*

Proposition 6.1.4. *Let M be a manifold. If $\varphi = (x_1, \dots, x_n)$ is a local coordinate system on M, one can define a local coordinate system $(x_1, \dots, x_n, p_1, \dots, p_n)$ on T^*M, where x_i represents $x_i \circ \pi$ and $p_i(u) = u\left(\frac{\partial}{\partial x_i}\right)$.*

6.2 Bundles of exterior forms

Definition 6.2.1 (Bundle of exterior p-forms). *Let M be a n-dimensional manifold and let $p \in \{0, \dots, n\}$. The* bundle of exterior p-forms *of M is defined by:*

$$\Lambda^p T^*M = \bigsqcup_{x \in M} \Lambda^p T_x^*M,$$

where $\Lambda^p T_x^*M = \Lambda^p (T_xM)^*$. Similarly to T^*M, $\Lambda^p T^*M$ is equipped with a natural projection $\pi : \Lambda^p T^*M \to M$, and with a structure of manifold of dimension $\binom{n}{p} + n$. If $p > n$, we set $\Lambda^p T^*M = \{0\}$.

Remark 6.2.2. *If M is a manifold, then $\pi : \Lambda^p T^*M \to M$ is a vector bundle of rank $\binom{n}{p}$.*

Example 6.2.3. *Let M be a n-dimensional manifold.*

(i) *If $p = 0$, then $\Lambda^0 T^*M = M \times \mathbb{R}$.*

(ii) *If $p = 1$, then $\Lambda^1 T^*M = T^*M$.*

(iii) *If $p = n$, then $\Lambda^n T^*M$ is a line bundle; we shall see that it is isomorphic to $M \times \mathbb{R}$ (but not canonically)* iff *M is orientable.*

Definition 6.2.4 (Bundle of all exterior forms). *Let M be a n-dimensional manifold. We define the* bundle of all exterior forms *of M by:*

$$\Lambda^* T^*M = \bigoplus_{0 \leq p \leq \dim M} \Lambda^p T^*M = \bigoplus_{p \in \mathbb{N}} \Lambda^p T^*M.$$

This is a vector bundle over M with fibers $\pi^{-1}(\{x\}) = \Lambda^* T_x^*M = \Lambda^* (T_xM)^*$.

6.3 Differential forms and their algebra

Definition 6.3.1 (Differential form)**.** *Let M be a n-dimensional manifold and let $p \in \mathbb{N}$. A* differential p-form *(or differential form of degree p, or p-form) is a smooth section of the projection $\pi : \Lambda^p T^*M \to M$, i.e. a map $\alpha : x \in M \longmapsto \alpha_x \in \Lambda^p T^*M$ s.t.*

$$\forall x \in M,\ \alpha_x \in \Lambda^p T_x^* M.$$

We denote by $\Omega^p(M)$ the vector space of all p-forms.

Example 6.3.2. *Let M be a n-dimensional manifold.*

(i) *If $p = 0$, then $\Omega^0(M) = \mathcal{C}^\infty(M, \mathbb{R})$.*

(ii) *If $p > n$, then $\Omega^p(M) = \{0\}$.*

Definition 6.3.3 (Algebra of differential forms)**.** *Let M be a n-dimensional manifold. We define the* algebra of differential forms *of M by:*

$$\Omega^*(M) = \bigoplus_{0 \le p \le \dim M} \Omega^p(M) = \bigoplus_{p \in \mathbb{N}} \Omega^p(M).$$

It is a graded algebra over $\mathcal{C}^\infty(M, \mathbb{R}) = \Omega^0(M)$.

Example 6.3.4 (Differential of a function)**.** *If M is a manifold and $f \in \mathcal{C}^\infty(M, \mathbb{R})$, then $\mathrm{d}f \in \Omega^1(M)$. Therefore, we have a linear operator $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$.*

6.4 Evaluation of a differential form on tangent vectors and vector fields

Proposition 6.4.1. *Let M be a n-dimensional manifold and $p \in \{1, \dots, n\}$. Let $\alpha \in \Omega^p(M)$. If $x \in M$ and $v_1, \dots, v_p \in T_x M$, one can define:*

$$\alpha(v_1, \dots, v_p) = \alpha_x(v_1, \dots, v_p) \in \mathbb{R}.$$

Thus, we obtain a smooth function $\alpha : \Pi^p(TM) \to \mathbb{R}$, where $\Pi^p(TM)$ is the product of p copies of TM:

$$\Pi^p(TM) = \{(v_1, \dots, v_p) \in TM \times \cdots \times TM,\ \pi(v_1) = \cdots = \pi(v_p)\}.$$

This function $\alpha : \Pi^p(TM) \to \mathbb{R}$ is p-linear and alternating on each fiber $T_x M \times \cdots \times T_x M$. Conversely, any smooth map $\alpha : \Pi^p(TM) \to \mathbb{R}$ which is p-linear and alternating on each fiber defines an element $\alpha \in \Omega^p(M)$.

Proposition 6.4.2. *Let M be a n-dimensional manifold and $p \in \{1, \dots, n\}$. If $\alpha \in \Omega^p(M)$ and $X_1, \dots, X_p \in \mathcal{X}(M)$, one can define:*

$$\alpha(X_1, \dots, X_p) : x \in M \longmapsto \alpha_x(X_1(x), \dots, X_p(x)) \in \mathbb{R}.$$

Thus, $\alpha(X_1, \dots, X_p)$ is a smooth map $M \to \mathbb{R}$. In this way, one can interpret $\alpha \in \Omega^p(M)$ as a p-$\mathcal{C}^\infty(M, \mathbb{R})$-linear alternating map $\mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathbb{R}$.

6.5 Differential forms in local coordinates

Proposition 6.5.1. *Let M be a n-dimensional manifold. If (U, φ) is a local system of coordinates on M, with $\varphi = (x_1, \dots, x_n)$, then the differentials $\mathrm{d}x_1, \dots, \mathrm{d}x_n \in \Omega^1(M)$ form a basis of $T_x^* M$ at each $x \in U$. Thus, a differential 1-form $\alpha \in \Omega^1(U)$ can be written uniquely:*

$$\alpha = \sum_{i=1}^n f_i \, \mathrm{d}x_i,$$

with $f_1, \dots, f_n \in \mathcal{C}^\infty(U, \mathbb{R})$. In particular, if $\alpha = \mathrm{d}f$, with $f \in \mathcal{C}^\infty(U, \mathbb{R})$, then $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$.

Proposition 6.5.2. *Let M be a n-dimensional manifold. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be two local systems of coordinates on M (defined on an open set U). Let $\alpha \in \Omega^1(M)$ and write:*

$$\alpha = \sum_{i=1}^{n} f_i \, \mathrm{d}x_i = \sum_{j=1}^{n} g_j \, \mathrm{d}x_j,$$

with $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{C}^\infty(U, \mathbb{R})$. Then, for $i \in \{1, \dots, n\}$, we have:

$$f_i \left(x_1, \dots, x_n\right) = \sum_{j=1}^{n} g_j \left(y_1, \dots, y_n\right) \frac{\partial y_j}{\partial x_i}.$$

Remark 6.5.3. *Let M be a n-dimensional manifold. If (U, φ) is a local system of coordinates on M, with $\varphi = (x_1, \dots, x_n)$, then can construct a basis of $\Lambda^p T_x^* M$ by defining:*

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p},$$

for $I = (i_1, \dots, i_p) \in \mathcal{J}_{p,n}$. We can then generalise the results related to 1-forms.

6.6 Functoriality

Definition 6.6.1 (Pullbacks of differential forms)**.** *Let $\varphi : M \to N$ be a smooth map between manifolds. We define:*

$$\varphi^* : \left| \begin{array}{l} \Omega^*(N) \longrightarrow \Omega^*(M) \\ \beta \longmapsto \left| \begin{array}{l} M \longrightarrow \Lambda^* T^* M \\ x \longmapsto \mathrm{d}\varphi_x^* \left(\beta_{\varphi(x)}\right) \end{array} \right. \end{array} \right. .$$

Note that φ^* sends $\Omega^p(N)$ to $\Omega^p(M)$ for all $p \in \mathbb{N}$. For $\beta \in \Omega(N)$, the map $\varphi^* \beta$ is called the pullback *of β by φ.*

Proposition 6.6.2. Ω^* *is a contravariant functor from the category of manifolds to the category of graded $\mathbb{R}$-algebras.*

6.7 Tautological 1-form on T^*M

Definition 6.7.1 (Tautological 1-form)**.** *If M is a manifold, then the tautological 1-form on M is $\lambda_M \in \Omega_1 \left(T^* M\right)$ defined by:*

$$\left(\lambda_M\right)_{(x,p)} = p \circ \mathrm{d}\pi_x.$$

Remark 6.7.2. *Let M be a manifold. If $(x_1, \dots, x_n)$ is a local system of coordinates on M inducing a local system of coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ on T^*M, we have:*

$$\lambda_M = \sum_{1 \leq i \leq \dim M} p_i \, \mathrm{d}x_i = p \, \mathrm{d}x.$$

Proposition 6.7.3. *Let M be a manifold. Then:*

$$\forall \alpha \in \Omega^1(M), \; \alpha^* \lambda_M = \alpha.$$

6.8 Orientations and volume forms

Definition 6.8.1 (Orientation of a vector space)**.** *An* orientation *of a finite dimension real vector space V is an equivalence class of bases of V for the equivalence relation $\sim$ defined by:*

$$\beta \sim \beta' \Longleftrightarrow \det_\beta \left(\beta'\right) > 0.$$

A real vector space equipped with an orientation is said to be oriented*; its set of orientations is denoted by* $\mathrm{Or}(V)$*. If $\omega \in \mathrm{Or}(V)$, the opposite orientation is denoted by $(-\omega)$. If $V = \{0\}$, we define* $\mathrm{Or}(V) = \{\pm 1\}$.

Definition 6.8.2 (Orientation of a manifold)**.** *Let M be a manifold of positive dimension. An* orientation *of M is an atlas for which the transition maps have positive jacobians. Such an atlas is called* oriented*. A chart in this atlas is said to be* direct*. If M admits an orientation, it is said to be* orientable*. A diffeomorphism $f : M \to N$ between manifolds is said to* preserve the orientation *if it preserves a maximal oriented atlas. If M is discrete (i.e.* $\dim M = 0$*), an orientation of M is a map $M \to \{\pm 1\}$.*

Remark 6.8.3. *If M is an orientable manifold, then the set of orientations of M is in bijection with the set of maps $\Pi_0(M) \to \{\pm 1\}$, where $\Pi_0(M)$ is the set of connected components of M.*

Example 6.8.4. *The Möbius band $\mathbb{M}$ and the real projective $(2n)$-space $\mathbb{PR}^{2n}$ are not orientable.*

Definition 6.8.5 (Volume form)**.** *Let M be a manifold of positive dimension n. A* volume form *on M is an element $\nu \in \Omega^n(M)$ s.t. $\forall x \in M$, $\nu_x \neq 0$.*

Remark 6.8.6. *Let M be a manifold of positive dimension n. Then a volume form ν on M defines an isomorphism:*

$$\left| \begin{array}{rcl} M \times \mathbb{R} & \longrightarrow & \Lambda^n T^* M \\ (x, \lambda) & \longmapsto & \lambda \nu_x \end{array} \right. .$$

Conversely, such an isomorphism defines a volume form.

Proposition 6.8.7. *Let M be a manifold of positive dimension n.*

(i) *A volume form ν on M defines an orientation, in which a chart $(x_1, \dots, x_n)$ is direct* iff *there exists $\lambda > 0$ s.t. $\nu = \lambda \,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$, i.e.* iff $\nu\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) > 0$.

(ii) *Conversely, if M is orientable, then it admits a volume form (defined using partitions of unity subordinated to an oriented atlas of M).*

In particular, M is orientable iff *it admits a volume form.*

Example 6.8.8. *Let M be a manifold of dimension n that is the boundary of a domain Ω in $\mathbb{R}^{n+1}$. Assume that M admits a* Gauß map *$\mathcal{N} : M \to \mathbb{S}^n$ i.e. s.t. for all $x \in M$, $\mathcal{N}(x)$ is a unit vector in $(T_x M)^\perp$ s.t. $x + t\mathcal{N}(x) \notin \Omega$ for $t > 0$ small enough. Then we can define a volume form ν on M by:*

$$\nu_x\left(v_1, \dots, v_n\right) = \nu_0\left(\mathcal{N}(x), v_1, \dots, v_n\right),$$

where $\nu_0 = \mathrm{d}x_0 \wedge \dots \wedge \mathrm{d}x_n$ is the canonical volume form on $\mathbb{R}^{n+1}$.

7 Exterior differential calculus

7.1 Construction of the exterior differential

Lemma 7.1.1 (Localisation lemma)**.** *Let M be a manifold. If D is a derivation (resp. anti-derivation) on $\Omega^*(M)$ and $U \subseteq M$ is an open set, then D induces a unique derivation (resp. anti-derivation) D_U on $\Omega^*(U)$.*

Theorem 7.1.2. *There exists a unique operator* $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$ *of degree 1 (i.e. sending $\Omega^p(M)$ to $\Omega^{p+1}(M)$), which extends* $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ *and satisfying the following two properties:*

(i) Leibniz's property*. If $\alpha, \beta \in \Omega^*(M)$, then:*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta,$$

where $p = \deg \alpha$.

(ii) $\mathrm{d}(\mathrm{d}f) = 0$ *for all $f \in \Omega^0(M)$.*

Moreover, d *also satisfies the two following properties:*

(iii) $\mathrm{d}\circ\mathrm{d}=0$.

(iv) *In a chart* $(x_1,\dots,x_n)$*, for smooth functions* $(f_I)_{I\in\mathcal{J}_{p,n}}\in\Omega^0(M)^{\mathcal{J}_{p,n}}$*, we have:*

$$\mathrm{d}\left(\sum_{I\in\mathcal{J}_{p,n}} f_I\,\mathrm{d}x_I\right)=\sum_{I\in\mathcal{J}_{p,n}}\mathrm{d}f_I\wedge\mathrm{d}x_I.$$

Hence, d *makes* $\Omega^*(M)$ *a* differential graded algebra.

Proof. *Uniqueness.* Check that any operator d satisfying (i) and (ii) must also satisfy (iv). *Existence.* Define d using (iv) and check the other properties. □

Example 7.1.3. *Let M be a manifold. Recall the definition of the tautological* 1*-form* $\lambda_M\in\Omega^1(T^*M)$ *from Definition 6.7.1. In a chart* $(x_1,\dots,x_n,p_1,\dots,p_n)$*, we have* $\lambda_M=\sum_{i=1}^n p_i\,\mathrm{d}x_i$*, therefore:*

$$\mathrm{d}\lambda_M=\sum_{i=1}^n\mathrm{d}p_i\wedge\mathrm{d}x_i.$$

Thus, $\mathrm{d}\lambda_M$ *is a nondegenerate* 2*-form, and* $(\mathrm{d}\lambda_M)^n$ *is a volume form on* T^*M*. In particular,* T^*M *is canonically oriented. The form* $\mathrm{d}\lambda_M$ *is called a* symplectic form.

Proposition 7.1.4. *If* $f:M\to N$ *is a smooth map between manifolds, then the map* $f^*:\Omega^*(N)\to\Omega^*(M)$ *is a morphism of differential graded algebras, i.e.*

$$f^*\circ\mathrm{d}=\mathrm{d}\circ f^*.$$

7.2 Lie derivative of a differential form with respect to a vector field

Remark 7.2.1. *If X is a vector field on a manifold M, we have already defined two notions of Lie derivatives w.r.t. X:*

(i) *For* $f\in\Omega^0(M)$, $L_Xf=X\cdot f=\left(\frac{\mathrm{d}}{\mathrm{d}t}(X^t)^*f\right)_{t=0}$.

(ii) *For* $Y\in\mathcal{X}(M)$, $L_XY=[X,Y]=\left(\frac{\mathrm{d}}{\mathrm{d}t}(X^t)^*Y\right)_{t=0}$.

Definition 7.2.2 (Lie derivative of a differential form with respect to a vector field)**.** *Let M be a manifold and* $X\in\mathcal{X}(M)$*. For* $\alpha\in\Omega^*(M)$*, the* Lie derivative *of α w.r.t. X is defined by:*

$$L_X\alpha=\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(X^t\right)^*\alpha\right)_{t=0}.$$

Definition 7.2.3 (Interior product on a manifold)**.** *Let M be a manifold. If* $X\in\mathcal{X}(M)$ *and* $\alpha\in\Omega^p(M)$*, we define the* interior product $\iota_X\alpha\in\Omega^{p-1}(M)$ *by:*

$$(\iota_X\alpha)_x(v_1,\dots,v_{p-1})=\alpha_x(X(x),v_1,\dots,v_{p-1}).$$

We extend the map $\iota_X:\Omega^p(M)\to\Omega^{p-1}(M)$ *by linearity to a map* $\iota_X:\Omega^*(M)\to\Omega^*(M)$ *(with* $\Omega^{-1}(M)=0$*).*

Lemma 7.2.4. *Let M be a manifold. Two derivations of* $\Omega^*(M)$ *which coincide on functions and commute with* d *are equal.*

Proposition 7.2.5 (Lie-Cartan calculus)**.** *Let M be a manifold. If* $\alpha\in\Omega^p(M)$, $X,X_0,\dots,X_p\in\mathcal{X}(M)$*, we have the following properties:*

(i) $\mathrm{d} \circ L_X = L_X \circ \mathrm{d}$.

(ii) Cartan's formula.

$$L_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}.$$

(iii) Leibniz's rule.

$$L_X\left(\alpha\left(X_1, \dots, X_p\right)\right) = \left(L_X \alpha\right)\left(X_1, \dots, X_p\right) + \sum_{i=1}^{p} \alpha\left(X_1, \dots, \left[X, X_i\right], \dots, X_p\right).$$

(iv) *Expression of* d *in terms of* L_X.

$$\mathrm{d}\alpha\left(X_0, \dots, X_p\right) = \sum_{i=0}^{p}(-1)^i L_{X_i}\left(\alpha\left(X_0, \dots, \hat{X}_i, \dots, X_p\right)\right) + \sum_{0 \leq i < j \leq p}(-1)^{i+j} \alpha\left(\left[X_i, X_j\right], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p\right).$$

Proof. (ii) Use Lemma 7.2.4. □

8 De Rham cohomology

8.1 Definition and general properties

Definition 8.1.1 (Closed and exact forms)**.** *Let M be a manifold. Consider the differential operator $\mathrm{d} : \Omega^*(M) \to \Omega^*(M)$. An element of* $\operatorname{Ker} \mathrm{d}$ *is called a* closed form*; an element of* $\operatorname{Im} \mathrm{d}$ *is called an* exact form*. Since* $\mathrm{d} \circ \mathrm{d} = 0$*, we have* $\operatorname{Im} \mathrm{d} \subseteq \operatorname{Ker} \mathrm{d}$.

Definition 8.1.2 (De Rham cohomology)**.** *Let M be a manifold. The* de Rham cohomology *of M is defined by:*

$$H^*(M) = \operatorname{Ker} \mathrm{d} / \operatorname{Im} \mathrm{d}.$$

Since d *sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$, $H^*(M)$ is a graded vector space:*

$$H^*(M) = \bigoplus_{p \in \mathbb{N}} H^p(M),$$

where $H^p(M) = \operatorname{Ker} \mathrm{d}_{|\Omega^p(M)} / \operatorname{Im} \mathrm{d}_{|\Omega^{p-1}(M)}$.

Proposition 8.1.3. *Let M be a manifold. The exterior product $\wedge$ on $\Omega^*(M)$ induces a product $\wedge$ on $H^*(M)$. More explicitly:*

(i) *If $\alpha, \beta \in \Omega^*(M)$ are closed, then $\alpha \wedge \beta$ is also closed.*

(ii) *If $\alpha, \alpha', \beta, \beta' \in \Omega^*(M)$ are closed with $(\alpha - \alpha')$ and $(\beta - \beta')$ exact, then $(\alpha \wedge \beta - \alpha' \wedge \beta')$ is exact.*

Thus, $H^*(M)$ is a graded algebra over $\mathbb{R}$.

Vocabulary 8.1.4. *Let M be a manifold. If $\alpha, \alpha' \in \Omega^*(M)$ are closed differential forms s.t. $(\alpha - \alpha')$ is exact, we say that α and α' are* cohomologous*. Moreover, the image of α in $H^*(M)$ is called the* cohomology class *of α and denoted by $[\alpha]$.*

Proposition 8.1.5. *Let $f : M \to N$ be a smooth map between manifolds. Then $f^* : \Omega^*(N) \to \Omega^*(M)$ is a morphism of differential graded algebras, inducing a morphism $f^* : H^*(N) \to H^*(M)$ of graded algebras. Hence, H^* defines a functor from the category of smooth manifolds to the category of graded algebras.*

Remark 8.1.6. *Let M be a smooth manifold. We have:*

$$H^0(M) = \left\{f \in \mathcal{C}^\infty(M, \mathbb{R}),\ \mathrm{d}f = 0\right\} = \left\{f \in \mathcal{C}^\infty(M, \mathbb{R}),\ f \textit{ is locally constant}\right\} = \mathbb{R}^{\Pi_0(M)},$$

where $\Pi_0(M)$ is the set of connected components of M.

8.2 Invariance by homotopy

Definition 8.2.1 (Homotopy)**.** *Two maps $f, g : M \to N$ between topological spaces (resp. manifolds) are said to be* homotopic *(resp.* smoothly homotopic*) if there exists a continuous (resp. smooth) map $h : [0,1] \times M \to N$, called a* homotopy *from f to g, s.t.*

$$h(0, \cdot) = f \qquad \text{and} \qquad h(1, \cdot) = g.$$

Remark 8.2.2. *Using the density of $\mathcal{C}^\infty(M, N)$ in $\mathcal{C}^0(M, N)$, one can prove the following facts:*

(i) *If $f, g \in \mathcal{C}^\infty(M, N)$ are homotopic, then they are smoothly homotopic.*

(ii) *For all $f \in \mathcal{C}^0(M, N)$, there exists $\widetilde{f} \in \mathcal{C}^\infty(M, N)$ s.t. f is homotopic to $\widetilde{f}$.*

The set $\mathcal{C}^\infty(M, N)$ quotiented by the relation of smooth homotopy is denoted by $[M, N]$.

Theorem 8.2.3. *Let $f, g : M \to N$ be smooth maps between manifolds. If f and g are homotopic, then the morphisms $f^*, g^* : H^*(N) \to H^*(M)$ are equal.*

Proof. The idea of the proof is to construct a homotopy operator $H : \Omega^*(N) \to \Omega^*(M)$ of degree (-1) and s.t.

$$H \circ \mathrm{d} + \mathrm{d} \circ H = g^* - f^*, \tag{$*$}$$

where $f^*, g^* : \Omega^*(N) \to \Omega^*(M)$. Using H, we easily show that $f^* = g^*$ (as maps $H^*(N) \to H^*(M)$). To construct H, we set:

$$H : \beta \in \Omega^*(N) \longmapsto \int_0^1 (\{t\} \times \mathrm{id}_M)^* \left(\iota_T h^* \beta\right) \mathrm{d}t \in \Omega^*(M),$$

where $T = \frac{\partial}{\partial t} \in \mathcal{X}([0,1] \times M)$ and h is a smooth homotopy from f to g. We then show that H satisfies $(*)$. ☐

Corollary 8.2.4. *H^* defines a functor from the category of homotopy types of smooth manifolds to the category of graded algebras.*

Corollary 8.2.5 (Poincaré's Lemma)**.** *Let M be a manifold. Then the differential* d *is locally exact in positive degrees. More precisely, if $U \subseteq M$ is diffeomorphic to $\mathbb{R}^n$, then every closed form $\beta \in \Omega^p(U)$ is exact for $p \in \mathbb{N}^*$. Therefore, $H^*(U) = H^0(U) = \mathbb{R}$.*

Proof. Show that we can assume U to be a star-shaped open set in $\mathbb{R}^n$ and deduce that there exists a homotopy from a point to id_U and show that $\beta = \mathrm{d}(H\beta)$ for every $\beta \in \Omega^p(M)$ $(p \in \mathbb{N}^*)$, where H is the homotopy operator of the proof of Theorem 8.2.3. ☐

8.3 Elementary study of H^1

Definition 8.3.1 (Integration along a piecewise $\mathcal{C}^1$ path)**.** *Let M be a manifold. Let $\gamma : [a, b] \to M$ be a piecewise $\mathcal{C}^1$ path and let $\alpha \in \Omega^1(M)$. We define:*

$$\int_\gamma \alpha = \int_a^b \alpha_{\gamma(t)} \left(\gamma'(t)\right) \mathrm{d}t.$$

This integral is invariant under direct reparametrisation and changes sign under indirect reparametrisation.

Proposition 8.3.2. *Let M be a manifold. Let $\gamma : [a, b] \to M$ be a piecewise $\mathcal{C}^1$ path. For $f \in \Omega^0(M)$, we have:*

$$\int_\gamma \mathrm{d}f = f\left(\gamma(b)\right) - f\left(\gamma(a)\right).$$

In particular, if γ is a loop, then $\int_\gamma \alpha = 0$ for any exact form $\alpha \in \Omega^1(M)$.

Example 8.3.3. *On the manifold $M = \mathbb{R}^2 \backslash \{0\}$, the* 1*-form $\alpha = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$ is closed but not exact. In particular $H^*\left(\mathbb{R}^2 \backslash \{0\}\right) \neq 0$.*

Proposition 8.3.4. *Let M be a connected manifold. If $\alpha \in \Omega^1(M)$, we define the* integration morphism *$I(\alpha)$ on the set $\Lambda_{x_0}(M)$ of piecewise $\mathcal{C}^1$ loops $[0,1] \to M$ based at x_0 by:*

$$I(\alpha) : \gamma \in \Lambda_{x_0}(M) \longmapsto \int_\gamma \alpha \in \mathbb{R}.$$

We obtain a map $I : \Omega^1(M) \to \mathbb{R}^{\Lambda_{x_0}(M)}$. This map induces a map:

$$\overline{I} : H^1(M) \to \operatorname{Hom}\left(\Pi_1(M), \mathbb{R}\right),$$

where $\Pi_1(M)$ is the fundamental group of M. The map $\overline{I}$ is an isomorphism of vector spaces.

Corollary 8.3.5. *If M is a connected manifold, then:*

$$H^1(M) \simeq \operatorname{Hom}\left(\Pi_1(M), \mathbb{R}\right).$$

9 Integration on manifolds

9.1 Definitions

Notation 9.1.1. *Let M be a manifold. For $k \in \mathbb{N} \cup \{\infty\}$, we write $\mathcal{C}^k\Omega^p(M)$ for the set of $\mathcal{C}^k$ p-forms $M \to \Lambda^p T^*M$. Moreover, we write $\mathcal{C}^k_c\Omega^p(M)$ for the set of elements of $\mathcal{C}^k\Omega^p(M)$ with compact support.*

Definition 9.1.2 (Integral of a n-form on $\mathbb{R}^n$)**.** *Let $\alpha \in \mathcal{C}^0_c\Omega^n(U)$, where U is an open subset of $\mathbb{R}^n$. Then there exists $f \in \mathcal{C}^0_c(U, \mathbb{R})$ s.t. $\alpha = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$; we define:*

$$\int_U \alpha = \int_U f\left(x_1, \ldots, x_n\right)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Proposition 9.1.3. *Let $\varphi : U \to V$ be a $\mathcal{C}^1$ direct diffeomorphism (i.e. with $\operatorname{jac}\varphi > 0$) between open subsets of $\mathbb{R}^n$. Then, for $\beta \in \mathcal{C}^0_c\Omega^n(V)$, we have:*

$$\int_V \beta = \int_U \varphi^*\beta.$$

If φ is indirect, then $\int_V \beta = -\int_U \varphi^*\beta$.

Definition 9.1.4 (Integral of a n-form on a n-manifold)**.** *Let M be an oriented manifold of dimension n and $\alpha \in \mathcal{C}^0_c\Omega^n(M)$. If $\left(U_i, \varphi_i\right)_{1 \leq i \leq k}$ is a finite family of direct charts covering the support of α and if $\left(f_i\right)_{1 \leq i \leq k}$ is a partition of unity subordinated to $\left(U_i\right)_{1 \leq i \leq k}$, we set:*

$$\int_M \alpha = \sum_{i=1}^k \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* \left(f_i \alpha\right).$$

We check that $\int_M \alpha$ is well-defined, i.e. it is independent of the choice of the charts and the partition of unity.

Remark 9.1.5. *Let M be an oriented manifold of dimension n and $\alpha \in \mathcal{C}^0_c\Omega^n(M)$. Then Definition 9.1.4 induces a continuous linear form $I_\alpha : \mathcal{C}^0_c(M, \mathbb{R}) \to \mathbb{R}$ defined by $I_\alpha(f) = \int_M f\alpha$; hence, it induces a Radon measure μ_α on M. This allows one to extend the definition of $\int_B \alpha$ to any Borel set B s.t. $\mathbb{1}_B$ is integrable w.r.t. $|\mu_\alpha|$.*

9.2 Stokes' formula

Definition 9.2.1 (Topological domain)**.** *In a topological space X, a* domain *is a set Ω s.t. $\Omega = \overline{\mathring{\Omega}}$ (in particular, Ω is closed).*

Definition 9.2.2 (Domain with $\mathcal{C}^k$ boundary)**.** *Let M be a $\mathcal{C}^k$ manifold of dimension n. A* domain with $\mathcal{C}^k$ boundary *(or $\mathcal{C}^k$ domain) is a closed subset $\Omega \subseteq M$ s.t. the (topological) boundary $\partial\Omega$ is a $\mathcal{C}^k$ submanifold of M. Equivalently, each point $x \in \partial\Omega$ admits a centred chart (of M) $\varphi : (U, x) \to (\varphi(U), 0)$ s.t.*

$$\varphi\left(U \cap \partial\Omega\right) = \varphi(U) \cap \left(\{0\} \times \mathbb{R}^{n-1}\right).$$

If M is oriented, we always assume that φ is direct and that:

$$\varphi\left(U \cap \Omega\right) = \varphi(U) \cap \left(]-\infty, 0] \times \mathbb{R}^{n-1}\right).$$

Remark 9.2.3. *A $\mathcal{C}^k$ domain is a topological domain.*

Proposition 9.2.4. *If M is an oriented manifold and Ω is a $\mathcal{C}^1$ domain, then $\partial\Omega$ is oriented by the outward normal.*

Notation 9.2.5. *Let M be an oriented manifold and Ω be a $\mathcal{C}^1$ domain. For $\beta \in \mathcal{C}^0\Omega^{n-1}(\Omega)$ with* Supp $\beta \cap \Omega$ *compact, we define:*

$$\int_{\partial\Omega} \beta = \int_{\partial\Omega} i^*\beta,$$

where $i : \partial\Omega \to \Omega$ is the natural inclusion.

Theorem 9.2.6 (Stokes' formula)**.** *Let M be an oriented manifold and Ω be a $\mathcal{C}^1$ domain. Let $\alpha \in \mathcal{C}^1\Omega^{n-1}(M)$ with* Supp $\alpha \cap \Omega$ *compact. Then:*

$$\int_\Omega \mathrm{d}\alpha = \int_{\partial\Omega} \alpha.$$

Proof. We cover $\operatorname{Supp}\alpha \cap \Omega$ by a finite number of direct charts $(U_i, \varphi_i)_{1\le i\le k}$, together with a subordinated $\mathcal{C}^1$ partition of unity $(f_i)_{1\le i\le k}$. Thus, we can assume that $M = \mathbb{R}^n$, $\alpha \in \mathcal{C}^1_c\Omega^n(\mathbb{R}^n)$ and Ω is either $\mathbb{R}^n$ or $]-\infty, 0] \times \mathbb{R}^{n-1}$. If $\Omega = \mathbb{R}^n$, then we may write $\alpha = f(x_1, \dots, x_n)\, \mathrm{d}x_2 \wedge \dots \wedge \mathrm{d}x_n$ in a well-chosen system of coordinates. Thus:

$$\int_\Omega \mathrm{d}\alpha = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, \mathrm{d}x_1\right) \mathrm{d}x_2 \cdots \mathrm{d}x_n = 0.$$

If $\Omega =]-\infty, 0] \times \mathbb{R}^{n-1}$, then either $\alpha = f(x_1, \dots, x_n)\, \mathrm{d}x_2 \wedge \dots \wedge \mathrm{d}x_n$ or $\alpha = f(x_1, \dots, x_n)\, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1}$. In the first case:

$$\begin{aligned}\int_\Omega \mathrm{d}\alpha &= \int_{]-\infty,0]\times\mathbb{R}^{n-1}} \frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^0 \frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, \mathrm{d}x_1\right) \mathrm{d}x_2 \cdots \mathrm{d}x_n \\ &= \int_{\mathbb{R}^{n-1}} f(0, x_2, \dots, x_n)\, \mathrm{d}x_2 \cdots \mathrm{d}x_n = \int_{\partial\Omega} \alpha.\end{aligned}$$

In the second case:

$$\begin{aligned}\int_\Omega \mathrm{d}\alpha &= \int_{]-\infty,0]\times\mathbb{R}^{n-1}} (-1)^{n-1} \frac{\partial f}{\partial x_n}(x_1, \dots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_{]-\infty,0]\times\mathbb{R}^{n-2}} (-1)^{n-1} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_n}(x_1, \dots, x_n)\, \mathrm{d}x_n\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = 0 = \int_{\partial\Omega} \alpha.\end{aligned}$$

□

Remark 9.2.7. *Stokes' formula is a generalisation of the fact that $\int_a^b f'(x)\, \mathrm{d}x = f(b) - f(a)$.*

9.3 Application: Brouwer's Fixed Point Theorem

Notation 9.3.1. *We denote by $\mathbb{D}^n$ the closed unit n-disk; it is a compact domain with $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$.*

Theorem 9.3.2 (Brouwer's Fixed Point Theorem)**.**

(i) *If $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then f has a fixed point.*

(ii) *There is no continuous map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ s.t. $g_{|\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$.*

Proof. Note that (ii) $\Rightarrow$ (i): if $f : \mathbb{D}^n \to \mathbb{D}^n$ is a continuous map without a fixed point, define a map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ by choosing $g(x)$ to be the unique element of $\mathbb{S}^{n-1} \cap \mathbb{R}_+ (x - f(x))$; hence, g is continuous and $g_{|\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$. Now, let us prove (ii). Assume for contradiction that there is a continuous map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ with $g_{|\mathbb{S}^{n-1}} = \mathrm{id}_{\mathbb{S}^{n-1}}$. By approximation, we may assume g to be smooth. We now let $\alpha = x_1 \,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^{n-1}(\mathbb{R}^n)$. Note that $\operatorname{rk} \mathrm{d}g_x < n$ for all x, therefore $g^*(\mathrm{d}\alpha) = 0$. But, using Stokes' formula:

$$0 = \int_{\mathbb{D}^n} g^*(\mathrm{d}\alpha) = \int_{\mathbb{D}^n} \mathrm{d}(g^*\alpha) = \int_{\mathbb{S}^{n-1}} g^*\alpha = \int_{\mathbb{S}^{n-1}} \alpha = \int_{\mathbb{D}^n} \mathrm{d}\alpha = \mathrm{vol}(\mathbb{D}^n) > 0.$$

This is a contradiction. □

10 Cohomolohy in maximal degree

10.1 De Rham cohomology with compact supports

Definition 10.1.1 (Compactly supported differential forms)**.** *Let M be a manifold. We let $\Omega_c^*(M) = \bigcup_{p\in\mathbb{N}} \Omega_c^p(M)$ be the set of compactly supported differential forms on M. It is a differential graded nonunital subalgebra of $\Omega^*(M)$.*

Definition 10.1.2 (De Rham cohomology with compact supports)**.** *Let M be a manifold. For $p \in \mathbb{N}$, we define:*

$$H_c^*(M) = \operatorname{Ker} \mathrm{d}_{|\Omega_c^*(M)} / \operatorname{Im} \mathrm{d}_{|\Omega_c^*(M)}.$$

$H_c^*(M)$ is a graded vector space:

$$H_c^*(M) = \bigoplus_{p\in\mathbb{N}} H_c^p(M),$$

where $H_c^p(M) = \operatorname{Ker} \mathrm{d}_{|\Omega_c^p(M)} / \operatorname{Im} \mathrm{d}_{|\Omega_c^{p-1}(M)}$.

Definition 10.1.3 (Integration morphism)**.** *Let M be an oriented manifold of dimension n. Then we define the* integration morphism $I_M : \alpha \in \Omega_c^n(M) \longmapsto \int_M \alpha \in \mathbb{R}$. *By Stokes' formula (Theorem 9.2.6), I_M induces a surjective morphism $\overline{I}_M : H_c^n(M) \to \mathbb{R}$.*

10.2 Computation of $H_c^n(\mathbb{R}^n)$

Proposition 10.2.1. *For $n \in \mathbb{N}^*$, $\overline{I} : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ is an isomorphism.*

Proof. It suffices to prove that $\overline{I}$ is injective, i.e. if $\alpha \in \Omega_c^n(\mathbb{R}^n)$ is s.t. $\int_{\mathbb{R}^n} \alpha = 0$, then there exists $\beta \in \Omega_c^{n-1}(\mathbb{R}^n)$ s.t. $\alpha = \mathrm{d}\beta$. By induction on n, we prove the following stronger statement: if $k \in \mathbb{N}^*$ and $(\alpha_t)_{t\in\mathbb{R}^k}$ is a smooth family of forms in $\Omega_c^n(\mathbb{R}^n)$ with $\operatorname{Supp} \alpha_t \subseteq [-R, +R]^n$ and $\int_{\mathbb{R}^n} \alpha_t = 0$ for all t, then there exists a smooth family $(\beta_t)_{t\in\mathbb{R}^k}$ in $\Omega_c^{n-1}(\mathbb{R}^n)$ s.t. $\operatorname{Supp} \beta_t \subseteq [-R, +R]^n$ and $\alpha_t = \mathrm{d}\beta_t$ for all t. For $n = 1$, write $\alpha_t = f_t \,\mathrm{d}x$ and take $\beta_t(x) = \int_{-\infty}^x f_t(u)\,\mathrm{d}u$. Assume we have proved the result for $(n-1)$ and write:

$$\alpha_t = \alpha_{t,x_n} \wedge \mathrm{d}x_n,$$

with $\alpha_{t,x_n} \in \Omega_c^{n-1}(\mathbb{R}^{n-1})$, $\operatorname{Supp} \alpha_{t,x_n} \subseteq [-R, +R]^{n-1}$. Now, set $I_t(x_n) = \int_{\mathbb{R}^{n-1}} \alpha_{t,x_n}$; thus $I_t \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ and $\operatorname{Supp} I_t \subseteq [-R, +R]$. By Fubini, $0 = \int_{\mathbb{R}^n} \alpha_t = \int_{\mathbb{R}} I_t$. Thus, the function $g_t(x_n) =$

$\int_{-\infty}^{x_n} I_t(u)\,\mathrm{d}u$ has support in $[-R,+R]$ and $\mathrm{d}g_t = I_t(x_n)\,\mathrm{d}x_n$. Choose a form $\sigma \in \Omega_c^{n-1}(\mathbb{R}^{n-1})$ with $\int_{\mathbb{R}^{n-1}} \sigma = 1$ and define:

$$\widetilde{\alpha}_{t,x_n} = \alpha_{t,x_n} - I_t(x_n)\,\sigma,$$

thus $\int_{\mathbb{R}^{n-1}} \widetilde{\alpha}_{t,x_n} = 0$. By induction, there exists $\beta_{t,x_n} \in \Omega_c^{n-2}(\mathbb{R}^{n-1})$ s.t. $\operatorname{Supp}(\beta_{t,x_n}) \subseteq [-R,+R]^{n-1}$ and $\mathrm{d}\beta_{t,x_n} = \widetilde{\alpha}_{t,x_n}$. Now, we obtain $\alpha_t = \mathrm{d}(\beta_{t,x_n} \wedge \mathrm{d}x_n + (-1)^{n-1} g_t(x_n)\,\sigma)$, which proves the result. □

10.3 Computation of $H_c^n(M)$

Theorem 10.3.1. *Let M be a connected oriented manifold of dimension $n \in \mathbb{N}^*$. Then $\overline{I} : H_c^n(M) \to \mathbb{R}$ is an isomorphism.*

Proof. Let $\alpha \in \Omega_c^n(M)$ s.t. $\int_M \alpha = 0$. Let us show that there exists $\beta \in \Omega_c^{n-1}(M)$ s.t. $\alpha = \mathrm{d}\beta$. We cover $\operatorname{Supp}\alpha$ by relatively compact open sets $(U_i)_{1 \le i \le k}$ diffeomorphic to $\mathbb{R}^n$. We order these sets in such a way that $U_i \cap U_{i+1} \neq \varnothing$ for all i. Now, we construct β by induction on k. If $k=1$, we just apply Proposition 10.2.1. Assume we have proved the result for $(n-1)$. Set $V = U_1 \cup \cdots \cup U_{k-1}$, so that $M \subseteq V \cup U_k$. Let (ρ_1, ρ_2) be a partition of unity with $\operatorname{Supp}\rho_1 \subseteq V$, $\operatorname{Supp}\rho_2 \subseteq U_k$, $\rho_1 + \rho_2 = \mathbb{1}_{U_1 \cup \cdots \cup U_k}$. We have $\alpha = \rho_1\alpha + \rho_2\alpha$, but $\int_M \rho_1\alpha = -\int_M \rho_2\alpha \neq 0$ a priori. Since $V \cap U_k \neq \varnothing$, there exists $\sigma \in \Omega_c^n(M)$ with $\operatorname{Supp}\sigma \subseteq V \cap U_k$ and $\int_M \sigma = \int_M \rho_1\alpha$. Now, write:

$$\alpha = \underbrace{(\rho_1\alpha - \sigma)}_{\widetilde{\alpha}_1} + \underbrace{(\rho_2\alpha + \sigma)}_{\widetilde{\alpha}_2}.$$

Thus, $\widetilde{\alpha}_1 \in \Omega_c^n(V)$, $\int_V \alpha_1 = 0$, $\widetilde{\alpha}_2 \in \Omega_c^n(U_k)$ and $\int_{U_k} \widetilde{\alpha}_2 = 0$. By induction, there exist $\beta_1 \in \Omega_c^n(V)$ and $\beta_2 \in \Omega_c^n(U_k)$ s.t. $\widetilde{\alpha}_1 = \mathrm{d}\beta_1$ and $\widetilde{\alpha}_2 = \mathrm{d}\beta_2$. Hence, $\alpha = \mathrm{d}(\beta_1 + \beta_2)$, which proves the result. □

Corollary 10.3.2. *If M is a compact, connected, oriented manifold of dimension $n \in \mathbb{N}^*$, then $\overline{I} : H^n(M) \to \mathbb{R}$ is an isomorphism.*

Remark 10.3.3. *If M is a manifold that is not connected, let $(M_i)_{i \in I}$ be the connected components of M. Then:*

(i) $\Omega^*(M) = \prod_{i \in I} \Omega^*(M_i)$ *and* $H^*(M) = \prod_{i \in I} \Omega^*(M_i)$.

(ii) $\Omega_c^*(M) = \bigoplus_{i \in I} \Omega_c^*(M_i)$ *and* $H_c^*(M) = \bigoplus_{i \in I} H_c^*(M_i)$.

Remark 10.3.4. *Let M be a connected manifold of dimension $n \in \mathbb{N}^*$.*

(i) *If M is not compact, then $H^n(M) = 0$.*

(ii) *If M is not orientable, then $H_c^n(M) = 0$.*

10.4 Degree of a map

Definition 10.4.1 (Degree of a smooth (proper) map)**.** *Let $f : M \to N$ be a smooth proper map between two connected oriented manifolds of the same dimension $n \in \mathbb{N}^*$. Then $f^* : H_c^n(N) \to H_c^n(M)$ induces a linear map $\mathbb{R} \to \mathbb{R}$; this map is the multiplication by a real number which we call the* degree *of f and which we denote by* $\deg f$. *In other words,* $\deg f$ *is the real number which makes the following diagram commute:*

$$N \xleftarrow{\;f\;} M$$

$$\begin{array}{ccc} H_c^n(N) & \xrightarrow{f^*} & H_c^n(M) \\ \overline{I}_N \big\downarrow & & \big\downarrow \overline{I}_M \\ \mathbb{R} & \xrightarrow{\times \deg f} & \mathbb{R} \end{array}$$

The degree is invariant under proper homotopy.

Remark 10.4.2. *If M and N are two compact, connected, oriented manifolds of the same dimension, then the degree is well-defined for any smooth map $f : M \to N$. It is invariant under homotopy.*

Remark 10.4.3. *Let $f : M \to N$ be a smooth proper map between two connected oriented manifolds of the same dimension $n \in \mathbb{N}^*$. Then, for any $\beta \in \Omega_c^n(N)$ s.t. $\int_N \beta = 1$, we have:*

$$\deg f = \int_M f^*\beta.$$

Remark 10.4.4. *Let M be a compact, connected, oriented manifold. Then $\deg(\mathrm{id}_M) = 1$, so id_M is not homotopic to a constant.*

Definition 10.4.5 (Local degree)**.** *Let $f : M \to N$ be a smooth proper map between two connected oriented manifolds of the same dimension $n \in \mathbb{N}^*$. If $x \in M$ is a regular point of f, we define the local degree $\deg_x(f) \in \{\pm 1\}$ of f at x by $\deg_x(f) = 1$ iff $\mathrm{d}f_x : T_xM \to T_xN$ preseves the orientation.*

Theorem 10.4.6. *Let $f : M \to N$ be a smooth proper map between two connected oriented manifolds of the same dimension $n \in \mathbb{N}^*$. For any regular value $y \in N$ of f, we have:*

$$\deg f = \sum_{x \in f^{-1}(\{y\})} \deg_x(f).$$

In particular, $\deg f \in \mathbb{Z}$ (because Sard's Theorem implies that f has at least one regular value).