# Hyperbolic Geometry & Discrete Groups

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# 1 The hyperbolic plane and its geometry

## 1.1 The upper half-plane model

**Definition 1.1** (Upper half-plane)**.** *The **upper half-plane** is the set $\mathcal{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.*

*Given $z \in \mathcal{H}$, the **hyperbolic norm** at $z$ of a vector $v \in \mathbb{C}$ is $\|v\|_z^{\mathrm{hyp}} = \frac{\|v\|^{\mathrm{eucl}}}{\Im(z)}$.*

*Given a $\mathcal{C}^1$ path $c : [a, b] \to \mathcal{H}$, its **hyperbolic length** is*

$$\ell_{\mathrm{hyp}}(c) = \int_a^b \|c'(t)\|_{c(t)}^{\mathrm{hyp}} \, \mathrm{d}t.$$

*The subscripts and superscripts* hyp *will be dropped from the notation.*

**Example 1.2.** *Consider the path $c : t \in [y_1, y_2] \mapsto it$, with $y_1, y_2 \in \mathbb{R}_{>0}$. Then $\ell(c) = \left|\log\left(\frac{y_2}{y_1}\right)\right|$.*

**Proposition 1.3.** *The hyperbolic length of a path is invariant under (increasing) reparametrisation, i.e. if $\varphi : [a', b'] \to [a, b]$ is $\mathcal{C}^1$ and increasing, then $\ell_{\mathrm{hyp}}(c \circ \varphi) = \ell_{\mathrm{hyp}}(c)$.*

**Definition 1.4** (Hyperbolic distance)**.** *Given $z_1, z_2 \in \mathcal{H}$, the **hyperbolic distance** between $z_1$ and $z_2$ is*

$$d_{\mathrm{hyp}}(z_1, z_2) = \inf\left\{\ell_{\mathrm{hyp}}(c),\ c \text{ is a continuous piecewise } \mathcal{C}^1 \text{ path from } z_1 \text{ to } z_2 \text{ in } \mathcal{H}\right\}.$$

**Proposition 1.5.** *$d_{\mathrm{hyp}}$ is a metric on $\mathcal{H}$, and it induces the Euclidean topology.*

*Proof.* The only nonobvious fact is the separation property, namely $d(z_1, z_2) > 0$ for $z_1 \neq z_2$. Consider a piecewise $\mathcal{C}^1$ curve $c : [a, b] \to \mathcal{H}$ from $z_1$ to $z_2$. Write $c(t) = x(t) + iy(t)$.

- If $y(t) \leqslant 2y(a) = 2\Im(z_1)$ for all $t$, then we have

$$\ell_{\mathrm{hyp}}(c) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} \, \mathrm{d}t \geqslant \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{2y(a)} \, \mathrm{d}t \geqslant \frac{\ell_{\mathrm{eucl}}(c)}{2y(a)} \geqslant \frac{d_{\mathrm{eucl}}(z_1, z_2)}{2\Im(z_1)}.$$

- Otherwise, there exists $t_0$ (chosen minimal) s.t. $y(t_0) = 2y(a)$. Let $c' = c_{|[a,t_0]}$. Then $\ell_{\text{eucl}}(c')$ is at least the Euclidean distance between $z_1$ and the line $\{y = 2y(a)\}$, i.e. $\ell_{\text{eucl}}(c') \geqslant y(a)$. Therefore
$$\ell_{\text{hyp}}(c) \geqslant \ell_{\text{hyp}}(c') \geqslant \frac{\ell_{\text{eucl}}(c')}{2y(a)} \geqslant \frac{1}{2}.$$

This proves that $d_{\text{hyp}}(z_1, z_2) \geqslant \min\left\{\frac{1}{2}, \frac{d_{\text{eucl}}(z_1,z_2)}{2\Im(z_1)}\right\} > 0$ if $z_1 \neq z_2$. ☐

**Proposition 1.6.** *Let $p : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection on $i\mathbb{R}_{>0}$: $z \mapsto \Im(z)i$. Then $p$ does not increase the length of paths, i.e. $\ell(p \circ c) \leqslant \ell(c)$, with equality* iff *$c$ is vertical.*

*Proof.* Let $c : [a, b] \to \mathcal{H}$, $c(t) = x(t) + iy(t)$. Then
$$\ell_{\text{hyp}}(p \circ c) = \int_a^b \frac{|y'(t)|}{y(t)} \,\mathrm{d}t \leqslant \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} \,\mathrm{d}t = \ell_{\text{hyp}}(c).$$
☐

**Corollary 1.7.** (i) *Given $z_1, z_2 \in \mathcal{H}$, we have $d_{\text{hyp}}(z_1, z_2) \geqslant \left|\log\left(\frac{\Im(z_2)}{\Im(z_1)}\right)\right|$.*

(ii) *If $z_1, z_2 \in i\mathbb{R}_{>0}$, then the vertical segment is the unique shortest path between $z_1$ and $z_2$.*

**Definition 1.8** (Isometries)**.** *Let $f : \mathcal{H} \to \mathcal{H}$ be a bijection.*

- *$f$ is an **isometry of metric spaces** if $d(f(z_1), f(z_2)) = d(z_1, z_2)$ for all $z_1, z_2 \in \mathcal{H}$.*
- *$f$ is a **Riemannian isometry** if it is $\mathcal{C}^1$ and $\mathrm{d}f_z$ preserves the hyperbolic norm for all $z \in \mathcal{H}$.*

*Riemannian isometries preserve the lengths of curves and therefore they are isometries of metric spaces.*

**Example 1.9.** *Horizontal translations, reflections along a vertical line, and homotheties, are all examples of Riemannian isometries.*

**Corollary 1.10.** *$\mathcal{H}$ is homogeneous, i.e. the action* $\operatorname{Isom}(\mathcal{H}) \curvearrowright \mathcal{H}$ *is transitive.*

## 1.2 Inversions and the Möbius group

**Definition 1.11** (Inversion)**.** *The **standard inversion** in $\mathbb{R}^n$ is the map $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $x \mapsto \frac{x}{\|x\|^2}$. It extends to a map $\hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$, where $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.*

*The standard inversion is an involution fixing $\mathbb{S}^{n-1}$ pointwise.*

*In general, given $a \in \mathbb{R}^n$ and $r > 0$, the **inversion across** $S(a, r)$ is given by*
$$x \mapsto a + \left(\frac{r}{\|x - a\|}\right)^2 (x - a).$$

**Definition 1.12** (Möbius group)**.** *The **Möbius group** of $\hat{\mathbb{R}}^n$ is the group* $\text{Möb}\left(\hat{\mathbb{R}}^n\right)$ *generated by all inversions. We denote by* $\text{Möb}^+\left(\hat{\mathbb{R}}^n\right)$ *its index-2 subgroup consisting of orientation-preserving maps.*

**Proposition 1.13.** *All inversions (and therefore all Möbius transforms) send spheres to spheres (hyperplanes are seen as spheres containing $\infty$).*

*Proof.* It is enough to prove it for the standard inversion. Use the cartesian equation of a sphere and a hyperplane. ☐

**Remark 1.14.** *In dimension 2, the standard inversion is given by $z \mapsto \frac{1}{\bar{z}}$ in complex coordinates. In particular, it is conformal.*

**Proposition 1.15.** *All inversions (and therefore all Möbius transforms) are conformal.*

**Remark 1.16.** *The group* $\text{Möb}\left(\hat{\mathbb{C}}\right)$ *contains (Euclidean) reflections along a line, and therefore translations and rotations. It also contains homotheties (which can be decomposed as a product of two inversions). Therefore,* $\text{Möb}\left(\hat{\mathbb{C}}\right)$ *contains all maps of the form* $z \mapsto \frac{az+b}{cz+d}$*, with* $a, b, c, d \in \mathbb{C}$*. Those maps are called* ***homographies*** *or* ***linear fractional maps****.*

**Theorem 1.17.** $\text{Möb}\left(\hat{\mathbb{C}}\right)$ *is the group of linear and antilinear fractional maps, namely* $z \mapsto \frac{az+b}{cz+d}$ *and* $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$*, for* $a, b, c, d \in \mathbb{C}$*.*

## 1.3 Projective geometry and the projective linear group

**Definition 1.18** (Projective spaces)**.** *Given a* $\mathbb{K}$*-vector space* $V$*,* $\mathbb{P}(V)$ *is the set of lines of* $V$*. When* $V$ *has a topology (for instance if* $\mathbb{K}$ *is* $\mathbb{R}$ *or* $\mathbb{C}$*),* $\mathbb{P}(V) = V/\mathbb{K}^*$ *is endowed with the quotient topology.*

*A map* $\mathbb{P}(V) \to \mathbb{P}(W)$ *is called* ***projective*** *if it is induced by an injective linear map* $V \to W$*.*

*In particular,* $GL(V) \curvearrowright \mathbb{P}(V)$*. The kernel of this action is* $\mathbb{K}^* \,\text{id}$*; we set* $PGL(V) = GL(V)/\mathbb{K}^* \,\text{id}$*. More generally, if* $G \leqslant GL(V)$*, we set* $PG = G/G \cap \mathbb{K}^* \,\text{id}$*.*

*We write* $\mathbb{P}^n\mathbb{R} = \mathbb{P}\left(\mathbb{R}^{n+1}\right)$*,* $PGL_n\mathbb{K} = PGL\left(\mathbb{K}^n\right)$ *and* $PSL_n\mathbb{K} = PGL\left(\mathbb{K}^n\right)$*.*

**Remark 1.19.** *There is a homeomorphism* $\psi : \mathbb{K} \cup \{\infty\} \to \mathbb{P}^1\mathbb{K}$ *given by* $\psi(x) = (x : 1)$ *and* $\psi(\infty) = (1 : 0)$ *(where* $\mathbb{K}$ *is* $\mathbb{R}$ *or* $\mathbb{C}$*).*

*Under this identification, linear fractional maps correspond to the action* $PGL_2\mathbb{K} \curvearrowright \mathbb{P}^1\mathbb{K}$*: the matrix* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2\mathbb{K}$ *induces a map* $\mathbb{K} \cup \{\infty\} \to \mathbb{K} \cup \{\infty\}$ *given by* $z \mapsto \frac{az+b}{cz+d}$*.*

**Notation 1.20.** *From now on, we assume that* $\dim V = 2$*, i.e.* $\mathbb{P}(V)$ *is a* ***projective line****.*

**Proposition 1.21.** $PGL(V)$ *acts simply transitively on the set* $\mathbb{P}(V)^{3*}$ *of triples of distinct points of* $\mathbb{P}(V)$*, i.e. for all* $x, y \in \mathbb{P}(V)^{3*}$*, there is a unique* $g \in PGL(V)$ *s.t.* $gx = y$*.*

*Proof.* Assume $V = \mathbb{K}^2$ with its canonical basis $(e_1, e_2)$. Given distinct points $p_1, p_2, p_3$ in $\mathbb{P}(V)$, we want to find $g \in PGL(V)$ s.t. $g\left(p_1, p_2, p_3\right) = (0, 1, \infty)$. Let $p_i = [v_i]$, $v_i \in V$. Note that there exists $g \in GL(V)$ s.t. $gv_1 = e_2$ and $gv_3 = e_1$. Write $w_2 = gv_2 = ae_1 + be_2$. Composing with $h = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$, we obtain $hgp_1 = 0$, $hgp_2 = 1$ and $hgp_3 = \infty$. ☐

**Definition 1.22** (Cross-ratio)**.** *Let* $p, x, y, q$ *be four distinct points in* $\mathbb{P}(V)$*. The* ***cross-ratio*** $[p, x, y, q]$ *is the unique* $a \in \mathbb{K}$ *s.t.* $(p, x, y, q)$ *can be sent to* $(0, 1, a, \infty)$ *by a projective isomorphism* $\mathbb{P}(V) \to \mathbb{P}^1\mathbb{K}$*.*

*Equivalently, we have the formula*

$$[p, x, y, q] = \frac{y-p}{x-p} \cdot \frac{x-q}{y-q}.$$

*Proof.* To see that the two definitions are equivalent, consider the linear fractional map

$$t \mapsto \frac{t-p}{x-p} \cdot \frac{x-q}{t-q}.$$

☐

**Proposition 1.23.** *Let* $p, x, y, q$ *be four distinct points in* $\mathbb{P}(V)$*.*

(i) $[p, y, x, q] = [p, x, y, q]^{-1}$,

(ii) $[q, x, y, p] = [p, x, y, q]^{-1}$,

(iii) $[x, p, y, q] = 1 - [p, x, y, q]$.

**Proposition 1.24.** *If $V, W$ are two $\mathbb{K}$-vectors spaces of dimension 2, then projective maps $\mathbb{P}(V) \to \mathbb{P}(W)$ preserve the cross-ratio.*

*In particular, the action $PGL(V) \curvearrowright \mathbb{P}(V)$ preserves the cross-ratio.*

**Proposition 1.25.** *Assume that $\mathbb{K} = \mathbb{C}$ and $\mathbb{P}(V) = \mathbb{P}^1\mathbb{C}$.*

(i) *Three distinct points $z, \alpha, \beta \in \mathbb{C}$ are aligned* iff $[z, \alpha, \infty, \beta] \in \mathbb{R}$.

(ii) *Four distinct points $z, \alpha, \omega, \beta \in \mathbb{C}$ are cocyclic* iff $[z, \alpha, \omega, \beta] \in \mathbb{R}$.

*Proof.* For the first assertion, note that $z, \alpha, \beta$ are aligned iff the angle at $z$ is in $\pi\mathbb{Z}$, iff $\frac{z-\alpha}{z-\beta} \in \mathbb{R}$. For the second one, use an element of $PGL(V)$ sending $\omega$ to $\infty$. □

## 1.4 The Poincaré disk model

**Definition 1.26** (Poincaré disk). *The* ***Poincaré disk*** *is the set $\mathcal{B} = \{z \in \mathbb{C},\ |z| < 1\}$.*

*Given $z \in \mathcal{B}$, the* ***hyperbolic norm*** *at $z$ of a vector $v \in \mathbb{C}$ is $\|v\|_z^{\text{hyp}} = \frac{2}{1-|z|^2} \|v\|^{\text{eucl}}$.*

*Given a $\mathcal{C}^1$ path $c : [a, b] \to \mathcal{H}$, its* ***hyperbolic length*** *is*

$$\ell_{\text{hyp}}(c) = \int_a^b \|c'(t)\|_{c(t)}^{\text{hyp}}\, \mathrm{d}t.$$

**Definition 1.27** (Hyperbolic distance). *Given $z_1, z_2 \in \mathcal{B}$, the* ***hyperbolic distance*** *between $z_1$ and $z_2$ is*

$$d_{\text{hyp}}(z_1, z_2) = \inf\left\{\ell_{\text{hyp}}(c),\ c \text{ is a continuous piecewise } \mathcal{C}^1 \text{ path from } z_1 \text{ to } z_2 \text{ in } \mathcal{H}\right\}.$$

**Proposition 1.28.** *$d_{\text{hyp}}$ is a metric on $\mathcal{B}$.*

*Proof.* For separation, note that $d_{\text{hyp}} \geqslant 2d_{\text{eucl}}$. □

**Example 1.29.** *Rotations centred at 0 and reflections along diameters are Riemannian isometries of $\mathcal{B}$.*

**Theorem 1.30.** *The map*

$$\phi : z \in \mathcal{H} \longmapsto \frac{z-i}{z+i} \in \mathcal{B}$$

*is an isometry sending $(0, 1, \infty)$ to $(-1, -i, 1)$. The map $\phi$ is called the* ***Cayley transform****.*

*Therefore, $\mathcal{B}$ and $\mathcal{H}$ are equivalent models of the hyperbolic plane.*

## 1.5 Geodesics

**Definition 1.31** (Geodesics). *A* ***unit speed geodesic path*** *of $\mathcal{H}$ is a path $c : I \to \mathcal{H}$ (where $I$ is an interval) which locally minimises the distance: for all $t \in I$, there is an open interval $t \in J \subseteq I$ s.t. $c_{|J}$ is an isometric embedding.*

*A geodesic path defined on $\mathbb{R}$ will be called a* ***geodesic****.*

**Proposition 1.32.** $\text{Isom}(\mathcal{H})$ *acts transitively on pairs of points of $\mathcal{H}$ at a given distance.*

*Proof.* We show this in $\mathcal{B}$. Let $z_1 \neq z_2$ in $\mathcal{B}$. Since $\mathcal{B}$ is homogeneous (as $\mathcal{H}$), there exists $g \in \text{Isom}(\mathcal{B})$ s.t. $gz_1 = 0$. Now choose a rotation $\rho$ centred at 0 s.t. $\rho g z_2 \in \mathbb{R}_{\geqslant 0}$. Therefore, $\rho g$ sends $(z_1, z_2)$ to $(0, d)$ with $d = d(z_1, z_2)$. □

**Proposition 1.33.** *For all $z_1, z_2 \in \mathcal{H}$, there exists a unique shortest path between $z_1$ and $z_2$.*

*Proof.* By Proposition 1.32, we may assume that $z_1$ and $z_2$ lie on the $y$-axis. But then the result is known (c.f. Corollary 1.7). □

**Theorem 1.34.** *The geodesics of $\mathcal{H}$ are the circular arcs orthogonal to the boundary and the straight vertical rays.*

*Moreover, all geodesics in $\mathcal{H}$ are globally minimising, i.e. they are shortest paths.*

*Proof.* We know that the $y$-axis is a shortest path (c.f. Corollary 1.7). It follows that its image under the isometry $\phi : \mathcal{H} \to \mathcal{B}$ of Theorem 1.30 is a shortest path of $\mathcal{B}$. Its image is the diameter $[-1, 1]$ of $\mathcal{B}$. Since rotations centred at 0 are isometries of $\mathcal{B}$, all diameters are shortest paths of $\mathcal{B}$. Applying $\phi^{-1}$, we see that all circular arcs through $i$ and orthogonal to the boundary are shortest paths of $\mathcal{H}$. We then use horizontal translations and homotheties in $\mathcal{H}$ and obtain that all straight vertical rays and all circular arcs orthogonal to the boundary are shortest paths (and therefore geodesics).

Conversely, those are the only shortest paths of $\mathcal{H}$ by Proposition 1.33.

Moreover, geodesics and shortest paths coincide: if $\sigma : \mathbb{R} \to \mathcal{H}$ is a geodesic, we may assume (using Proposition 1.32) that $i = \sigma(t_0) \in \sigma(\mathbb{R})$ and there exists $\varepsilon > 0$ s.t. $\sigma_{|]t_0-\varepsilon,t_0+\varepsilon[}$ is vertical. Let $\eta(t) = ie^{(t-t_0)}$ be the unit speed parametrisation of the vertical line and assume for contradiction that $\eta \neq \sigma$. Let $t_1 = \max\{t \geqslant 0,\ \sigma(t) = \eta(t)\}$. Hence $\sigma = \eta$ on $[t_0, t_1]$ and the geodesic $\sigma$ forks at $t_1$; this contradicts Proposition 1.33. □

**Corollary 1.35.** *The geodesics of $\mathcal{B}$ are the arcs of circles (and line segments) orthogonal to the boundary.*

## 1.6 The full isometry group

**Proposition 1.36.** *We have an action $PGL_2\mathbb{C} \curvearrowright \hat{\mathbb{C}}$ by homographies. Under this action,*

(i) $\operatorname{Stab}\left(\hat{\mathbb{R}}\right) = PGL_2\mathbb{R}$,

(ii) $\operatorname{Stab}(\mathcal{H}) = PSL_2\mathbb{R}$.

*In particular, $PSL_2\mathbb{R} \curvearrowright \mathcal{H}$.*

*Proof.* If $g \in \operatorname{Stab}\left(\hat{\mathbb{R}}\right)$, then by Proposition 1.21, there exists $h \in PGL_2\mathbb{R}$ s.t. $g(0, 1, \infty) = h(0, 1, \infty)$, and the uniqueness in the context of $PGL_2\mathbb{C}$ implies that $g = h$, so $g \in PGL_2\mathbb{R}$. The reverse inclusion is clear.

The stabiliser of $\mathcal{H}$ is the stabiliser of the oriented circle $\hat{\mathbb{R}}$. We show that $PSL_2\mathbb{R} \subseteq \operatorname{Stab}(\mathcal{H})$ (for instance because $PSL_2\mathbb{R} \cdot i \subseteq \mathcal{H}$), and since $\left[\operatorname{Stab}\left(\hat{\mathbb{R}}\right) : \operatorname{Stab}(\mathcal{H})\right] = 2 = [PGL_2\mathbb{R} : PSL_2\mathbb{R}]$, we conclude that $\operatorname{Stab}(\mathcal{H}) = PSL_2\mathbb{R}$. □

**Proposition 1.37.** *The action $PSL_2\mathbb{R} \curvearrowright \mathcal{H}$ is by isometries.*

*Proof.* Note that $PSL_2\mathbb{R}$ is generated by inversions in circles containing $i$ (which correspond to reflections in line through 0 in $\mathcal{B}$, so they are isometries) and the translation $z \mapsto z + 1$, which is also an isometry. □

**Lemma 1.38.** *The stabiliser of $i$ under the action $PSL_2\mathbb{R} \curvearrowright \mathcal{H}$ acts transitively on hyperbolic geodesics through $i$.*

*Proof.* Given a hyperbolic geodesic $c$ through $i$ that meets $\partial\mathcal{H}$ at $\omega_1$ and $\omega_2$, find $g \in PSL_2\mathbb{R}$ s.t. $g\omega_1 = 0$ and $g\omega_2 = \infty$, and compose with a homothety so that $gi = i$. This sends $c$ to the $y$-axis. □

**Proposition 1.39.** *Hyperbolic circles are Euclidean circles (but generally not of the same centre!) in the upper half-plane and in the Poincaré disk.*

*Proof.* This is clear for hyperbolic circles centred at 0 in $\mathcal{B}$. But by transitivity of the action $PSL_2\mathbb{R} \curvearrowright \mathcal{B}$, any hyperbolic circle can be sent to a hyperbolic (and therefore Euclidean) circle centred at 0. Now the result follows from the fact that $PSL_2\mathbb{R}$ sends (Euclidean) circles to (Euclidean) circles. □

**Theorem 1.40.** $\mathrm{Isom}^+(\mathcal{H}) = PSL_2\mathbb{R}$.

*Proof.* We have seen that $PSL_2\mathbb{R} \subseteq \mathrm{Isom}^+(\mathcal{H})$. Conversely, let $f \in \mathrm{Isom}^+(\mathcal{H})$. We may compose $f$ with an element of $PSL_2\mathbb{R}$ to assume that $f$ fixes $i$ and $\infty$. It follows that $f$ fixes $i\mathbb{R}_{>0}$ pointwise. Now use the Cayley transform to get a corresponding isometry $\widetilde{f} \in \mathrm{Isom}^+(\mathcal{B})$ fixing the horizontal diameter. Therefore, given $z \in \mathcal{B}$, we have

$$d\left(0, \widetilde{f}(z)\right) = d\left(\widetilde{f}(0), \widetilde{f}(z)\right) = d(0, z).$$

Therefore, $\widetilde{f}$ stabilises all hyperbolic circles centred at 0 in $\mathcal{B}$. This remains true when 0 is replaced by any point on the horizontal diameter, so $\widetilde{f}$ stabilises all hyperbolic circles centred on the horizontal diameter.

Now let $z \in \mathcal{B}$. Consider the hyperbolic circle $\mathcal{C}$ through $z$ with centre 0, and choose a point $w \neq 0$ on the horizontal diameter and the hyperbolic circle $\mathcal{C}'$ through $z$ with centre $w$. Then $\mathcal{C}$ and $\mathcal{C}'$ are both stabilised by $\widetilde{f}$, and they are Euclidean circles by Proposition 1.39. Hence $\mathcal{C} \cap \mathcal{C}' = \{z, \overline{z}\}$, so $\widetilde{f}(z) \in \{z, \overline{z}\}$. By continuity, we see that $\widetilde{f} \in \{\mathrm{id}, \bar{\cdot}\}$; but $\widetilde{f}$ is orientation-preserving, so $\widetilde{f} = \mathrm{id}$. □

**Corollary 1.41.** $\mathrm{Isom}^-(\mathcal{H}) = \left\{\varphi \circ \sigma,\ \varphi \in \mathrm{Isom}^+(\mathcal{H})\right\}$, *where $\sigma : z \mapsto -\overline{z}$ is the reflection along the $y$-axis.*

**Proposition 1.42.** (i) $\mathrm{Stab}_{PSL_2\mathbb{R}}(i) = PSO_2\mathbb{R}$.

(ii) *$\mathcal{H}$ is in bijection with $PSL_2\mathbb{R}/PSO_2\mathbb{R}$ via $g \mapsto gi$.*

## 1.7 Isometries in the Poincaré disk model

**Proposition 1.43.** *The orientation-preserving isometries of $\mathcal{B}$ are the linear fractional maps $z \mapsto \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}$, with $\alpha, \beta \in \mathbb{C}$ and $1 = |\alpha|^2 - |\beta|^2$.*

*Proof.* $\mathrm{Isom}^+(\mathcal{B}) = \phi\, \mathrm{Isom}^+(\mathcal{H})\, \phi^{-1} = \phi PSL_2\mathbb{R}\phi^{-1}$, where $\phi : \mathcal{H} \to \mathcal{B}$ is the Cayley transform (recall that its matrix is $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$). □

**Notation 1.44.** *We define:*

- $U(1,1) = \left\{ \begin{pmatrix} \alpha & \theta\beta \\ \overline{\beta} & \theta\overline{\alpha} \end{pmatrix},\ 1 = |\alpha|^2 - |\beta|^2 \text{ and } |\theta| = 1 \right\} \leqslant GL_2\mathbb{C}$ *the subgroup preserving the standard hermitian form $h(z,w) = z_1\overline{w}_1 - z_2\overline{w}_2$ of signature $(1,1)$.*
- $SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},\ 1 = |\alpha|^2 - |\beta|^2 \right\} = U(1,1) \cap SL_2\mathbb{C}$.

*Note that $PU(1,1) = PSU(1,1)$.*

**Corollary 1.45.** $\mathrm{Isom}^+(\mathcal{B}) = PU(1,1)$.

**Remark 1.46.** *Let $h(z,w) = z_1\overline{w}_1 - z_2\overline{w}_2$ be the standard hermitian form of signature $(1,1)$. Then $U(1,1)$ preserves the negative cone $V_- = \{z \in \mathbb{C}^2,\ h(z,z) < 0\}$ of $h$. Hence, $PU(1,1)$ preserves $\mathbb{P}(V_-)$; but in the standard affine chart $z_2 = 1$, $\mathbb{P}(V_-)$ identifies to $\mathcal{B}$; this gives a direct proof of the fact that $PU(1,1)$ preserves $\mathcal{B}$.*

## 1.8 Classification of isometries

**Definition 1.47** (Boundary at infinity)**.** *The* ***boundary*** *$\partial\mathcal{H}$ (resp. $\partial\mathcal{B}$) of $\mathcal{H}$ (resp. $\mathcal{B}$) is defined by $\partial\mathcal{H} = \hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$ (resp. by $\partial\mathcal{B} = \mathbb{S}^1 \subseteq \hat{\mathbb{C}}$). We denote the* ***closure*** *$\overline{\mathcal{H}} = \mathcal{H} \cup \partial\mathcal{H}$ (resp. $\overline{\mathcal{B}} = \mathcal{B} \cup \partial\mathcal{B}$). Points on the boundary are called* ***ideal points****.*

*Note that isometries of $\mathcal{H}$ (resp. of $\mathcal{B}$) extend to the boundary.*

**Definition 1.48** (Classification of isometries)**.** *Let $g \in PSL_2\mathbb{R}$.*

(i) *We say that $g$ is* ***elliptic*** *if $g$ has a fixed point in $\mathcal{H}$ (or $g = e$).*

(ii) *We say that $g$ is* ***parabolic*** *if $g$ has no fixed point in $\mathcal{H}$ and exactly one in $\partial\mathcal{H}$.*

(iii) *We say that $g$ is* ***hyperbolic*** *if $g$ has no fixed point in $\mathcal{H}$ and exactly two in $\partial\mathcal{H}$.*

*In this case, the geodesic $c$ joining the two fixed points of $g$ is called the* ***axis*** *of $g$; $g$ acts on $c$ by translation. The positive real number $\ell(g) = d(z, gz)$ for $z \in c$ is called the* ***translation length*** *of $g$.*

**Example 1.49.** *The homothety $z \mapsto \lambda z$ (for $\lambda > 0$) is hyperbolic with axis $i\mathbb{R}_{>0}$ and translation length $|\log \lambda|$.*

**Proposition 1.50.** *Let $g \in PSL_2\mathbb{R} \smallsetminus \{e\}$. Then $\operatorname{tr} g$ is well-defined up to a sign, and we have:*

(i) *$g$ is elliptic* iff $(\operatorname{tr} g)^2 < 4$. *In this case, $g$ is conjugated in $PSL_2\mathbb{R}$ to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R} \smallsetminus 2\pi\mathbb{Z}$.*

(ii) *$g$ is parabolic* iff $(\operatorname{tr} g)^2 = 4$. *In this case, $g$ is conjugated in $PSL_2\mathbb{R}$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.*

(iii) *$g$ is hyperbolic* iff $(\operatorname{tr} g)^2 > 4$. *In this case, $g$ is conjugated in $PSL_2\mathbb{R}$ to $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ for some $\lambda \in \mathbb{R}_{>0} \smallsetminus \{1\}$.*

*Proof.* Write $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2\mathbb{R} \smallsetminus \{e\}$. Then $g$ fixes a point $[v] \in \mathbb{P}^1\mathbb{C}$, i.e. $v$ is an eigenvector for $g$ in $\mathbb{C}^2$. But the characteristic polynomial of $g$ is

$$\chi_g = X^2 - (\operatorname{tr} g)\, X + 1;$$

so its discriminant is $\Delta = (\operatorname{tr} g)^2 - 4$. Looking at the sign of $\Delta$ yields the three cases. Then use the reduction of endomorphisms over $\mathbb{R}$ to find the conjugacy classes. $\square$

## 1.9 Some geometric properties

**Proposition 1.51.** *Let $x \neq y \in \mathcal{H}$ and denote by $c$ the geodesic through $x$ and $y$. If $p, q \in \partial\mathcal{H} \cap c$, in such a way that $p, x, y, q$ occur on $c$ in this order, then*

$$d(x, y) = \log\,[p, x, y, q]\,.$$

*Proof.* Let $g \in PSL_2\mathbb{R}$ s.t. $g(p, q) = (0, \infty)$. Then $g\,(x, y) = (i\lambda, i\mu)$ for some $0 < \lambda < \mu$ in $\mathbb{R}$. Now

$$d\,(\lambda, \mu) = \log\left(\frac{\mu}{\lambda}\right) = \log\,[0, i\lambda, i\mu, \infty] = \log\,[gp, gx, gy, gq] = \log\,[p, x, y, q]\,,$$

because $PSL_2\mathbb{R}$ preserves the cross-ratio. $\square$

**Remark 1.52.** *The upper half-plane model and the Poincaré model are conformal, i.e. hyperbolic angles are equal to Euclidean angles in those models.*

*Proof.* This is because the Riemannian metrics of $\mathcal{H}$ and $\mathcal{B}$ are rescalings of the Euclidean metric. $\square$

**Definition 1.53** (Hyperbolic area)**.** *The **hyperbolic area** of a region $D \subseteq \mathcal{H}$ is given by*

$$\operatorname{Area}(D) = \iint_D \frac{1}{y^2} \,\mathrm{d}x \,\mathrm{d}y.$$

*Likewise, the hyperbolic area of $D \subseteq \mathcal{B}$ is given by*

$$\operatorname{Area}(D) = \iint_D \left(\frac{2}{1-r^2}\right)^2 r \,\mathrm{d}r \,\mathrm{d}\theta.$$

*Note that isometries preserve the area.*

**Definition 1.54** (Hyperbolic triangle)**.** *A **hyperbolic triangle** consists of three noncolinear points in $\overline{\mathcal{H}}$. A vertex in $\partial\mathcal{H}$ is called an **ideal vertex**.*

*Note that all ideal triangles are congruent.*

**Theorem 1.55** (Gauß-Bonnet)**.** *If $\Delta$ is a hyperbolic triangle with interior angles $\alpha, \beta, \gamma$, then*

$$\operatorname{Area}(\Delta) = \pi - (\alpha + \beta + \gamma).$$

*In particular, $\alpha + \beta + \gamma < \pi$.*

*Proof. First case*: there is at least one ideal vertex, say $c$. Using $PSL_2\mathbb{R}$, we may assume that $c = \infty$ in $\mathcal{H}$ and that the other two vertices $a, b$ are on the geodesic from $-1$ to $1$. Writing $a_1 = \Re(a) = \cos(\pi - \alpha)$ and $b_1 = \Re(b) = \cos\beta$, we get

$$\operatorname{Area}(\Delta) = \iint_\Delta \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{x=a_1}^{b_1} \mathrm{d}x \int_{y=\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}y}{y^2} = \int_{a_1}^{b_1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_{\pi-\alpha}^{\beta} \frac{-\sin\theta \,\mathrm{d}\theta}{\sin\theta} = \pi - (\alpha + \beta).$$

*General case.* Using $PSL_2\mathbb{R}$ we may assume that $a, b$ are on the geodesic from $-1$ to $1$ and that $c$ is on the geodesic from $a$ to $\infty$. Consider the triangle $\Delta'$ with vertices $c, b, \infty$, and with interior angles $\pi - \gamma, \beta', 0$. Then $\Delta \cup \Delta'$ is a triangle with vertices $a, b, \infty$ and with interior angles $\alpha, \beta + \beta', 0$. Therefore, by the first case,

$$\operatorname{Area}(\Delta) = \operatorname{Area}(\Delta \cup \Delta') - \operatorname{Area}(\Delta') = \pi - (\alpha + \beta + \beta') - \pi + (\pi - \gamma + \beta') = \pi - (\alpha + \beta + \gamma). \quad \square$$

**Proposition 1.56.** *Given $\alpha, \beta, \gamma \geqslant 0$ with $\alpha + \beta + \gamma < \pi$, there is a hyperbolic triangle with interior angles $\alpha, \beta, \gamma$. Moreover, this triangle is unique up to isometry.*

**Proposition 1.57** (Orthogonal projection)**.**

(i) *Let $c$ be a geodesic in $\mathcal{H}$ and let $p \in \mathcal{H}$. Then there is a unique $q \in c$ s.t.*

$$d(p, q) = d(p, c).$$

*Equivalently, $q$ is the unique point of $c$ satisfying $(qp) \perp c$.*

*This defines a map $\pi_c : \mathcal{H} \to c$ given by $p \mapsto q$.*

(ii) *Given two geodesics $c_1, c_2$ in $\mathcal{H}$ which are disjoint in $\overline{\mathcal{H}}$, there exists a unique couple $(x_1, x_2) \in c_1 \times c_2$ s.t.*

$$d(x_1, x_2) = \min_{(y_1, y_2) \in c_1 \times c_2} d(y_1, y_2).$$

*Moreover, the geodesic through $x_1$ and $x_2$ is the unique common perpendicular to $c_1$ and $c_2$.*

(iii) *The map $\pi_c : \mathcal{H} \to c$ is 1-Lipschitz.*

*Proof.* (i) Use $PSL_2\mathbb{R}$ to reduce to the case where $c$ is the geodesic from $-1$ to $1$ in $\mathcal{H}$ and $p \in i\mathbb{R}_{\geqslant 1}$.

(ii) Reduce to the case where $c_1$ is the $y$-axis in the upper half-plane and use Euclidean geometry to construct a common perpendicular to $c_1$ and $c_2$. Use Gauß-Bonnet to show that the common perpendicular is unique, and then show that its intersection points with $c_1$ and $c_2$ realise the minimal distance.

(iii) Use (ii). □

**Proposition 1.58** (Composition of inversions)**.** *Let $c_1 \neq c_2$ be two geodedics in $\mathcal{H}$ and let $\sigma_i$ be the inversion through $c_i$.*

(i) *If $c_1, c_2$ intersect in $\mathcal{H}$, then $\sigma_2 \circ \sigma_1$ is elliptic.*

(ii) *If $c_1, c_2$ intersect in $\partial\mathcal{H}$, then $\sigma_2 \circ \sigma_1$ is parabolic.*

(iii) *If $c_1, c_2$ do not intersect in $\overline{\mathcal{H}}$, then $\sigma_2 \circ \sigma_1$ is hyperbolic.*

## 1.10 Other models

**Definition 1.59** (Hyperboloid)**.** *Consider the* ***lorentzian form****, i.e. the standard quadratic form in $\mathbb{R}^3$ of signature $(2, 1)$:*

$$q(x) = x_1^2 + x_2^2 - x_3^2.$$

*The* ***hyperboloid*** *is the set $\mathbb{H} = \{x \in \mathbb{R}^3,\ q(x) = -1 \text{ and } x_3 > 0\}$. For $x \in \mathbb{H}$, we have $T_x\mathbb{H} = x^\perp$ (in the sense of $q$); the Riemannian metric on $\mathbb{H}$ is given by the restriction of $q$ to $T_x\mathbb{H}$, which is positive definite.*

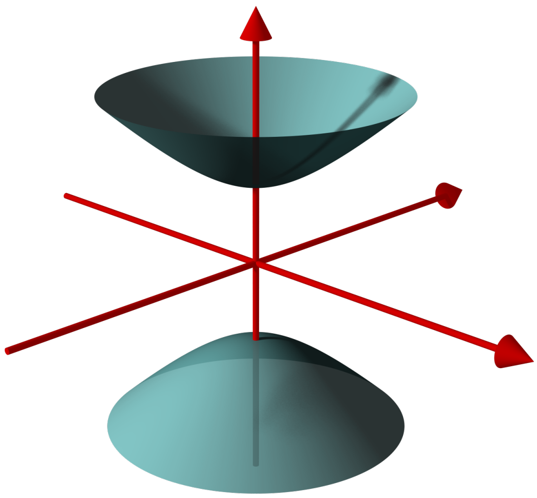

Figure 1: The hyperboloid of two sheets

**Proposition 1.60.** $\operatorname{Isom}(\mathbb{H})$ *is the index-2 subgroup $O^+(q)$ of $O(q)$ preserving the upper sheet of $\{q(x) = -1\}$. Elements of $O^+(q)$ are called* ***positive lorentzian transformations****.*

**Remark 1.61.** (i) *Isometries of $\mathbb{H}$ are linear automorphisms of $\mathbb{R}^3$, so they preserve linear subspaces of $\mathbb{R}^3$.*

(ii) *The action $\operatorname{Isom}(\mathbb{H}) \curvearrowright \mathbb{H}$ is transitive.*

(iii) $\operatorname{Isom}(\mathbb{H}) = O^+(q)$ *contains some orientation-reversing isometries, for instance all reflections in planes containing the $x_3$-axis.*

**Proposition 1.62.** *Geodesics of $\mathbb{H}$ are intersections of $\mathbb{H}$ with linear planes. Moreover, the unit speed parametrisation of the geodesics $c_{x,v}$ starting at $x$ and with initial speed $v$ is*

$$c_{x,v}(t) = (\cosh t)\, x + (\sinh t)\, v.$$

*Proof.* Let $c : [0,1] \to \mathbb{H}$ be a geodesic. Since $\mathbb{H}$ is homogeneous, we may assume that $c(0) = x_0 = (0,0,1)$; moreover, $\operatorname{Stab}(x_0)$ acts transitively on directions at $x_0$ so we may assume that $c'(0) = e_1 = (1,0,0)$.

Now consider the reflection $\sigma \in \operatorname{Isom}(\mathbb{H})$ in the plane $\mathbb{R}e_1 \oplus \mathbb{R}e_3$. Then $\sigma \circ c$ is a geodesic starting at $x_0$ and with initial speed $\mathrm{d}\sigma_{x_0} \cdot c'(0) = \sigma(e_1) = e_1$. By uniqueness of geodesics through a point with a given initial speed, it follows that $\sigma \circ c = c$. In particular, $c \subseteq \mathbb{H} \cap (\mathbb{R}e_1 \oplus \mathbb{R}e_3)$.

For the parametrisation, it suffices to check that $c_{x,v}(t) \in \mathbb{H}$, $c'_{x,v}(0) = v$ and $q\left(c'_{x,v}(t)\right) = 1$. □

**Proposition 1.63.** *The stereographic projection $\pi$ on the plane $\{x_3 = 0\}$ centred at $(0,0,-1)$ induces an isometry $\mathbb{H} \to \mathcal{B}$.*

**Lemma 1.64.** *Let $\alpha \leqslant \alpha' \leqslant x \leqslant y \leqslant \beta' \leqslant \beta$ be points on a projective line $\mathbb{P}(V)$. Then*

$$[\alpha', x, y, \beta'] \geqslant [\alpha, x, y, \beta].$$

*Proof.* It suffices to prove the result in the two special cases $\alpha = \alpha'$ and $\beta = \beta'$.

Let us prove it when $\beta = \beta'$. If $a = [\alpha, x, y, \beta]$, then by definition of the cross-ratio, there is a projective isomorphism $g : \mathbb{P}(V) \to \mathbb{P}^1\mathbb{K}$ s.t. $g(\alpha, x, y, \beta) = (0, 1, a, \infty)$. Therefore

$$[\alpha', x, y, \beta] = [g\alpha', gx, gy, g\beta] = [u, 1, a, \infty] = \frac{a-u}{1-u},$$

for some $0 < u < 1$. But $a > 1$, so $a - u > a(1-u)$, i.e. $\frac{a-u}{1-u} > a$. □

**Definition 1.65** (Projective disk)**.** *The **projective disk** (or **Klein disk**) is the set*

$$\mathbb{D} = \left\{[x_1 : x_2 : x_3] \in \mathbb{P}^2\mathbb{R},\ x_1^2 + x_2^2 - x_3^2 < 0\right\} = p(\mathbb{H}) \subseteq \mathbb{P}^2\mathbb{R},$$

*where $p : \mathbb{R}^3 \smallsetminus \{0\} \to \mathbb{P}^2\mathbb{R}$ is the projection. Note that, in the affine chart $x_3 = 1$, $\mathbb{D}$ identifies with the unit disk.*

*The metric on $\mathbb{D}$ is defined as follows: given $x, y \in \mathbb{D}$, consider the unique projective line through $x, y$. This line intersects $\partial\mathbb{D}$ at $\alpha, \beta$ s.t. $\alpha, x, y, \beta$ occur in that order. Then*

$$d_{\mathbb{D}}(x,y) = \frac{1}{2}\log[\alpha, x, y, \beta].$$

*Proof.* Let us prove that $d_{\mathbb{D}}$ satisfies the triangle inequality.

First note that, if $y \in [x,z]$, then $d_{\mathbb{D}}(x,z) = d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z)$ because

$$[\alpha, x, z, \beta] = [\alpha, x, y, \beta]\,[\alpha, y, z, \beta].$$

Now let $x, y, z \in \mathbb{D}$. Let $\alpha, \beta$ be the boundary points of the line through $x, z$. Consider the lines tangent to $\partial\mathbb{D}$ through $\alpha, \beta$ respectively. These lines meet at some point $q \in \mathbb{P}^2\mathbb{R}$. Let $\alpha_1, \beta_1$ be the boundary points of the line through $x, y$. Consider the lines $(\alpha_1, q)$ and $(\beta_1, q)$. They meet the line $(\alpha, \beta)$ at points $\alpha_1', \beta_1'$ respectively. Likewise, the line $(y, q)$ meets $(\alpha, \beta)$ at some point $y'$. Hence (using Lemma 1.64),

$$d_{\mathbb{D}}(x,y) = \frac{1}{2}\log[\alpha_1, x, y, \beta_1] = \frac{1}{2}\log[\alpha_1', x, y', \beta_1'] \geqslant \frac{1}{2}\log[\alpha, x, y', \beta] = d_{\mathbb{D}}(x, y').$$

Similarly, $d_{\mathbb{D}}(y,z) \geqslant d_{\mathbb{D}}(y', z)$. Since $y' \in [x,z]$, we have

$$d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z) \geqslant d_{\mathbb{D}}(x,y') + d_{\mathbb{D}}(y',z) = d_{\mathbb{D}}(x,z).$$ □

**Corollary 1.66.** *The fact that $d_{\mathbb{D}}(x,z) = d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z)$ if $y \in [x,z]$ implies that geodesics of $\mathbb{D}$ are projective lines, and hence affine lines in the standard affine chart.*

**Remark 1.67.** *The construction of the projective disk generalises to **Hilbert geometry**: the unit disk can be replaced by any open bounded convex subset $\Omega$ of $\mathbb{R}^2$, and we can still define a metric on $\Omega$ using the cross-ratio.*

**Proposition 1.68.** *The projection $p : \mathbb{R}^3 \smallsetminus \{0\} \to \mathbb{P}^2\mathbb{R}$ induces an isometry $\mathbb{H} \to \mathbb{D}$.*

*In particular,* $\operatorname{Isom}(\mathbb{D})$ *is the projective group $PO^+(2,1)$ of matrices preserving the standard quadratic form of signature $(2,1)$.*

# 2 Fuchsian groups

## 2.1 Discrete subgroups

**Definition 2.1** (Discrete group)**.** *A topological group $\Gamma$ is called **discrete** if it satisfies one of the following equivalent conditions:*

(i) *The topology of $\Gamma$ is discrete.*

(ii) *$\{\gamma\}$ is open in $\Gamma$ for all $\gamma \in \Gamma$.*

(iii) *$\{e\}$ is open in $\Gamma$.*

*If $\Gamma$ is metrisable, then it is discrete if and only if every convergent sequence in $\Gamma$ is eventually constant.*

**Example 2.2.** (i) *$\mathbb{Z} \leqslant \mathbb{R}$ is discrete.*

(ii) *$\mathbb{Z}[i] \leqslant \mathbb{C}$ is discrete.*

(iii) *$\Gamma \leqslant \mathbb{R}^n$ is discrete* iff *$\Gamma = \bigoplus_{i=1}^{k} \mathbb{Z}v_i$, where $v_1, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$.*
*If $k = n$, we say that $\Gamma$ is a **lattice**.*

(iv) *$\langle k \rangle \leqslant \mathbb{C}^\times$ is discrete* iff *$k \in \exp(2i\pi\mathbb{Q})$ or $|k| \neq 1$.*

(v) *$GL_n\mathbb{Z} \leqslant GL_n\mathbb{R}$ is discrete.*

(vi) *$GL_n\mathbb{Z}[i] \leqslant GL_n\mathbb{C}$ is discrete.*

**Proposition 2.3.** *Let $G$ be a metrisable topological group. Then any discrete subgroup $\Gamma \subseteq G$ is closed in $G$.*

*Proof.* Let $(\gamma_n)_{n\geqslant 0} \in \Gamma$ s.t. $\gamma_n \to g \in G$. Then $\gamma_{n+1}\gamma_n^{-1} \to 1 \in \Gamma$. But $\{1\}$ is open in $\Gamma$, so there exists $n_0 \geqslant 0$ s.t. $\gamma_{n+1}\gamma_n^{-1} = 1$ for all $n \geqslant n_0$. In other words, $(\gamma_n)_{n\geqslant 0}$ is eventually constant, so $g \in \Gamma$. ☐

**Remark 2.4.** *It is false in general that a discrete subspace of a topological space is closed.*

## 2.2 Discrete subgroups of matrix groups

**Proposition 2.5.** *Let $X$ be Hausdorff and locally compact. Then $Y \subseteq X$ is closed and discrete* iff *for all $K \sqsubseteq X$ compact, $Y \cap K$ is finite.*

*Proof.* ($\Rightarrow$) This is a consequence of the fact that a discrete compact space is finite. ($\Leftarrow$) Let $y \in X$ and let $U$ be a relatively compact neighbourhood of $y$ in $X$ (i.e. $\overline{U}$ is compact). Then $Y \cap U$ is finite by assumption. Therefore, $U' = U \smallsetminus \{u \in Y \cap U,\ u \neq y\}$ is a neighbourhood of $y$ in $X$, and $Y \cap U'$ is $\{y\}$ or $\varnothing$. ☐

**Corollary 2.6.** *Let $\widehat{SL_n}\mathbb{C} = \{g \in GL_n\mathbb{C},\ \det g \in \{\pm 1\}\}$. Then a subgroup $\Gamma \leqslant \widehat{SL_n}\mathbb{C}$ is discrete* iff *the set $\{\gamma \in \Gamma,\ \|\gamma\| \leqslant R\}$ is finite for all $R > 0$.*

*In particular, discrete subgroups of $\widehat{SL_n}\mathbb{C}$ are countable.*

*Proof.* The sets $\left\{g \in \widehat{SL_n}\mathbb{C},\ \|g\| \leqslant R\right\}$ for $R > 0$ form a basis of compact subsets of $\widehat{SL_n}\mathbb{C}$. ☐

**Remark 2.7.** *The argument of Corollary 2.6 would not work with $GL_n\mathbb{C}$ since $\{g \in GL_n\mathbb{C},\ \|g\| \leqslant R\}$ is not compact. However, we can use Corollary 2.6 to find discrete subgroups of $GL_n\mathbb{C}$ by noting that $GL_n\mathbb{C}$ can be embedded as a closed subgroup of $SL_{n+1}\mathbb{C}$ via*

$$A \in GL_n\mathbb{C} \longmapsto \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \in SL_{n+1}\mathbb{C}.$$

**Definition 2.8** (Fuchsian group)**.** *A **Fuchsian group** is a discrete subgroup of $PSL_2\mathbb{R}$.*

**Remark 2.9.** *$PSL_2\mathbb{R}$ is $SL_2\mathbb{R}/\{\pm I\}$ equipped with the quotient topology. The associated projection $p : SL_2\mathbb{R} \to PSL_2\mathbb{R}$ is continuous and proper, so a subgroup $\Gamma \leqslant SL_2\mathbb{R}$ is discrete* iff *$p(\Gamma) \leqslant PSL_2\mathbb{R}$ is discrete.*

**Corollary 2.10.** *$PSL_2\mathbb{Z}$ is a Fuchsian group.*

**Remark 2.11.** *Let $X$ be a proper metric space (i.e. closed balls are compact). Then* $\mathrm{Isom}(X)$ *can be endowed with the topology of uniform convergence on compact subsets. The topology induced by that of* $\mathrm{Isom}(\mathcal{H})$ *on $PSL_2\mathbb{R}$ is the same as the quotient topology.*

**Proposition 2.12.** *Let $X$ be a proper metric space. Given $x \in X$ and $K \sqsubseteq X$ compact, the set $\{g \in \mathrm{Isom}(X),\ gx \in K\}$ is compact in* $\mathrm{Isom}(X)$.

*Proof.* Use the Arzelà-Ascoli Theorem. □

## 2.3 Proper discontinuous actions

**Definition 2.13** (Proper discontinuous action)**.** *Let $\Gamma \curvearrowright X$ be an action by isometries of a group on a proper metric space. The following assertions are equivalent:*

(i) *Every point $x \in X$ has a neighbourhood $U$ s.t. $\{\gamma \in \Gamma,\ \gamma U \cap U \neq \varnothing\}$ is finite.*

(ii) *For all $x \in X$, the orbit $\Gamma x$ is closed and discrete and the stabiliser $\Gamma_x$ is finite.*

(iii) *For all $x \in X$ and for all $K \sqsubseteq X$ compact, the set $\{\gamma \in \Gamma,\ \gamma x \in K\}$ is finite.*

(iv) *For all $K \sqsubseteq X$ compact, the set $\{\gamma \in \Gamma,\ \gamma K \cap K \neq \varnothing\}$ is finite.*

*We then say that the action $\Gamma \curvearrowright X$ is **properly discontinuous**.*

*Proof.* (iv) ⇒ (i) and (ii) ⇔ (iii) can be proven using arguments from point-set topology, i.e. the proof remains valid for an action by homeomorphisms of a group on a Hausdorff locally compact topological space.

(i) ⇒ (ii) can be proven using the sequential characterisation of closedness for metric spaces.

(ii) ⇒ (iv) can be proven using the sequential characterisation of compactness, using moreover the fact that the action is by isometries. □

**Lemma 2.14.** *Let $\Gamma \curvearrowright X$ be a continuous action of a Hausdorff topological group on a topological space. If there is a point $x \in X$ s.t. $\Gamma x$ is discrete and $\Gamma_x$ is finite, then $\Gamma$ is discrete.*

*Proof.* We have a continuous map $\theta_x : \Gamma \to \Gamma x$ given by $\gamma \mapsto \gamma x$. Therefore, the subgroup $\Gamma_x = \theta_x^{-1}(\{x\})$ is open in $\Gamma$, and finite by assumption. It follows that $\{e\} \subseteq \Gamma_x$ is open in $\Gamma$. □

**Theorem 2.15.** *A subgroup $\Gamma \leqslant PSL_2\mathbb{R}$ is discrete* iff *$\Gamma$ acts properly discontinuously on $\mathcal{H}$.*

*Proof.* (⇐) Let $x \in \mathcal{H}$. Since $\Gamma$ acts properly discontinuously, the orbit $\Gamma x$ is discrete and $\Gamma_x$ is finite. Lemma 2.14 implies that $\Gamma$ is discrete. (⇒) Let $x \in \mathcal{H}$ and $K \sqsubseteq \mathcal{H}$ compact. Then

$$\Gamma_K = \{\gamma \in \Gamma,\ \gamma x \in K\} = \Gamma \cap G_K,$$

where $G = \mathrm{Isom}(\mathcal{H})$. But $\Gamma$ is discrete (and therefore closed by Proposition 2.3), and $G_K$ is compact by Proposition 2.12. Therefore, $\Gamma_K$ is finite, proving that $\Gamma$ acts properly discontinuously. □

## 2.4 Fundamental domains

**Definition 2.16** (Fundamental domain)**.** *Let $\Gamma \curvearrowright X$ be a continuous action of a topological group on a topological space. A* ***fundamental domain*** *for $\Gamma$ in $X$ is a subset $D \subseteq X$ s.t.*

(i) $\bigcup_{\gamma\in\Gamma} \gamma\overline{D} = \mathcal{H}$,

(ii) $\gamma\mathring{D} \cap \mathring{D} = \varnothing$ *for all* $\gamma \neq e$.

*Hence, $\{\gamma D,\ \gamma \in \Gamma\}$ is a* ***tessellation*** *of $X$ under $\Gamma$.*

**Example 2.17.** (i) *$[0,1]^2$ is a fundamental domain for the action $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ by translations.*

(ii) *Let $\gamma \in PSL_2\mathbb{R}$.*

- *If $\gamma$ is hyperbolic with axis $\sigma$, then for fixed $x \in \sigma$, the set $\{y \in \mathcal{H},\ \pi_\sigma(y) \in [x, \gamma x]\}$ is a fundamental domain for the action $\langle\gamma\rangle \curvearrowright \mathcal{H}$, where $\pi_\sigma : \mathcal{H} \to \sigma$ is the orthogonal projection (c.f. Proposition 1.57).*
- *If $\gamma$ is parabolic with fixed point $\xi \in \partial\mathcal{H}$, then for fixed $x \in \mathcal{H}$, the convex hull of the geodesics $(x, \xi)$ and $(\gamma x, \xi)$ is a fundamental domain for the action $\langle\gamma\rangle \curvearrowright \mathcal{H}$.*
- *If $\gamma$ is elliptic with centre $x$ and angle $\theta$, then $\langle\gamma\rangle$ is discrete* iff *$\gamma$ is torsion* iff *$\theta \in 2\pi\mathbb{Q}$. In this case, the sector of angle $\frac{2\pi}{q}$ at $x$, with $q = |\langle\gamma\rangle|$, is a fundamental domain for the action $\langle\gamma\rangle \curvearrowright \mathcal{H}$.*

**Proposition 2.18.** *Consider*

$$D = \left\{ z \in \mathcal{H},\ |z| > 1 \text{ and } |\Re(z)| < \frac{1}{2} \right\}.$$

*In other words, $D$ is the hyperbolic triangle with one ideal vertex at $\infty$ and two vertices at $e^{i\frac{\pi}{3}}$ and $e^{i\frac{2\pi}{3}}$.*

*Then $D$ is a fundamental domain for $\Gamma = PSL_2\mathbb{Z}$.*

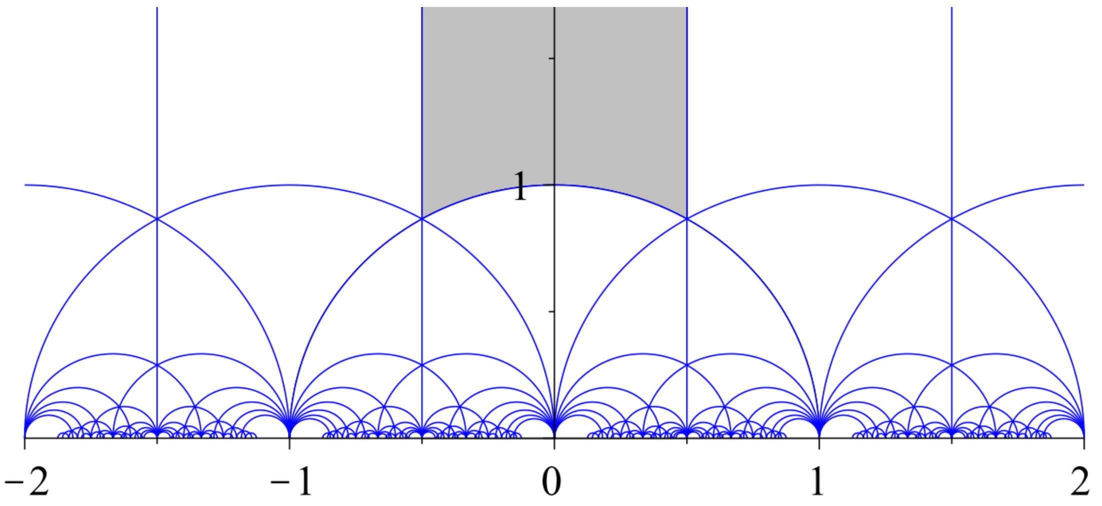


Figure 2: Tessellation of $\mathcal{H}$ by $PSL_2\mathbb{Z}$ with fundamental domain $D$

*Proof.* Let $z \in D$ and $\gamma \in \Gamma$. If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2\mathbb{Z}$, then

$$\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}.$$

Now assume that $z, \gamma z \in D$.

- Assume first that $c = 0$, so $ad = \det\gamma = 1$. Therefore $a = d = \pm 1$. We may assume that $a = d = 1$ by multiplying by $-I$. Hence $\gamma z = z + b$ with $b \in \mathbb{Z}$. But $|\Re(\gamma z)|, |\Re(z)| < \frac{1}{2}$, so we must have $b = 0$ and therefore $\gamma = 1$.

- If $c \neq 0$, then
$$|cz+d|^2 = |c|^2|z|^2 + |d|^2 + 2cd\Re(z) > |c|^2 + |d|^2 - |cd| = (|c| - |d|)^2 + |cd| \geqslant 1,$$
so that $\Im(\gamma z) < \Im(z)$. But the same argument shows that $\Im(z) = \Im(\gamma^{-1}(\gamma z)) < \Im(\gamma z)$; this is a contradiction.

This proves that, if $\gamma D \cap D \neq \varnothing$, then $\gamma = 1$.

Now let $z \in \mathcal{H}$; we want to show that $\Gamma z$ meets $\overline{D}$. If $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2\mathbb{Z}$, then we have $\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}$. But $z\mathbb{Z} \oplus \mathbb{Z}$ is a discrete subgroup of $\mathbb{Z}^2$, so the set $\left\{|cz+d|^2,\ c, d \text{ coprime}\right\}$ has a minimum and therefore $\{\Im(\gamma z), \gamma \in \Gamma\}$ has a maximum, i.e. we may choose $\gamma_0 \in \Gamma$ s.t.
$$\Im(\gamma_0 z) = \max_{\gamma \in \Gamma} \Im(\gamma z).$$
Write $z_0 = \gamma_0 z$. After applying some power of $z \mapsto z+1$, we may assume that $|\Re(z_0)| \leqslant \frac{1}{2}$. Moreover, if $|z_0| < 1$, then apply $z \mapsto -\frac{1}{z}$; we have $\Im\left(-\frac{1}{z_0}\right) = \frac{\Im(z_0)}{|z_0|^2} > \Im(z_0)$, contradicting the choice of $z_0$. Therefore, $|z_0| \geqslant 1$, so $z_0 \in \overline{D} \cap \Gamma z$. □

## 2.5 Existence of fundamental domains

**Proposition 2.19.** *If $\Gamma \leqslant PSL_2\mathbb{R}$ has an open fundamental domain $D$, then $\Gamma$ is discrete.*

*Proof.* Let $x \in D$. Then $D \cap \Gamma x = \{x\}$ is open in $\Gamma x$ because $D$ is open in $\mathcal{H}$, so $\Gamma x$ is discrete, and $\Gamma_x = \{e\}$. By Lemma 2.14, $\Gamma$ is discrete. □

**Definition 2.20** (Locally finite fundamental domain)**.** *A fundamental domain $D$ for an action $\Gamma \curvearrowright X$ is called **locally finite** if the set $\{\gamma D,\ \gamma \in \Gamma\}$ is locally finite, i.e. for all compact $K \sqsubseteq X$, the set $\{\gamma \in \Gamma,\ \gamma D \cap K \neq \varnothing\}$ is finite.*

**Theorem 2.21.** *Let $\Gamma$ be a Fuchsian group. Let $x \in \mathcal{H}$ such that $\Gamma_x = \{e\}$. Then the set*
$$D(x) = \{y \in \mathcal{H},\ \forall \gamma \in \Gamma \smallsetminus \{e\},\ d(y,x) < d(y,\gamma x)\}$$
*is an open, convex and locally finite fundamental domain for $\Gamma$ in $\mathcal{H}$. It is called the **Dirichlet domain** of $\Gamma$ at $x$.*

*Proof.* Note first that
$$D(x) = \bigcap_{\gamma \in \Gamma \smallsetminus \{e\}} D_\gamma,$$
where $D_\gamma = \{y \in \mathcal{H},\ d(y,x) < d(y,\gamma x)\}$. This set $D_\gamma$ is a half-plane bounded by the perpendicular bisector $\sigma_\gamma$ of $[x, \gamma x]$ (which is well-defined because $\gamma x \neq x$ since $\Gamma_x = \{e\}$). In particular, $D_\gamma$ is convex for all $\gamma$, and therefore $D(x)$ is convex.

If $\gamma \in \Gamma$, note that $\gamma D(x) = D(\gamma x)$; it follows that $D(x) \cap \gamma D(x) = D(x) \cap D(\gamma x) = \varnothing$ if $\gamma \neq e$.

Now given $y \in \mathcal{H}$, since the orbit $\Gamma x$ is locally finite, there exists $\gamma \in \Gamma$ s.t. $d(y, \Gamma x) = d(y, \gamma x)$, so $y \in \widetilde{D}(\gamma x) = \gamma\widetilde{D}(x)$, with
$$\widetilde{D}(x) = \{z \in \mathcal{H},\ d(z,x) = d(z, \Gamma x)\}.$$
The set $\widetilde{D}(\gamma x)$ is closed (it can be written as an intersection of closed half-planes) and contains $D(x)$, so $\overline{D(x)} \subseteq \widetilde{D}(x)$. Conversely, if $y \in \widetilde{D}(x)$, then $[x,y] \subseteq \widetilde{D}(x)$ and $[x,y) \subseteq D(x)$, so $y \in \overline{[x,y)} \subseteq \overline{D(x)}$. This proves that $\widetilde{D}(x) = \overline{D(x)}$, so the translates of $\overline{D(x)}$ cover $\mathcal{H}$.

Let us now prove that $D(x)$ is a locally finite fundamental domain. Let $R>0$ and consider the finite set

$$\Gamma_R = \{\gamma \in \Gamma \smallsetminus \{e\},\ d\left(x, \gamma x\right) \leqslant 2R\}.$$

If $\gamma D(x) \cap B(x,R) \neq \varnothing$ then, taking $y \in \gamma D(x) \cap B(x,R)$, we have $d(x,y) < R$, and $y \in D\left(\gamma x\right)$, so $d\left(y, \gamma x\right) < d\left(y, x\right) < R$. Hence $d\left(x, \gamma x\right) < 2R$, so $\gamma \in \Gamma_R$. This proves that

$$\{\gamma \in \Gamma,\ \gamma D(x) \cap B(x,R) \neq \varnothing\} \subseteq \Gamma_R,$$

so $D(x)$ is locally finite.

The above also proves that, for all $R>0$,

$$D(x) \cap B(x,R) = B(x,R) \cap \bigcap_{\gamma \in \Gamma_R \smallsetminus \{e\}} D_\gamma.$$

Therefore, $D(x) \cap B(x,R)$ is open as a finite intersection of open half-spaces, from which it follows that $D(x) = \bigcup_{R>0} \left(D(x) \cap B(x,R)\right)$ is open. $\square$

**Example 2.22.** *If $\Gamma = PSL_2\mathbb{Z}$, then the Dirichlet domain $D\left(2i\right)$ is the fundamental domain given by Proposition 2.18.*

**Proposition 2.23.** *If $\Gamma$ is a Fuchsian group, then the set $E$ of fixed points of elliptic elements of $\Gamma$ is closed and discrete.*

*In particular, $\{x \in \mathcal{H},\ \Gamma_x = \{e\}\}$ is open and dense.*

*Proof.* Let $\left(x_n\right)_{n \geqslant 0} \in E$ with $x_n \to x \in \mathcal{H}$. For $n \geqslant 0$, there exists $\gamma_n \in \Gamma$ fixing $x_n$. By discreteness of $\Gamma$, let $U$ be a neighbourhood of $x$ s.t. $S_U = \{\gamma \in \Gamma,\ \gamma U \cap U \neq \varnothing\}$ is finite. There is a rank $n_0 \geqslant 0$ s.t. $x_n \in U$ for $n \geqslant n_0$. Therefore $x_n \in \gamma_n U \cap U$, so $\gamma_n \in S_U$. It follows that $\{\gamma_n,\ n \geqslant 0\}$ is a finite subset of $\Gamma$, so the sequence $\left(\gamma_n\right)_{n \geqslant 0}$ has a constant subsequence. Since each elliptic isometry has exactly one fixed point, $\left(x_n\right)_{n \geqslant 0}$ also has a constant subsequence. This proves that $E$ is closed and discrete. $\square$

**Corollary 2.24.** *Every Fuchsian group has an open, convex and locally finite fundamental domain.*

## 2.6 Convex fundamental polygons and side-pairings

**Definition 2.25** (Convex fundamental polygon)**.** *If $\Gamma$ is a Fuchsian group, then a* ***convex fundamental polygon*** *for $\Gamma$ is a convex locally finite fundamental domain $P$ for the action $\Gamma \curvearrowright \mathcal{H}$.*

*A* ***side*** *of $P$ is a nontrivial geodesic segment of the form $\overline{P} \cap \gamma\overline{P}$ (with $\gamma \neq e$). A* ***vertex*** *of $P$ is a point of the form $\overline{P} \cap \gamma_1\overline{P} \cap \gamma_2\overline{P}$ (with $e, \gamma_1, \gamma_2$ all distinct).*

**Proposition 2.26.** *Let $P$ be a convex fundamental polygon for some Fuchsian group $\Gamma$.*

(i) *$\partial P$ is the union of all sides and vertices of $P$.*

(ii) *The collections of sides and vertices of $P$ are locally finite.*

(iii) *Every vertex of $P$ lies on exactly two sides of $P$ and is the common endpoint of those two sides.*

(iv) *If $s, s'$ are nondisjoint sides of $P$, then $s \cap s' = \{v\}$, where $v$ is the common endpoint of $s, s'$.*

**Remark 2.27.** *Some vertices of convex fundamental polygons may have interior angle $\pi$.*

**Definition 2.28** (Side-pairing)**.** *Let $P$ be a convex fundamental polygon for some Fuchsian group $\Gamma$. If $S$ is the set of sides of $P$, then for each $s \in S$, there is a unique $\gamma_s \in \Gamma$ s.t. $s = \overline{P} \cap \gamma_s\overline{P}$. Therefore, $\gamma_s^{-1}(s)$ is a side $\sigma(s)$ of $P$.*

*This defines an involution $\sigma : S \to S$ with the property that $\gamma_{\sigma(s)} = \gamma_s^{-1}$. This involution is called a* ***side-pairing****.*

**Remark 2.29.** *Note that a side-pairing may have fixed points (e.g. for $PSL_2\mathbb{Z}$). In fact, $\sigma(s) = s$* iff *$\gamma_s$ is the half-turn around the middle of $s$. We may therefore add the middle of $s$ as a vertex if we want to ensure that $\sigma$ is fixed-point free.*

**Proposition 2.30.** *With the notations of Definition 2.28, $\Gamma$ is generated by $\{\gamma_s,\ s \in S\}$.*

*Proof.* Let $h \in \Gamma$ and $x \in \mathring{P}$. Choose a $\mathcal{C}^1$ path $c$ from $x$ to $hx$ avoiding the vertices of the translates of $P$ and transverse to all sides. Then there is a sequence of adjacent tiles $P = P_0, \dots, P_n$, with $P_k = h_k P$. Since $hx \in P_n$, we must have $h_n = h$. Such a sequence of translates of $P$ is called a **gallery**.

Now, $P_k = h_k P$ is adjacent to $P_{k+1} = h_{k+1} P$, so $P = h_k^{-1} P_k$ is adjacent to $h_k^{-1} h_{k+1} P$. Hence, there exists $s_{k+1} \in S$ s.t.

$$h_k^{-1} h_{k+1} = \gamma_{s_{k+1}}.$$

Therefore $h = h_n = \left(h_0^{-1} h_1\right)\left(h_1^{-1} h_2\right) \cdots \left(h_{n-1}^{-1} h_n\right) = \gamma_{s_1} \cdots \gamma_{s_n}$. □

**Definition 2.31** (Vertex cycles). *Let $P$ be a convex fundamental polygon for some Fuchsian group $\Gamma$, and let $v$ be a vertex of $P$. Let $P = P_0, \dots, P_n = P$ be the gallery of tiles obtained by turning counter-clockwise around $v$. For all $k$, there exists $h_k \in \Gamma$ s.t. $P_k = h_k P$. By the argument used in the proof of Proposition 2.30, we see that $h_{k+1} = h_k \gamma_{s_{k+1}}$ for some side $s_{k+1}$ of $P$; hence $h_k = \gamma_{s_1} \cdots \gamma_{s_k}$ for all $k$.*

*Another way to construct the* ***edge cycle*** *$\{s_1, \dots, s_n\}$ is to note that $s_{k+1}$ is the side preceeding $\sigma(s_k)$ when $\partial P$ is endowed with the counter-clockwise orientation.*

*Now consider the vertex $v_k = h_k^{-1} v$ of $P$ (this is the initial endpoint of $\sigma(s_k)$), and denote by $\alpha_k$ the angle at $v_k$ of $P$ (or equivalently, the angle at $v$ of $P_k = h_k P$). The set $C_v = \{v_0, \dots, v_n\}$ is called the* ***vertex cycle*** *of $v$. Note that*

$$C_v = \Gamma v \cap \overline{P}.$$

*The* ***angle sum*** *of $C_v$ is the sum of the angles of $P$ at each vertex in the cycle.*

*Let $m = \min\{k \geqslant 1,\ h_k \in \Gamma_v\} = \min\{k \geqslant 1,\ v_k = v\}$. Then $n = qm$, where $q = |\Gamma_v|$, and $h_m$ is a generator of $\Gamma_v$. It follows that $C_v = \{v_0, \dots, v_{m-1}\}$, so the angle sum is*

$$\sum_{k=0}^{m-1} \alpha_k = \frac{2\pi}{q}.$$

*Moreover, we have the following* ***vertex cycle relation****:*

$$\left(\gamma_{s_1} \cdots \gamma_{s_{m-1}}\right)^q = 1.$$

## 2.7 Poincaré's Theorem

**Theorem 2.32** (Poincaré). *Let $P \subseteq \mathcal{H}$ be a compact, convex, finitely-sided polygon. We consider a fixed-point free involution $\sigma : S \to S$ s.t. the sides $s$ and $\sigma(s)$ have the same length. For each $s \in S$, we consider the unique $\gamma_s \in PSL_2\mathbb{R}$ s.t. $\gamma_s^{-1} s = \sigma(s)$, and $\gamma_s$ reverses the orientation of $s$ (with sides oriented s.t. $P$ lies to the left).*

*We suppose that the angle sum of each vertex cycle is of the form $\frac{2\pi}{q}$ for some $q \geqslant 1$.*

*Then, if $\Gamma = \langle \gamma_s,\ s \in S \rangle \leqslant PSL_2\mathbb{R}$, we have:*

(i) *$P$ is a fundamental domain for $\Gamma$,*

(ii) *$\Gamma$ is discrete,*

(iii) *$\Gamma$ has the presentation $\langle \gamma_s,\ s \in S \mid R \rangle$, where $R$ is the set of vertex cycle relations, together with the relations $\gamma_{\sigma(s)} = \gamma_s^{-1}$ for $s \in S$.*

*Proof.* See [3] or [1]. □

**Example 2.33.** *Consider a hyperbolic quadrilateral with opposite sides of the same length, and with angles $\alpha_0, \ldots, \alpha_3$ such that $\sum_{i=0}^{3} \alpha_i = \frac{2\pi}{q}$ for $q \geqslant 2$. Note that such a quadrilateral can be constructed as soon as $\alpha_0 = \alpha_2$ and $\alpha_1 = \alpha_3$, using Proposition 1.56. Pairing each side with the opposite one, Poincaré's Theorem yields a subgroup $\Gamma \leqslant PSL_2\mathbb{R}$ with presentation*

$$\Gamma = \langle \gamma_1, \gamma_2 \mid [\gamma_1, \gamma_2]^q = 1 \rangle .$$

*Note that, by changing the values of $\alpha_0$ and $\alpha_1$, we get different Fuchsian groups with the same isomorphism type. This phenomenon is specific to dimension 2.*

**Example 2.34.** *Let $P$ be a regular hyperbolic octogon, with angles $\alpha$ such that $\alpha = \frac{\pi}{4q}$ for some $q \geqslant 1$, with side-pairing given by Figure 3. Then Poincaré's Theorem yields a subgroup $\Gamma \leqslant PSL_2\mathbb{R}$ with presentation*

$$\Gamma = \langle a_1, a_2, b_1, b_2 \mid ([a_1, a_2]\,[b_1, b_2])^q = 1 \rangle .$$

*The quotient $\mathcal{H}/\Gamma$ is the genus 2 surface when $q = 1$.*

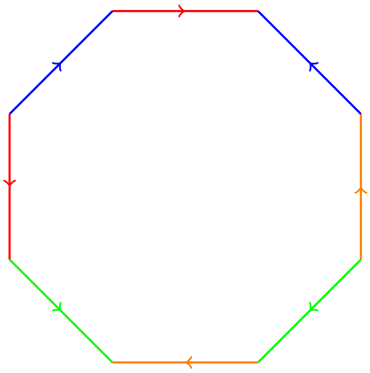

Figure 3: A genus 2 surface obtained as a quotient of an octogon

**Theorem 2.35.** *Let $P \subseteq \mathcal{H}$ be a compact, convex, finitely-sided polygon. For each side $s$ of $P$, let $\sigma_s$ be the reflection along $s$.*

*Suppose that the angle at each vertex $v$ of $P$ is of the form $\frac{\pi}{q_v}$ for some $q_v \geqslant 1$.*

*Then, if $\Gamma = \langle \gamma_s,\ s \in S \rangle \leqslant \mathrm{Isom}\,(\mathcal{H})$, we have:*

(i) *$P$ is a fundamental domain for $\Gamma$,*

(ii) *$\Gamma$ is discrete,*

(iii) *$\Gamma$ has the presentation $\langle \gamma_s,\ s \in S \mid \sigma_s^2 = 1,\ (\sigma_s \sigma_{s'})^{q_v} = 1,\ s \cap s' = \{v\} \rangle$.*

*The group $\Gamma$ is a* ***Coxeter group*** *(or* ***triangle group*** *if $P$ is a triangle).*

## 2.8 Quotients and hyperbolic surfaces

**Definition 2.36** (Quotient space)**.** *Given a Fuchsian group $\Gamma \leqslant PSL_2\mathbb{R}$, the* ***quotient space*** *(or* ***orbit space****) $\mathcal{H}/\Gamma$ is the quotient of $\mathcal{H}$ by the equivalence relation associated to the action of $\Gamma$. It is endowed with the quotient topology. We have a projection $p : \mathcal{H} \to \mathcal{H}/\Gamma$ that is continuous, surjective and open.*

**Definition 2.37** (Quotient metric)**.** *Given a Fuchsian group $\Gamma$, we define a map $\overline{d} : \mathcal{H}/\Gamma \times \mathcal{H}/\Gamma \to \mathbb{R}_{\geqslant 0}$ by $\overline{d}\,(px, py) = d\,(\Gamma x, \Gamma y)$. Then*

(i) *$\overline{d}\,(px, py) = d\,(x, \Gamma y)$ for all $x, y \in \mathcal{H}$,*

(ii) *$\overline{d}$ is a metric on $\mathcal{H}/\Gamma$ and induces the quotient topology.*

*Proof.* (i) This comes from the fact that $\Gamma$ acts by isometries. (ii) The only nontrivial fact is the separation property, which is a consequence of (i), together with the fact that $\Gamma y$ is closed in $\mathcal{H}$. $\square$

**Proposition 2.38.** *Let $\Gamma$ be a Fuchsian group. For all $x \in \mathcal{H}$, there exists $\varepsilon > 0$ s.t., if $\gamma B(x,\varepsilon) \cap B(x,\varepsilon) \neq \varnothing$, then $\gamma \in \Gamma_x$.*

*It follows that*

$$p^{-1}\left(B\left(px,\varepsilon\right)\right) = \coprod_{\overline{\gamma} \in \Gamma/\Gamma_x} B\left(\gamma x, \varepsilon\right),$$

*where $p : \mathcal{H} \to \mathcal{H}/\Gamma$ is the projection.*

*Proof.* Let $\varepsilon = \frac{1}{2} d\left(x, \Gamma x \smallsetminus \{x\}\right) > 0$. If $y \in \gamma B(x,\varepsilon) \cap B(x,\varepsilon)$, then $d\left(x,\gamma x\right) \leqslant d\left(x,y\right) + d\left(y,\gamma x\right) < 2\varepsilon$, so $\gamma \in \Gamma_x$ by definition of $\varepsilon$. $\square$

**Proposition 2.39.** *Let $\Gamma$ be a Fuchsian group. Given $x \in \mathcal{H}$, there exists $\varepsilon > 0$ s.t. the projection $p : B(x,\varepsilon) \to B\left(px,\varepsilon\right)$ induces an isometry*

$$\overline{p} : B\left(x,\varepsilon\right)/\Gamma_x \xrightarrow{\cong} B\left(px,\varepsilon\right).$$

*Proof.* Take $\varepsilon = \frac{1}{4} d\left(x, \Gamma x \smallsetminus \{x\}\right) > 0$. Given $y, z \in B\left(x,\varepsilon\right)$, we have $\overline{d}\left(py,pz\right) = d\left(y,\Gamma z\right)$. If $\gamma \in \Gamma$ and $d\left(y,\gamma z\right) < d\left(y,z\right) < 2\varepsilon$, then we show that $d\left(x,\gamma x\right) < 4\varepsilon$, so $\gamma \in \Gamma_x$. It follows that $d\left(y,\Gamma z\right) = d\left(y,\Gamma_x z\right)$, proving that $\overline{p}$ is an isometry. $\square$

**Corollary 2.40.** *If a Fuchsian group $\Gamma$ acts freely on $\mathcal{H}$, then the projection $p : \mathcal{H} \to \mathcal{H}/\Gamma$ is a covering map and a local isometry. The quotient $\mathcal{H}/\Gamma$ is a **hyperbolic surface** (i.e. a metric space that is locally isometric to $\mathcal{H}$); $p$ is its universal cover and $\Gamma$ is its fundamental group.*

*Conversely, if $S$ is a complete oriented hyperbolic surface, then its universal cover $\widetilde{S}$ is isometric to $\mathcal{H}$, and the action $\pi_1 S \curvearrowright \widetilde{S}$ by isometries induces an embedding $\pi_1 S \hookrightarrow \mathrm{Isom}\left(\mathcal{H}\right)$, whose image is a Fuchsian group s.t. $S \cong \mathcal{H}/\pi_1 S$*

**Proposition 2.41.** *If $D$ is a locally finite convex fundamental domain for a Fuchsian group $\Gamma$, then*

$$\mathcal{H}/\Gamma \cong \overline{D}/\Gamma.$$

**Remark 2.42.** *Proposition 2.41 need not hold if $D$ is not locally finite. For instance, consider $\gamma : z \in \mathbb{C}^* \mapsto 2z \in \mathbb{C}^*$. Then the annulus $\{z \in \mathbb{C}^*,\ 1 \leqslant |z| \leqslant 2\}$ is a locally finite fundamental domain for the action $\langle\gamma\rangle \curvearrowright \mathbb{C}^*$, from which we see that $\mathbb{C}^*/\left\langle\gamma\right\rangle$ is, topologically, a torus. However, if we consider*

$$D' = \{z \in \mathbb{C}^*,\ 1 \leqslant |z| \leqslant 2 \ \text{and}\ \left(\Re(z) \leqslant 0 \ \text{or}\ \Im(z) \leqslant 0\right)\} \cup \left\{x + iy,\ x \in \mathbb{R}_{>0} \ \text{and}\ e^{-x} \leqslant y \leqslant 2e^{-x}\right\},$$

*then $D'$ is a non-locally finite fundamental domain for $\langle\gamma\rangle \curvearrowright \mathbb{C}^*$, but $\overline{D}'/\left\langle\gamma\right\rangle$ is a cylinder that is infinite on one side, so it is not homeomorphic to $\mathbb{C}^*/\left\langle\gamma\right\rangle$.*

**Definition 2.43** (Hyperbolic cone)**.** *Given $\theta \in [0, 2\pi)$, the **hyperbolic cone** $C(\theta, r)$ of angle $\theta$ and radius $r$ is defined by glueing the edges of a hyperbolic disk sector $S$ of radius $r$ and angle $\theta$.*

*For instance, if $r_{2\pi/q}$ is the rotation around $0$ of angle $\frac{2\pi}{q}$ in the Poincaré disk, then*

$$B\left(0,\varepsilon\right)/\left\langle r_{2\pi/q}\right\rangle \cong C\left(\frac{2\pi}{q}, \varepsilon\right).$$

**Proposition 2.44.** *If $\Gamma$ is a Fuchsian group, then $\mathcal{H}/\Gamma$ is a hyperbolic surface with conical singularities at points $v \in \mathcal{H}/\Gamma$ of angle $\frac{2\pi}{q_v}$, with $q_v \geqslant 1$. Moreover, the conical singularities of $\mathcal{H}/\Gamma$ correspond to the vertex cycles with angle sum less than $2\pi$ in the fundamental domain.*

## 2.9 Glueing constructions

**Definition 2.45** (Glueing constructions)**.** *Let $(P_j)_{j\in J}$ be a finite collection of finitely-sided convex hyperbolic polygons. Assume we have a* ***glueing data*** *$\phi$, i.e. an involution $\sigma : S \to S$, with $S \subseteq \bigcup_{j\in J} \operatorname{Sides}(P_j)$, and orientation-reversing isometries $\gamma'_s : s \to \sigma(s)$ (note that this $\gamma'_s$ corresponds to $\gamma_s^{-1}$ with the convention used for Poincaré's Theorem). Let $\sim_\phi$ be the equivalence relation on $X = \coprod_{j\in J} \overline{P}_j$ generated by $x \sim_\phi \gamma'_s(x)$ for $x \in s$, $s \in S$.*

*Then the surface obtained by glueing the $(P_j)_{j\in J}$ along $\phi$ is $\overline{X} = X/\sim_\phi$, endowed with the quotient topology. The surface $\overline{X}$ can be equipped with a metric $\overline{d}$ defined as follows: a* ***chain*** *$w : x \rightsquigarrow y$ is a sequence $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y$ of points in $X$ s.t. $y_k \sim_\phi x_{k+1}$; its length is*

$$\ell(w) = \sum_{k=0}^{n} d(x_k, y_k),$$

*and we set $\overline{d}(\overline{x}, \overline{y}) = \inf_{w:x\rightsquigarrow y} \ell(w)$. Then $\overline{d}$ defines a metric on $\overline{X}$.*

*Proof.* See [3] for the fact that $\overline{d}$ is a metric. □

**Proposition 2.46.** *Let $\overline{X}$ be a surface obtained by a glueing construction as in Definition 2.45. Then $\overline{X}$ is a hyperbolic surface with conical singularities of angles given by the angle sums of vertex cycles (note that those sums may be more than $2\pi$).*

## 2.10 Hyperbolic genus 2 surfaces

**Definition 2.47** (Pair of pants)**.** *A* ***pair of pants*** *is the topological surface with boundary obtained by removing three disjoint open disks from a sphere.*

**Lemma 2.48.** *Given $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_{>0})^3$, there exists a unique (up to isometry) hyperbolic right-angled hexagon with side lengths $\ell_1, k_1, \ell_2, k_2, \ell_3, k_3$, in that order, for some $(k_1, k_2, k_3) \in (\mathbb{R}_{>0})^3$.*

*Proof.* Start with a segment $[x, y]$ of length $\ell_1$. Let $\alpha$ (resp. $\eta$) be the geodesic orthogonal to $[x, y]$ at $x$ (resp. $y$), oriented s.t. $[x, y]$ lies to the right (resp. left). Let $z_t$ be the point on $\eta$ at distance $t$ from $y$; let $\beta_t$ be the geodesic orthogonal to $\eta$ at $z_t$. Let $w_t$ be the point on $\beta_t$ at distance $\ell_2$ from $z_t$; let $\gamma_t$ be the geodesic orthogonal to $\beta_t$ at $w_t$. It suffices to show that there is a unique value of $t$ for which $d_t = d(\alpha, \gamma_t) = \ell_3$. First note that there exists $t_0 > 0$ (chosen minimal) s.t. $\alpha$ and $\gamma_t$ do not intersect in $\overline{\mathcal{H}}$ for $t > t_0$. Then show that $\lim_{t_0} d_t = 0$, $\lim_\infty d_t = \infty$, and $d_t$ increases with $t$. The result follows. □

**Proposition 2.49.** *Given $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_{>0})^3$, there exists a unique (up to isometry) hyperbolic pair of pants whose boundary components have respective lengths $\ell_1, \ell_2, \ell_3$.*

*Proof.* Take two hyperbolic right-angled hexagons with side lengths $\frac{1}{2}\ell_1, k_1, \frac{1}{2}\ell_2, k_2, \frac{1}{2}\ell_3, k_3$ and glue them along the sides of lengths $k_1, k_2, k_3$. □

**Corollary 2.50.** *Given $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_{>0})^3$ and $(\theta_1, \theta_2, \theta_3) \in (\mathbb{R}/2\pi\mathbb{Z})^3$, there is a unique (up to isometry) genus 2 surface with handlebodies of diameter $\ell_1, \ell_3$ and central curve of diameter $\ell_2$, and with twists $\theta_1, \theta_2, \theta_3$.*

*Proof.* Glue two hyperbolic pairs of pants, one with boundary components of lengths $\ell_1, \ell_1, \ell_2$, and the other with boundary components of lengths $\ell_3, \ell_3, \ell_2$. The glueing isometries between the boundary components are determined by the twist parameters. □

**Definition 2.51** (Teichmüller space)**.** *The* ***Teichmüller space*** *$\mathcal{T}(S)$ of a surface $S$ is the set of all discrete embeddings $\pi_1(S) \hookrightarrow PSL_2\mathbb{R}$ up to conjugacy.*

**Example 2.52.** *If $S$ is the genus 2 surface, then $\mathcal{T}(S)$ has dimension 6, with parameters $\ell_i, \theta_i$ as in Corollary 2.50. Those parameters are called* ***Fenchel-Nielsen parameters****.*

## 2.11 Poincaré's Theorem for noncompact polygons

**Theorem 2.53** (Poincaré)**.** *Let $P \subseteq \mathcal{H}$ be a finitely-sided polygon. We consider a fixed-point free involution $\sigma : S \to S$ together with elements $\gamma_s \in PSL_2\mathbb{R}$ s.t. $\gamma_s^{-1}s = \sigma(s)$, and $\gamma_s$ reverses the orientation of $s$ (with sides oriented s.t. $P$ lies to the left).*

*We suppose:*

($VCC$) *Vertex cycle condition: the angle sum of each ordinary vertex cycle is of the form $\frac{2\pi}{q}$ for some $q \geqslant 1$.*

($PCC$) *Parabolic cycle condition: if $v_0, v_1, \dots, v_n = v_0$ is a cycle of ideal vertices, then the corresponding return map $\gamma_1 \cdots \gamma_n$ is parabolic.*

*Then, if $\Gamma = \langle \gamma_s,\ s \in S\rangle \leqslant PSL_2\mathbb{R}$, we have:*

(i) *$P$ is a fundamental domain for $\Gamma$,*

(ii) *$\Gamma$ is discrete,*

(iii) *$\Gamma$ has the presentation $\langle \gamma_s,\ s \in S \mid R\rangle$, where $R$ is the set of ordinary vertex cycle relations, together with the relations $\gamma_{\sigma(s)} = \gamma_s^{-1}$ for $s \in S$.*

**Remark 2.54.** *In Poincaré's Theorem for noncompact polygons, condition ($PCC$) is necessary. Indeed, let $P$ be a locally finite fundamental domain for $\Gamma$. Take a cycle $v_0, v_1, \dots, v_n = v_0$ of ideal vertices of $P$. We may assume for instance that $v_0 = \infty$ in the upper half-plane model. This gives a gallery $P = P_0, P_1, \dots, P_n$ of adjacent polygons, with $P_k = \gamma_1 \cdots \gamma_k P$. The associated first return map is $\gamma_{v_0} = \gamma_1 \cdots \gamma_n$; it fixes $\infty$. Therefore, if $\gamma_{v_0}$ is not parabolic, then it is hyperbolic, with axis $(\omega, \infty)$ for some $\omega \in \partial\mathcal{H} \smallsetminus \{\infty\}$. In this case, copies of $P$ will accumulate near the geodesic $(\omega, \infty)$, contradicting the local finiteness of $P$.*

# 3 The hyperbolic $n$-space

## 3.1 The hyperboloid model

**Definition 3.1** ($\mathbb{R}^{n,1}$)**.** *Equip $\mathbb{R}^{n+1}$ with the* ***standard Lorentzian form****:*

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

*The matrix of $\langle \cdot, \cdot\rangle$ in the canonical basis is*

$$J = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix}.$$

*We denote by $\mathbb{R}^{n,1}$ the vector space $\mathbb{R}^{n+1}$ together with $\langle \cdot, \cdot\rangle$.*

**Definition 3.2** (Hyperboloid model)**.** *The* ***hyperboloid model*** *of the hyperbolic $n$-space is*

$$\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n,1},\ \langle x, x\rangle = -1 \text{ and } x_{n+1} > 0 \right\}.$$

**Remark 3.3.** *The quadratic form $\langle \cdot, \cdot\rangle$ divides $\mathbb{R}^{n,1}$ into three cones:*

- *Elements of $V^+ = \{x \in \mathbb{R}^{n,1},\ \langle x, x\rangle > 0\}$ are called* ***space-like****.*
- *Elements of $V^- = \{x \in \mathbb{R}^{n,1},\ \langle x, x\rangle < 0\}$ are called* ***time-like****.*
- *Elements of $V^0 = \{x \in \mathbb{R}^{n,1},\ \langle x, x\rangle = 0\}$ are called* ***light-like****.*

**Proposition 3.4.** *Let $x, y \in \mathbb{R}^{n,1}$ be two non-space elements (i.e. $x, y$ are time-like or light-like) with $x_{n+1}, y_{n+1} > 0$. Then $\langle x, y\rangle \leqslant 0$ with equality* iff *$x, y$ are colinear light-like vectors.*

*Proof.* Note that, if $(\cdot,\cdot)$ is the Euclidean inner product on $\mathbb{R}^{n+1}$, then for any two vectors $u, v \in \mathbb{R}^{n,1}$, we have

$$\langle u, v\rangle = (u, v) - 2u_{n+1}v_{n+1}.$$

In particular, the assumption that $x, y$ are non-space means that $(x, x) \leqslant 2x_{n+1}^2$ and $(y, y) \leqslant 2y_{n+1}^2$. Hence, using the Cauchy-Schwarz inequality,

$$\langle x, y\rangle = (x, y) - 2x_{n+1}y_{n+1} \leqslant \sqrt{(x, x)}\sqrt{(y, y)} - 2x_{n+1}y_{n+1} \leqslant \sqrt{2x_{n+1}^2}\sqrt{2y_{n+1}^2} - 2x_{n+1}y_{n+1} = 0. \quad \square$$

**Proposition 3.5** (Coordinates in $\mathbb{H}^n$)**.** *Let $x \in \mathbb{H}^n \smallsetminus \{e_{n+1}\}$. Then there is a unique $(\ell, u) \in \mathbb{R}_{>0} \times \mathbb{S}^{n-1}$ s.t.*

$$x = \begin{bmatrix} (\sinh \ell)\, u \\ \cosh \ell \end{bmatrix}.$$

*Proof.* Write $x = \begin{bmatrix} X \\ x_{n+1} \end{bmatrix}$ with $X \in \mathbb{R}^n$. Because $x \in \mathbb{H}$, we have $\|X\|^2 - x_{n+1}^2 = -1$, so there exists $\ell > 0$ s.t. $\|X\| = \sinh \ell$ and $x_{n+1} = \cosh \ell$. Then set $u = \frac{X}{\|X\|}$. $\square$

**Proposition 3.6.** *Let $x \in \mathbb{H}$.*

(i) *The tangent space of $\mathbb{H}^n$ at $x$ is given by $T_x\mathbb{H}^n = \operatorname{Ker}\langle x, \cdot\rangle$.*

(ii) *The restriction of $\langle\cdot,\cdot\rangle$ to $T_x\mathbb{H}^n$ is positive definite.*

*Hence, we can endow $\mathbb{H}^n$ with a Riemannian metric given by the restriction of $\langle\cdot,\cdot\rangle$ to tangent spaces.*

## 3.2 Isometries of $\mathbb{H}^n$

**Definition 3.7** ($O(n,1)$)**.** *We denote by $O(n,1)$ the group of linear transformations preserving $\langle\cdot,\cdot\rangle$. Equivalently,*

$$O(n,1) = \left\{A \in GL_{n+1}\mathbb{R},\ {}^tAJA = J\right\}.$$

*In particular, if $A \in O(n,1)$, then $\det A \in \{\pm 1\}$.*

*We also define:*

- $SO(n,1) = \{A \in O(n,1),\ \det A = 1\}$,
- $O^+(n,1) = \{A \in O(n,1),\ A(\mathbb{H}^n) = \mathbb{H}^n\}$,
- $SO^+(n,1) = O^+(n,1) \cap SO(n,1)$.

*Note that $O^+(n,1)$ acts on $\mathbb{H}^n$ by Riemannian isometries.*

**Theorem 3.8.** $\operatorname{Isom}(\mathbb{H}^n) = O^+(n,1)$ *and* $\operatorname{Isom}^+(\mathbb{H}^n) = SO^+(n,1)$.

*Proof.* See [2]. $\square$

**Proposition 3.9.** (i) *$\mathbb{H}^n$ is* ***homogeneous****, i.e. $O^+(n,1)$ (and also $SO^+(n,1)$) acts on $\mathbb{H}^n$ transitively.*

(ii) *For $x \in \mathbb{H}^n$, $\operatorname{Stab}(x) \cong O(n)$.*

(iii) *$\mathbb{H}^n$ is* ***isotropic****, i.e. for all $x, y \in \mathbb{H}^n$ and for all $u \in T_x\mathbb{H}^n$ and $v \in T_y\mathbb{H}^n$ s.t. $\langle u, u\rangle = \langle v, v\rangle$, there exists $g \in O^+(n,1)$ s.t. $gx = y$ and $\mathrm{d}g_x u = v$.*

(iv) *$\mathbb{H}^n$ is a* ***Riemannian symmetric space****, i.e. for all $x \in \mathbb{H}^n$, there exists $\iota_x \in O^+(n,1)$ s.t. $\iota_x(x) = x$ and $(\mathrm{d}\iota_x)_x = -\mathrm{id}_{T_x\mathbb{H}^n}$.*

*Proof.* (ii) By homogeneity of $\mathbb{H}^n$, the stabilisers of different points are conjugate, so it suffices to prove the result for $x = e_{n+1}$. But note that

$$\mathrm{Stab}\,(e_{n+1}) = \left\{ \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix},\ A' \in O(n) \right\}.$$

(iii) Using (i) and (ii), this amounts to proving that $O(n)$ acts transitively on the unit sphere of $T_{e_{n+1}}\mathbb{H}^n$.

(iv) Using homogeneity, we may assume that $x = e_{n+1}$. Then it suffices to take

$$\iota_{e_{n+1}} = \begin{bmatrix} -I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

□

**Remark 3.10.** *Since $SO^+(n,1)$ acts transitively on $\mathbb{H}^n$ with point stabilisers isomorphic to $SO(n)$, it follows that*

$$\mathbb{H}^n \cong SO^+(n,1)/SO(n).$$

*This is a bijection, but also a homeomorphism. Likewise, we have seen (in Proposition 1.42) that the upper half-plane is homeomorphic to $PSL_2\mathbb{R}/PSO_2\mathbb{R}$.*

*These are special cases of the following: if $G$ is a Lie group and $K$ is a maximal compact subgroup of $G$, then $G/K$ is a Riemannian symmetric space, called the* ***symmetric space*** *of $G$. Moreover, $G = \mathrm{Isom}^+(G/K)$.*

## 3.3 Geodesics and totally geodesic subspaces

**Definition 3.11** (Hyperbolic linear subspace)**.** *A $(k+1)$-dimensional linear subspace $V \subseteq \mathbb{R}^{n,1}$ is called* ***hyperbolic*** *(resp.* ***elliptic****) if the restriction of $\langle\cdot,\cdot\rangle$ to $V$ has signature $(k,1)$ (resp. $(k+1,0)$).*

**Theorem 3.12.** *The geodesics of $\mathbb{H}^n$ are the intersections $\mathbb{H}^n \cap V$, where $V$ is a hyperbolic 2-plane.*

**Proposition 3.13.** (i) *For all $x \neq y \in \mathbb{H}^n$, there is a unique geodesic through $x$ and $y$, namely $\mathbb{H}^n \cap \mathrm{Span}(x,y)$.*

(ii) *For all $x \in \mathbb{H}^n$ and $u \in T_x\mathbb{H}^n$ with $\langle u,u\rangle = 1$, the geodesic starting at $x$ and directed by $u$ is given by*

$$\gamma(t) = (\cosh t)\,x + (\sinh t)\,u.$$

*Proof.* (i) The 2-plane $V = \mathrm{Span}(x,y)$ is hyperbolic since $\langle\cdot,\cdot\rangle$ has signature $(n,1)$ and $V$ contains at least one vector of negative type, so $\mathbb{H}^n \cap V$ is a geodesic by Theorem 3.12. Uniqueness is clear.

(ii) Let $\gamma(t) = (\cosh t)\,x + (\sinh t)\,y$. Then $\langle\gamma(t),\gamma(t)\rangle = -1$ and $\langle\dot\gamma(t),\dot\gamma(t)\rangle = 1$, so $\gamma$ is a parametrisation of $\mathbb{H}^n \cap \mathrm{Span}(x,y)$ by arc length. □

**Corollary 3.14.** *Let $x \neq y \in \mathbb{H}^n$.*

(i) $\langle x,y\rangle < -1$,

(ii) $\cosh\left(d(x,y)\right) = -\langle x,y\rangle$.

*Proof.* (i) The restriction of $\langle \cdot, \cdot \rangle$ to $V = \mathrm{Span}(x, y)$ has matrix

$$M = \begin{pmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle & \langle y, y \rangle \end{pmatrix} = \begin{pmatrix} -1 & \langle x, y \rangle \\ \langle x, y \rangle & -1 \end{pmatrix}.$$

But $V$ is hyperbolic, so $0 > \det M = 1 - \langle x, y \rangle^2$. It follows that $\langle x, y \rangle^2 > 1$, so $\langle x, y \rangle < -1$ by Proposition 3.4.

(ii) Using Proposition 3.13, we may write

$$y = (\cosh \ell)\, x + (\sinh \ell)\, u$$

with $\ell = d(x, y)$, for some $u \in T_x\mathbb{H}^n$. It follows that $\langle x, y \rangle = -\cosh \ell$. □

**Remark 3.15.** *To complete the above arguments, one should prove that geodesics are distance-minimising. One way to do this is to define the distance $d$ on $\mathbb{H}^n$ via the formula $\cosh(d(x, y)) = -\langle x, y \rangle$ and to prove that the geodesics of $(\mathbb{H}^n, d)$ coincide with the Riemannian geodesics. For more details, see [12, 3.2].*

**Definition 3.16** (Totally geodesic subspace)**.** *Let $N \subseteq \mathbb{H}^n$ be a submanifold. We say that $N$ is* ***totally geodesic*** *if it satisfies the following condition:*

(i) *For any geodesic $\gamma : \mathbb{R} \to \mathbb{H}^n$ s.t. $\gamma(0) \in N$ and $\dot\gamma(0) \in T_{\gamma(0)}N$, then $\gamma(\mathbb{R}) \subseteq N$.*

*Since $\mathbb{H}^n$ is* ***uniquely geodesic*** *(i.e. any two points are joined by a unique geodesic segment), this is equivalent to:*

(ii) *For all $x, y \in N$, the unique complete geodesic through $x$ and $y$ is contained in $N$.*

**Proposition 3.17.** *The totally geodesic subspaces of $\mathbb{H}^n$ are the sets $\mathbb{H}^n \cap W$, where $W$ is a hyperbolic linear subspace of $\mathbb{R}^{n,1}$.*

*In particular, all totally geodesic subspaces of $\mathbb{H}^n$ are isometrically embedded copies of $\mathbb{H}^k$ for some $1 \leqslant k \leqslant n$.*

**Example 3.18.** *If $x \in \mathbb{H}^n$ and $u, v \in T_x\mathbb{H}^n$, then $\mathrm{Span}_{\mathbb{R}}(x, u, v) \cap \mathbb{H}^n$ is an embedded copy of $\mathbb{H}^2$ containing $x$ and tangent to $u, v$.*

**Proposition 3.19.** $\mathrm{Isom}^+(\mathbb{H}^n)$ *acts transitively on the set of $k$-dimensional totally geodesic subspaces of $\mathbb{H}^n$.*

## 3.4 Boundary at infinity

**Remark 3.20.** *Consider a geodesic $\gamma(t) = x(\cosh t) + u(\sinh t)$ in $\mathbb{H}^n$. We may rewrite the expression of $\gamma$ as*

$$\gamma(t) = e^t (x + u) + \underbrace{e^{-t}(x - u)}_{\to 0}.$$

*Therefore, we would like to view $x + u$ as the point of $\partial\mathbb{H}^n$ that is the limit of $\gamma$ at infinity. We say that $x + u$ is the* ***point at infinity*** *of $\gamma$.*

**Definition 3.21** (Boundary at infinity of $\mathbb{H}^n$)**.** *We denote by $S$ the set of geodesic rays $\mathbb{R}_{\geqslant 0} \to \mathbb{H}^n$ parametrised by arc-length. Two geodesic rays $\gamma_1, \gamma_2 \in S$ are said to be equivalent (which we denote $\gamma_1 \sim \gamma_2$) if one of the following two equivalent conditions is satisfied:*

(i) *$\gamma_1, \gamma_2$ have the same point at infinity, i.e. $\gamma_1(0) + \dot\gamma_1(0) = \gamma_2(0) + \dot\gamma_2(0)$.*

(ii) $\sup_{t \geqslant 0} d(\gamma_1(t), \gamma_2(t)) < +\infty$.

*Hence, $\sim$ is an equivalence relation on $S$, and the* ***boundary at infinity*** *of $\mathbb{H}^n$, denoted by $\partial\mathbb{H}^n$, is the set $S/\sim$.*

**Remark 3.22.** (i) *Definition 3.21 is a good definition because it works for negatively curved manifolds and Gromov-hyperbolic spaces.*

(ii) *Fix $x \in \mathbb{H}^n$. Given a unitary tangent vector $u \in UT_x\mathbb{H}^n$, set $\gamma_u(t) = x(\cos t) + u(\sinh t)$. Then $u \mapsto \gamma_u$ gives an identification between the boundary at infinity (seen from $x$) and the unit sphere in $T_x\mathbb{H}^n$. Therefore, $\partial\mathbb{H}^n$ is a $(n-1)$-sphere.*

## 3.5 Projective models and conformal models

**Definition 3.23** (Projective models)**.** *Consider a quadratic form $q$ of signature $(n,1)$ on $\mathbb{R}^{n+1}$. Then $q$ induces a partition of $\mathbb{R}^{n+1}$ into three cones: $V_q^+, V_q^0, V_q^-$. The* ***projective model*** *of $\mathbb{H}^n$ associated with $q$ is the subset $\mathbb{P}\left(V_q^-\right) \subseteq \mathbb{P}^n\mathbb{R}$, equipped with the distance given by*

$$\cosh^2\left(d\left([x],[y]\right)\right) = \frac{\langle x,y\rangle_q^2}{\langle x,x\rangle_q \langle y,y\rangle_q}.$$

*This is well-defined because the above formula is invariant under $x \mapsto \lambda x$ and $y \mapsto \mu y$.*

**Remark 3.24.** *The classification of quadratic forms on $\mathbb{R}^{n+1}$ says that two quadratic forms $q_1, q_2$ have the same signature if and only if there exists $A \in GL_{n+1}\mathbb{R}$ s.t. $q_2 = q_1 \circ A$. This implies that $AV_{q_2}^- = V_{q_1}^-$, and therefore, if $[A] \in PGL_{n+1}\mathbb{R}$ is the projective linear transformation associated to $A$,*

$$[A]\,\mathbb{P}\left(V_{q_2}^-\right) = \mathbb{P}\left(V_{q_1}^-\right).$$

*Hence, if $q_1, q_2$ are two quadratic forms of signature $(n,1)$, then their projective models are projectively equivalent.*

**Example 3.25.** (i) *Let $q_1(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. In the affine chart $\{x_{n+1} = 1\}$, the set $\mathbb{P}\left(V_{q_1}^-\right)$ is the unit ball in $\mathbb{R}^n$. This is the* ***Klein model****.*

(ii) *Let $q_2(x) = 2x_1x_{n+1} + x_2^2 + \cdots + x_n^2$. In the affine chart $\{x_{n+1} = 1\}$, the set $\mathbb{P}\left(V_{q_2}^-\right)$ is the paraboloid $\{2v_1 + v_2^2 + \cdots + v_n^2 < 0\}$. This is the* ***paraboloid model*** *or* ***Siegel model****.*

**Definition 3.26** (Poincaré ball)**.** *Consider the central projection $\pi$ onto $\mathbb{R}^n \times \{0\}$ centred at $-e_{n+1}$:*

$$\pi : (x_1, \ldots, x_{n+1}) \longmapsto \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n, 0).$$

*Then $\pi$ induces a diffeomorphism $\mathbb{H}^n \to \mathcal{B}^n$, where $\mathcal{B}^n$ is the unit ball of $\mathbb{R}^n \times \{0\}$.*

*The metric on $\mathcal{B}^n$ is obtained by pushing forward the metric on $\mathbb{H}^n$:*

$$\mathrm{d}s^2 = \frac{4\,\mathrm{d}x^2}{\left(1 - |x|^2\right)^2}.$$

*Since this Riemannian metric on $\mathcal{B}^n$ is a rescaling of the Euclidean metric, the Poincaré ball model is* ***conformal****, i.e. it preserves angles.*

**Definition 3.27** (Poincaré upper half-space)**.** *Consider the inversion $\iota$ with respect to the sphere centred at $-e_n$ and with radius $\sqrt{2}$ in $\mathbb{R}^n$:*

$$\iota : x \longmapsto 2\frac{x + e_n}{\|x + e_n\|^2} - e_n.$$

*Then $\iota$ induces a diffeomorphism $\mathcal{B}^n \to \mathcal{H}^n$, where $\mathcal{H}^n = \{x \in \mathbb{R}^n,\ x_n > 0\}$.*

*The metric on $\mathcal{H}^n$ is obtained by pushing forward the metric on $\mathcal{B}^n$:*

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{x_n^2}.$$

*The Poincaré upper half-space model is also conformal.*

# 4 Hyperbolic manifolds

## 4.1 $(X, G)$-manifolds

**Definition 4.1** ($(X, G)$-manifold)**.** *Let $X$ be a connected and simply connected, oriented differentiable $n$-manifold and let $G$ be a group of diffeomorphisms of $X$. An $(X, G)$**-manifold** is a differentiable manifold $M$ equipped with an atlas $\{\varphi : \mathcal{U} \to X\}$ with transition maps that are restrictions of elements of $G$.*

*Namely, there is an open cover $(\mathcal{U}_i)_{i \in I}$ of $M$ together with charts $(\varphi_i : \mathcal{U}_i \to X)_{i \in I}$ that are diffeomorphisms onto their image, and such that, if $\mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$, then $\varphi_i \circ \varphi_j^{-1}$ is the restriction of some element $g_{ij} \in G$.*

*The set $\{\varphi_i : \mathcal{U}_i \to X,\ i \in I\}$ is called an $(X, G)$**-atlas**. We say that $M$ has an $(X, G)$**-structure**, or a **geometric structure in the sense of Ehresmann and Thurston**.*

**Example 4.2.** *Consider the topological torus $\mathbb{T}^2$.*

(i) *$\mathbb{T}^2$ can be constructed as the quotient of a rectangle with opposite sides identified. This is an $(X, G)$-structure, with $X = \mathbb{R}^2$ (seen as a Euclidean space) and $G = \mathbb{R}^2$ is the group of translations of $\mathbb{R}^2$. This is a **Euclidean structure**.*

(ii) *$\mathbb{T}^2$ can also be constructed as the quotient of any quadrilateral, glueing each side to the opposite side by the unique similarity between them. This is an $(X, G)$-structure, with $X = \mathbb{R}^2$ (seen as an affine space) and $G = \mathrm{Aff}(X)$. This is an **affine structure**.*

*These two examples are very different: in the first case, we get a tessellation of $\mathbb{R}^2$ and a covering map $\mathbb{R}^2 \to \mathbb{T}^2$, but not in the second case.*

## 4.2 A drop of Riemannian geometry

**Definition 4.3** (Riemannian manifold)**.** *A **Riemannian manifold** is the data of a differentiable manifold $M$ together with a **metric tensor** $g$, i.e. a scalar product on each tangent space $T_x M$ that varies smoothly with $x$. For $v \in T_x M$, we write $|v|_x = \sqrt{g_x(v, v)}$*

**Definition 4.4** (Lengths of curves)**.** *A **curve** on a Riemannian manifold $(M, g)$ is a differentiable map $\gamma : I \to M$; its **length** is*

$$\ell(\gamma) = \int_I |\dot{\gamma}(t)|_{\gamma(t)}\, \mathrm{d}t.$$

*This is invariant under reparametrisation. The **Riemannian distance** between two points $p, q \in M$ is given by*

$$d(p, q) = \inf_{\gamma : p \rightsquigarrow q} \ell(\gamma).$$

*This is a distance on $M$ inducing the topology of $M$.*

**Definition 4.5** (Geodesics)**.** *A **geodesic** is a curve $\gamma : I \to M$ having constant speed (i.e. $|\dot{\gamma}(t)|_{\gamma(t)} = k$ for all $t$), and such that $\gamma$ locally realises the distance, i.e. for all $t \in I$, there exist $t_0 < t < t_1$ s.t. for all $s, s' \in [t_0, t_1]$,*

$$d(\gamma(s), \gamma(s')) = \ell\left(\gamma_{|[s,s']}\right) = k\,|s - s'|\,.$$

**Remark 4.6.** *We give Definition 4.5 to avoid introducing the technical language of tensors; however, in Riemannian geometry, a geodesic is normally defined as satisfying a certain differential equation (which says roughly that $\gamma$ has no acceleration), which implies in turn that $\gamma$ locally realises the distance.*

**Theorem 4.7.** *Let $p \in M$ and $u \in T_p M$. Then there exists a unique maximal geodesic $\gamma : I_u \to M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = u$. The interval $I_u$ is the longest interval containing $0$ on which such a geodesic can be defined.*

**Lemma 4.8.** *Let $p \in M$. Then there exists a neighbourhood $N_p$ of $p$ in $M$, positive real numbers $\varepsilon, \delta > 0$, and a smooth map $\gamma : D \to M$, where*

$$D = \left\{ (t, q, v) \in (-\delta, \delta) \times N_p \times TM,\ v \in T_q M \text{ and } |v|_q < \varepsilon \right\},$$

*and s.t. $\gamma(t, q, v)$ is the point of parameter $t$ on the geodesic $\gamma$ defined by $\gamma(0) = q$ and $\dot\gamma(0) = v$.*

**Definition 4.9** (Exponential map)**.** *Let $p \in M$. Consider $U_p = \{u \in T_p M,\ 1 \in I_u\}$. The **exponential map** at $p$ is the smooth map*

$$\exp_p : U_p \subseteq T_p M \longrightarrow M$$

*defined by $\exp_p(u) = \gamma(1, p, u)$.*

**Example 4.10.** (i) *Let $\mathbb{R}^n_{\text{aff}}$ be the affine space $\mathbb{R}^n$ together with the Euclidean metric. Then for all $A \in \mathbb{R}^n_{\text{aff}}$ and $\vec{u} \in T_A \mathbb{R}^n_{\text{aff}} = \mathbb{R}^n_{\text{eucl}}$, we have $\exp_A(\vec{u}) = A + \vec{u}$.*

(ii) *Consider the manifold $\mathbb{R}^*_+$ together with the metric $\frac{\mathrm{d}x}{x}$. Then $\exp_1(u) = e^u$.*

(iii) *Consider the Riemannian manifold $\mathbb{H}^n$. If $p = e_{n+1}$ in the hyperboloid model and $u \in T_p \mathbb{H}^n = p^\perp$, then $\gamma(t, p, u) = (\cosh t)\, p + (\sinh t)\, u$.*

**Proposition 4.11.** *If $p \in M$, then there exists $\varepsilon > 0$ s.t. $\exp_p : B(0, \varepsilon) \to M$ is a diffeomorphism onto its image.*

*Proof.* Prove that

$$\mathrm{d}\left(\exp_p\right)_0 = \mathrm{id}_{T_p M},$$

and conclude using the Inverse Function Theorem. □

**Definition 4.12** (Normal neighbourhoods and normal balls)**.** *Let $p \in M$.*

- *A **normal neighbourhood** of $p$ in $M$ is a neighbourhood of the form $\exp_p(V)$ s.t. $\exp_p : V \to M$ is a diffeomorphism onto its image.*
- *A **normal ball** at $p$ is a ball $B_p(\varepsilon) = \exp_p(B(0, \varepsilon))$, where $\exp_p : B(0, \varepsilon) \to M$ is a diffeomorphism onto its image.*

**Remark 4.13.** *If $B$ is a normal ball at $p$ in $M$ and $q \in B$, then there exists a unique geodesic $\sigma : [0, 1] \to B$ s.t. $\sigma(0) = p$ and $\sigma(1) = q$. Such a geodesic is called **radial**.*

**Proposition 4.14** (Local length minimisation)**.** *Let $B$ be a normal ball at $p$ in $M$. Let $\gamma : [0, 1] \to B$ be a geodesic with $\gamma(0) = p$. Then for any curve $c$ on $M$ s.t. $\gamma(0) = c(0)$ and $\gamma(1) = c(1)$, we have*

$$\ell(\gamma) \leqslant \ell(c),$$

*with equality* iff *$\gamma([0, 1]) = c([0, 1])$.*

*Proof.* See [5, 3.6]. □

**Proposition 4.15** (Global length minimisation)**.** *If $\gamma : [a, b] \to M$ is a piecewise differentiable curve with parameter proportional to arc length and s.t. $\ell(\gamma) \leqslant \ell(c)$ for any curve $c$ s.t. $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$, then $\gamma$ is a geodesic.*

## 4.3 The Hopf-Rinow Theorem

**Definition 4.16** (Geodesically complete)**.** *A Riemannian manifold is called **geodesically complete** if all its geodesics are defined over* $\mathbb{R}$*.*

**Example 4.17.** (i) *The Riemannian manifolds* $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n$ *(with their usual Riemannian metrics) are geodesically complete.*

(ii) $\mathbb{R}^n \setminus \{0\}$ *(with the Euclidean Riemannian metric) is not geodesically complete.*

(iii) *Consider an ideal square* $S$ *in* $\mathbb{H}^2$ *with torus-like side-pairings given by two isometries* $a, b \in PSL_2\mathbb{R}$*. If the commutator* $[a, b]$ *is not parabolic, then it must be hyperbolic (one can check that it fixes one of the ideal vertices of* $S$*), so it has an axis* $\alpha$*. It follows that, for all* $\gamma \in \Gamma = \langle a, b\rangle$*, the tiles* $\gamma S$ *remain on the same side of* $\alpha$*. Hence* $\Gamma S$ *tessellates a proper convex subset* $\mathcal{R}$ *of* $\mathbb{H}^2$ *delimited by some geodesics; the quotient* $\mathcal{R}/\Gamma$ *is a non-complete Riemannian manifold.*

**Lemma 4.18.** *Given* $p \in M$*, there exists an* $\eta > 0$ *and a neighbourhood* $V$ *of* $p$ *s.t. for all* $q \in V$*,* $\exp_q : B(0, \eta) \subseteq T_qM \longrightarrow M$ *is a diffeomorphism onto its image. We say that* $V$ *is a **totally normal neighbourhood** of* $p$*.*

*Proof.* See [5, 3.7]. □

**Theorem 4.19** (Hopf-Rinow, 1931)**.** *Let* $M$ *be a Riemannian manifold and* $p \in M$*. Then the following are equivalent:*

(i) $\exp_p$ *is defined on* $T_pM$*.*

(ii) *Closed and bounded subsets of* $M$ *are compact.*

(iii) $M$ *is Cauchy-complete.*

(iv) $M$ *is geodesically complete.*

*Moreover, conditions* (i)-(iv) *imply:*

(v) *For all* $q \in M$*, there is a geodesic* $\gamma$ *on* $M$ *from* $p$ *to* $q$ *with* $\ell(\gamma) = d(p, q)$*.*

*Proof.* (i) $\Rightarrow$ (v) Let $q \in M$ and write $r = d(p, q)$. Consider a closed normal sphere $\partial B_p(\varepsilon)$; it is compact, and $d(\cdot, q)$ is continuous, so there exists $x_0 \in \partial B_p(\varepsilon)$ with

$$d(x_0, q) = \min_{x \in \partial B_p(\varepsilon)} d(x, q).$$

We can write $x_0 = \exp_p(\varepsilon v)$ with $v \in T_pM$ and $\|v\| = 1$. Consider the geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Our goal is to prove that $\gamma(r) = q$.

Consider the set

$$\mathcal{E} = \{t \in \mathbb{R},\ d(\gamma(t), q) = r - t\}.$$

It is clear that $\mathcal{E} \ni 0$ and $\mathcal{E}$ is closed; if we prove that $\mathcal{E}$ is open, then it will follow by connectedness that $\mathcal{E} = \mathbb{R}$, and in particular $r \in \mathcal{E}$, so $\gamma(r) = q$.

Let $t_0 \in \mathcal{E}$. Let $\alpha > 0$ be small enough so that the ball $B_{\gamma(t_0)}(\alpha)$ is normal. We want to show that $t_0 + \alpha \in \mathcal{E}$. Let $x_0' \in \partial B_{\gamma(t_0)}(\alpha)$ s.t.

$$d(x_0', q) = \min_{x \in \partial B_{\gamma(t_0)}(\alpha)} d(x, q).$$

First note that

$$d(\gamma(t_0), q) \leqslant d(\gamma(t_0), x_0') + d(x_0', q) = \alpha + d(x_0', q).$$

Moreover, given a curve $c : [0,1] \to M$ from $\gamma(t_0)$ to $q$, if $t_1 = \min\left\{t \in [0,1],\ c(t) \in \partial B_{\gamma(t_0)}(\alpha)\right\}$, then

$$\ell(c) = \ell\left(c_{|[0,t_1]}\right) + \ell\left(c_{|[t_1,1]}\right) \geqslant \alpha + d\left(c\left(t_1\right), q\right) \geqslant \alpha + d\left(x_0', q\right).$$

It follows that

$$r - t_0 = d\left(\gamma\left(t_0\right), q\right) = \alpha + d\left(x_0', q\right),$$

and therefore $d\left(x_0', q\right) = r - t_0 - \alpha$. Now we have

$$d\left(p, x_0'\right) \geqslant d(p,q) - d\left(x_0', q\right) = r - \left(r - t_0 - \alpha\right) = t_0 + \alpha,$$

and the piecewise geodesic curve $c = [p, \gamma(t_0)] \cup [\gamma(t_0), x_0']$ has length $t_0 + \alpha$, so $d\left(p, x_0'\right) = t_0 + \alpha$ and $c$ is a geodesic by Proposition 4.15. By uniqueness, $c = \gamma$, so $\gamma\left(t_0 + \alpha\right) = x_0'$ and therefore $t_0 + \alpha \in \mathcal{E}$.

(i) $\Rightarrow$ (ii) Let $E \subseteq M$ be closed and bounded. Then $E$ is contained in a ball $B$ of radius $r$ for the Riemannian distance $d$. Any point in $B$ is related to $p$ by a geodesic realising the distance (by (v), which is implied by (i)), so that

$$\exp_p\left(B(0,r)\right) \supseteq B \supseteq E.$$

Hence $E$ is compact as a closed subset of a compact set.

(ii) $\Rightarrow$ (iii) OK.

(iii) $\Rightarrow$ (iv) If $M$ is not geodesically complete, then there is a geodesic $\gamma$ which is defined on $[0, t_0)$ but not beyond. Now pick $t_n \to t_0$ with $0 \leqslant t_n < t_0$. The sequence $\left(\gamma\left(t_n\right)\right)_{n \geqslant 1}$ is Cauchy; use Lemma 4.18 to prove that this sequence does not converge. Therefore, $M$ is not Cauchy complete.

(iv) $\Rightarrow$ (i) OK. $\square$

## 4.4 Complete hyperbolic manifolds

**Remark 4.20.** *A hyperbolic manifold in the sense of Thurston is in particular a Riemannian manifold that is locally isometric to $\mathbb{H}^n$.*

**Definition 4.21** (Local isometry)**.** *A map $\varphi : M \to N$ between Riemannian manifolds is called a* ***local isometry*** *if every point $x \in M$ has a neighbourhood $U_x \subseteq M$ s.t. $\varphi_{|U_x} : U_x \to \varphi(U_x)$ is an isometry.*

**Lemma 4.22.** *Let $M, N$ be two connected Riemannian $n$-manifolds. Let $\phi_1, \phi_2 : M \to N$ be two local isometries s.t. there is a point $p \in M$ with*

$$\phi_1(p) = \phi_2(p) \qquad \text{and} \qquad \left(\mathrm{d}\phi_1\right)_p = \left(\mathrm{d}\phi_2\right)_p.$$

*Then $\phi_1 = \phi_2$.*

*Proof.* Consider

$$E = \left\{p \in M,\ \phi_1(p) = \phi_2(p) \text{ and } \left(\mathrm{d}\phi_1\right)_p = \left(\mathrm{d}\phi_2\right)_p\right\}.$$

Then $E$ is nonempty and closed. Let us prove that $E$ is open; by connectedness, this will imply that $E = M$. Let $m \in E$. Let $\varepsilon > 0$ s.t. the ball $B_\varepsilon(m) \subseteq M$ is normal. Then

$$\exp_m : B(0, \varepsilon) \subseteq T_m M \longrightarrow B_\varepsilon(m) \subseteq M$$

is a diffeomorphism. Now note that, since $\phi_1, \phi_2$ are local isometries, they map geodesics to geodesics. Hence, since $\phi_1(m) = \phi_2(m)$ and $\left(\mathrm{d}\phi_1\right)_m = \left(\mathrm{d}\phi_2\right)_m$, the image of $t \mapsto \exp_m(tu)$ is the same under $\phi_1$ and $\phi_2$, namely it is the geodesic starting at $\phi_1(m)$ and with initial tangent vector $\left(\mathrm{d}\phi_1\right)_m \cdot u$. Hence, if $q \in B_\varepsilon(m)$, then $q = \exp_m(u)$ for some $u \in B(0,\varepsilon) \subseteq T_m M$ and therefore $\phi_1(q) = \phi_2(q)$. This proves that $\left(\phi_1\right)_{|B_\varepsilon(m)} = \left(\phi_2\right)_{|B_\varepsilon(m)}$, so $B_\varepsilon(m) \subseteq E$ (using the fact that $B_\varepsilon(m)$ is open). Therefore, $E$ is open, and hence $E = M$. $\square$

**Notation 4.23.** *From now on, we consider a Riemannian manifold $X$ together with a group $G \leqslant \mathrm{Isom}(X)$. In this case, any $(X, G)$-manifold is also a Riemannian manifold and its $(X, G)$-atlas is composed of isometries.*

**Lemma 4.24** (Lebesgue's Number Lemma)**.** *Given a compact metric space $(X, d)$ together with an open covering, there exists some $\delta > 0$ s.t. every subset $A \subseteq X$ with $\mathrm{diam}(A) < \delta$ is contained in one of the open subsets covering $X$.*

**Proposition 4.25.** *Let $M$ be a connected and simply connected $(X, G)$-manifold. Consider an isometric embedding $\phi : U_0 \subseteq M \to X$, where $U_0$ is open in $M$. Then $\phi$ extends uniquely to a local isometry $D : M \to X$ with $D_{|U_0} = \phi$. We say that $D$ is a* ***developing map****.*

*Proof.* Uniqueness is a consequence of Lemma 4.22. To construct $D$, fix $p_0 \in U_0$ and choose $p \in M$ together with a curve $c : p_0 \rightsquigarrow p$ defined on $[0, 1]$. Since $\mathcal{C} = c([0, 1])$ is compact, we can cover it by a finite number of $(X, G)$-open charts $(\phi_k : U_k \to X)_{1 \leqslant k \leqslant m}$. We may assume that

- $c^{-1}(U_k)$ is an interval denoted by $I_k$,
- $I_k$ only intersects $I_{k-1}$ and $I_{k+1}$,
- $U_k \cap U_\ell$ is connected if nonempty.

For all $k$, there is some $g_k \in G$ s.t. $\phi_k \circ \phi_{k-1}^{-1} = g_k$. Now define $D$ by "following the curve":

$$D_{|U_k \cap \mathcal{C}} = g_1^{-1} \circ g_2^{-1} \circ \cdots \circ g_k^{-1} \circ (\phi_k)_{|c(I_k)} .$$

Check that this works on the intersections, i.e. $D$ is well-defined on $U_k \cap U_{k-1}$.

We need to prove that (i) $D(p)$ is independent of the choice of the covering of $c$, and (ii) $D(p)$ is independent of the choice of the path $c$. It will then be clear that $D : M \to X$ is a local isometry.

(i) We prove that $D(p)$ does not change when $U_1$ is replaced by some $U_1'$. Indeed, we have

$$D_{|U_1 \cap U_1' \cap \mathcal{C}} = g_1^{-1} \circ \phi_1 \qquad \text{and} \qquad D'_{|U_1 \cap U_1' \cap \mathcal{C}} = {g_1'}^{-1} \circ \phi_1'.$$

But on $U_0 \cap U_1 \cap U_1'$, we have

$$\phi_1 \circ {\phi_1'}^{-1} = \left(\phi_1 \circ \phi_0^{-1}\right) \circ \left(\phi_0 \circ {\phi_1'}^{-1}\right) = g_1 \circ {g_1'}^{-1};$$

it follows (using Lemma 4.22) that $D_{|U_1 \cap U_1' \cap \mathcal{C}} = D'_{|U_1 \cap U_1' \cap \mathcal{C}}$. Similarly, one can change all the open sets $U_k$.

(ii) Let $c_0, c_1$ be two choices of paths from $p_0$ to $p$. Assume first that $c_0 = c_1$ except on one of the $U_k$s, and define $D_0, D_1$ using $c_0, c_1$ respectively. In that situation, the transition maps used to defined $D_0, D_1$ are the same along $c_0$ and $c_1$, which implies that $D_0(p) = D_1(p)$. In the general case, recall that $M$ is simply connected, so there is a homotopy $c_\bullet$ from $c_0$ to $c_1$. Using Lemma 4.24, there is a subdivision $[0, 1]^2 = \bigcup_{i,j} [t_i, t_{i+1}] \times [s_j, s_{j+1}]$ s.t. each rectangle is contained in some $U_k$. Now moving along rectangles in this grid, and using the first case, we see that $D_0(p) = D_1(p)$. ☐

**Remark 4.26.** *If $X$ is not Riemannian, then Proposition 4.25 can remain true with the same proof if we assume for instance that $G$ acts analytically on $X$ (and any $g \in G$ is determined by its restriction to any open subset of $X$).*

**Corollary 4.27.** *If $M$ is a connected and simply connected $(X, G)$-manifold, and $D, D'$ are two developing maps, then there exists $g \in G$ s.t. $D' = g \circ D$.*

*Proof.* Assume that $D$ (resp. $D'$) is a developing map on $U_0 \ni p$ (resp. on $U_0' \ni p'$) and pick a path $c : p \rightsquigarrow p'$. Take a finite covering of $c$ by $(X, G)$-charts; if $g_1, \ldots, g_n$ are the transition maps, then check that $g_1^{-1} \cdots g_n^{-1} D'$ is a developing map on $U_0'$ which coincides with $D$ on $U_0$, so $D = g_1^{-1} \cdots g_n^{-1} D'$. □

**Theorem 4.28** (Ambrose)**.** *Let $\phi : M_1 \to M_2$ be a local isometry between two Riemannian manifolds with $M_1$ complete. Then $M_2$ is complete and $\phi$ is a covering map.*

**Theorem 4.29.** *Every complete, connected and simply connected hyperbolic manifold is isometric to $\mathbb{H}^n$.*

*Proof.* A hyperbolic manifold is a $(X, G)$-manifold with $X = \mathbb{H}^n$ and $G = \mathrm{Isom}\,(\mathbb{H}^n)$. Hence, given $M$ a complete, connected and simply connected hyperbolic manifold, applying Proposition 4.25 to a chart of $M$ yields a developing map $D : M \to X$. In particular, $D$ is a local isometry, so Ambrose's Theorem (Theorem 4.28) implies that $D$ is a covering. But $M, X$ are both simply connected, so $D$ must be bijective. Hence, $D$ is a bijective local isometry. This implies that $D$ preserves the length of curves, and so $D$ is 1-Lipschitz. The same is true for $D^{-1}$, so $D$ is an isometry. □

**Definition 4.30** (Free and properly discontinuous actions)**.** *An action $G \curvearrowright X$ by homeomorphisms is said to be*

- ***Free*** *if $gx \neq x$ for all $x \in X$ and $g \in G \setminus \{e\}$,*
- ***Properly discontinuous*** *if every $x \in X$ has a neighbourhood $U_x$ s.t. $gU_x \cap U_x \neq \varnothing$ for all $g \in G \setminus \{e\}$.*

**Theorem 4.31.** *If $M$ is a complete connected hyperbolic (resp. flat, spherical) manifold, then there exits a discrete subgroup $\Gamma \leqslant \mathrm{Isom}\,(\mathbb{H}^n)$ (resp. $\mathrm{Isom}\,(\mathbb{E}^n)$, $\mathrm{Isom}\,(\mathbb{S}^n)$) such that there is an isometry*

$$\Gamma \cong \mathbb{H}^n / \Gamma$$

*(resp. $\mathbb{E}^n/\Gamma$, $\mathbb{S}^n/\Gamma$).*

*Proof.* The universal cover $\widetilde{M}$ of $M$ is simply connected, so Theorem 4.29 implies that $\widetilde{M} \cong \mathbb{H}^n$. Now $\pi_1 M$ acts properly discontinuously by isometries on $\widetilde{M}$, so it is isomorphic to a discrete subgroup $\Gamma \leqslant \mathrm{Isom}\,(\mathbb{H}^n)$, in such a way that

$$M \cong \widetilde{M}/\pi_1 M \cong \mathbb{H}^n/\Gamma.$$

□

# 5 Discrete groups of hyperbolic isometries

**Theorem 5.1.** *Let $M$ be a Riemannian manifold. Then a subgroup $\Gamma \leqslant \mathrm{Isom}\,(M)$ is discrete* iff *it acts properly discontinuously on $M$.*

*Proof.* The Myers-Steenrod Theorem implies that $\mathrm{Isom}\,(M)$ is a Lie group (note that if $M = \mathbb{H}^n$, then $\mathrm{Isom}\,(M)$ is a Lie group because it is isomorphic to $O^+(n, 1)$ by Theorem 3.8). This allows one to do the same proof as for Theorem 2.15. □

## 5.1 Classification of hyperbolic isometries

**Proposition 5.2** (Classification of isometries)**.** *If $f \in \mathrm{Isom}\,(\mathbb{H}^n)$, then exactly one of the following holds:*

(i) *$f$ has exactly two fixed points in $\partial\mathbb{H}^n$ and none in $\mathbb{H}^n$. We then say that $f$ is* ***hyperbolic****.*

(ii) *$f$ has exactly one fixed point in $\partial\mathbb{H}^n$ and none in $\mathbb{H}^n$. We then say that $f$ is* ***parabolic****.*

(iii) *$f$ has at least one fixed point in $\mathbb{H}^n$. We then say that $f$ is **elliptic**.*

**Remark 5.3.** *In a projective model for $\mathbb{H}^n$, types of isometries correspond to different possible Jordan forms for elements of $SO^+(n,1)$.*

**Lemma 5.4** (Normalised forms for isometries)**.** *Any element of $\operatorname{Isom}(\mathbb{H}^n)$ is conjugate to one of the following:*

(i) *A hyperbolic isometry fixing $0$ and $\infty$ in the upper half-space $\mathcal{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$, given by*

$$(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \longmapsto (\lambda A x, \lambda t),$$

*with $A \in O(n-1)$ and $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$.*

*If $A = I$, we say that $f$ is **pure hyperbolic** (or a **translation**); otherwise, we say that $f$ is **loxodromic**.*

(ii) *A parabolic isometry fixing $\infty$ in the upper half-space $\mathcal{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$, given by*

$$(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \longmapsto (Ax+b, t),$$

*with $A \in O(n-1)$ and $b \in \mathbb{R}^{n-1} \setminus \{0\}$.*

(iii) *An elliptic isometry fixing $0$ in the Poincaré ball $\mathcal{B}^n$, given by*

$$x \in \mathcal{B}^n \longmapsto Ax,$$

*with $A \in O(n)$.*

**Remark 5.5.** *If $a, b \in \mathbb{H}^n$, then there exists a unique translation $\tau_{a,b} \in \operatorname{Isom}(\mathbb{H}^n)$ s.t. $\tau_{a,b}(a) = b$.*
*Hence, we get a homeomorphism $O(n) \times \mathbb{H}^n \xrightarrow{\simeq} \operatorname{Isom}(\mathbb{H}^n)$ given by $(A,a) \mapsto \tau_{0,a} \circ A$.*

**Remark 5.6** (Stable subsets for isometries)**.** *Let $f \in \operatorname{Isom}(\mathbb{H}^n)$.*

(i) *If $f$ is hyperbolic with axis $\mathcal{A}$, then $f$ preserves $\mathcal{A}$ as well as all equidistant hypersurfaces $\{d(x,\mathcal{A}) = k\}$ for $k \in \mathbb{R}_{>0}$.*

(ii) *If $f$ is parabolic with fixed point $p \in \partial\mathbb{H}^n$, then $f$ preserves all horospheres centred at $p$.*

(iii) *If $f$ is elliptic with fixed-point set $F$, then $f$ preserves all spheres with centre in $F$.*

## 5.2 Examples of hyperbolic manifolds: tubes and cusps

**Remark 5.7.** *If $f \in \operatorname{Isom}(\mathbb{H}^n)$ is hyperbolic, then the group $\langle f \rangle \leqslant \operatorname{Isom}(\mathbb{H}^n)$ is discrete.*

**Definition 5.8** (Tube)**.** *A **tube** is a quotient $\mathbb{H}^n / \langle f \rangle$, where $f \in \operatorname{Isom}(\mathbb{H}^n)$ is hyperbolic.*

**Remark 5.9.** *The restriction of the hyperbolic metric to any horosphere is Euclidean, and parabolic maps act on them as Euclidean isometries. Thus, any discrete subgroup of $\operatorname{Isom}(\mathbb{E}^{n-1})$ can be realised as a discrete subgroup of some parabolic stabiliser of $p \in \partial\mathbb{H}^n$.*
*Note however that horospheres are not totally geodesic.*

**Definition 5.10** (Cusp)**.** *A **cusp** is a quotient $\mathbb{H}^n/\Gamma$, where $\Gamma$ is a discrete subgroup of some parabolic stabiliser $\operatorname{Stab}(p)$ with $p \in \partial\mathbb{H}^n$.*
*Cusps look like $M \times \mathbb{R}_{>0}$, where $M$ is a Euclidean manifold.*

**Example 5.11.** *In dimension $3$, in the upper half-space $\mathcal{H}^3$, if $\Gamma \leqslant \operatorname{Stab}(\infty)$ is discrete and torsion-free, then $\Gamma$ is generated by one or two parabolic isometries. We say that the cusp $\mathcal{H}^3/\Gamma$ is of rank $1$ (resp. $2$).*

## 5.3 Nilpotent groups and commutators

**Notation 5.12.** *Given two elements $g, h$ in a group $G$, we denote $[g,h] = ghg^{-1}h^{-1}$.*

**Definition 5.13** (Commutator subgroup)**.** *Let $H, K \trianglelefteq G$. The* ***commutator subgroup*** *is defined by*

$$[H,K] = \langle [h,k]\,,\ h \in H,\ k \in K \rangle\,.$$

*$[H,K]$ is a normal subgroup of $G$ and is contained in $H \cap K$.*

**Definition 5.14** (Nilpotent group)**.** *Let $G$ be a group. The* ***lower central series*** *$(G_n)_{n\geqslant 0}$ of $G$ is defined by $G_0 = G$ and $G_{n+1} = [G_n, G]$ for $n \geqslant 0$.*

*We say that $G$ is* ***nilpotent*** *if $G_n = 0$ for some $n \geqslant 0$.*

**Example 5.15.**

- *Abelian groups are nilpotent.*
- *The Heisenberg group* $\mathrm{Heis}_3 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \leqslant GL_3\mathbb{R}$ *is nilpotent.*

**Proposition 5.16.** *If a nontrivial group $G$ is nilpotent, then $Z(G) \neq 1$.*

*Proof.* There exists $n \geqslant 0$ s.t. $G_n \neq 1$ but $G_{n+1} = 1$. In other words, $[G_n, G] = 1$, so $G_n \subseteq Z(G)$. □

**Lemma 5.17.** *Let $G$ be a group with generating set $S$. Denote by $(G_n)_{n\geqslant 0}$ the lower central series of $G$. Then*

$$G_n = \langle [a_1, [a_2, [\ldots, [a_n, b]]]]\,,\ a_1, \ldots, a_n \in S,\ b \in G \rangle\,.$$

*Proof.* Apply inductively the formula $[a, bc] = [a,b] \cdot [b, [a,c]] \cdot [a,c]$. □

## 5.4 The Zassenhaus Lemma

**Lemma 5.18** (Zassenhaus)**.** *If $G$ is a Lie group, then there exists a neighbourhood $U_Z$ of $e$ in $G$ s.t. any discrete group $\Gamma$ generated by elements of $U_Z$ is nilpotent.*

*We say that $U_Z$ is a* ***Zassenhaus neighbourhood****.*

*Proof.* We prove the lemma in the special case where $G$ is a matrix group.

Consider $\phi = [\cdot,\cdot] : G \times G \to G$. We first compute $\mathrm{d}\phi_{(I,I)}$:

$$\phi\left(I+H, I+K\right) = \left(I+H\right)\left(I+K\right)\left(I+H\right)^{-1}\left(I+K\right)^{-1} = I + HK - KH + o\left(\|H\|^2 + \|K\|^2\right);$$

it follows that $\mathrm{d}\phi_{(I,I)} = 0$. Therefore, by the Mean Value Inequality, there must exist a neighbourhood $U$ of $I$ in $G$ on which $\phi$ is a strict contraction, i.e.

$$\|\phi\left(A_1, B_1\right) - \phi\left(A_2, B_2\right)\| \leqslant \frac{1}{2}\left(\|A_1 - A_2\| + \|B_1 - B_2\|\right),$$

for $A_1, B_1, A_2, B_2 \in U$. It follows that $\|\phi\left(A,B\right) - I\| = \|\phi\left(A,B\right) - \phi\left(A,I\right)\| \leqslant \frac{1}{2}\|B - I\|$, and by symmetry,

$$\|\phi\left(A,B\right) - I\| \leqslant \frac{1}{2}\min\left\{\|A - I\|, \|B - I\|\right\}, \tag{$*$}$$

for $A, B \in U$.

Now let $V$ be any neighbourhood of $I$. Then $(*)$ implies the existence of $k \geqslant 0$ s.t. for any $A_1, \ldots, A_k \in U$, we have

$$[A_1, [A_2, [\ldots, [A_{k-1}, A_k]]]] \in V. \tag{$**$}$$

Finally, assume that $\Gamma$ is discrete and generated by some subset $S \subseteq U$. By discreteness, there is a neighbourhood $V$ of $I$ s.t. $\Gamma \cap V = 1$. By Lemma 5.17, if $\Gamma_k$ is the $k$-th term of the lower central series of $\Gamma$, then

$$\Gamma_k = \langle [A_1, [A_2, [\ldots, [A_k, B]]]]\,,\ A_1, \ldots, A_k \in S,\ B \in \Gamma \rangle\,.$$

Hence, $(**)$ implies that $\Gamma_k$ is generated by elements of $\Gamma \cap V = 1$, so $\Gamma_k = 1$ and $\Gamma$ is nilpotent. □

**Remark 5.19.** *The morale of the Zassenhaus Lemma is that, if you generate a discrete group with elements that are all close to e, then you have no choice but to generate a nilpotent group, i.e. in some sense an algebraically simple group.*

## 5.5 The Kazhdan-Margulis Lemma

**Lemma 5.20** (Kazhdan-Margulis)**.** *For any $n \geqslant 2$, there exists $\mu_n > 0$ s.t. for any discrete subgroup $\Gamma \leqslant \operatorname{Isom}(\mathbb{H}^n)$ and for any $x \in \mathbb{H}^n$, the group*

$$\Gamma_{\mu_n}(x) = \langle \gamma \in \Gamma,\ d(x, \gamma x) < \mu_n \rangle$$

*is virtually nilpotent.*

*Proof.* Write $G = \operatorname{Isom}(\mathbb{H}^n)$, and $K = \operatorname{Stab}(x)$. Note that $K$ is conjugate to $\operatorname{Stab}(0) \cong O(n)$ in the Poincaré ball model, so $K$ is compact. We also fix a Zassenhaus neighbourhood $U_Z$ (c.f. Lemma 5.18). We make the following choices independently of $\Gamma$:

- Let $W \subseteq U_Z$ be a neighbourhood of $e$ s.t. $W = W^{-1}$ and $W^2 = \{w_1 w_2,\ w_1, w_2 \in W\} \subseteq U_Z$.
- Since $K$ is compact, we may choose $V = \{g \in G,\ d(x, gx) < \alpha\}$ a relatively compact neighbourhood of $K$.
- Since $V$ is relatively compact, there are $N_W \geqslant 1$ and $g_1, \dots, g_{N_W} \in G$ s.t.
$$V \subseteq \bigcup_{i=1}^{N_W} g_i W.$$
- We set $V' = \{g \in G,\ d(x, gx) < \mu_n\}$. Hence, ${V'}^{N_W} \subseteq V$ if $\mu_n < \frac{\alpha}{N_W}$.

Now take a discrete group $\Gamma \leqslant \operatorname{Isom}(\mathbb{H}^n)$ and set

$$\begin{aligned} \Gamma_{\mu_n} &= \langle \gamma \in \Gamma,\ d(x, \gamma x) < \mu_n \rangle = \langle \Gamma \cap V' \rangle, \\ \Gamma_{\mu_n}^0 &= \langle \Gamma_{\mu_n} \cap U_Z \rangle. \end{aligned}$$

Note that $\Gamma_{\mu_n}$ is discrete (as a subgroup of $\Gamma$), so the Zassenhaus Lemma (Lemma 5.18) implies that $\Gamma_{\mu_n}^0$ is nilpotent. We will prove that

$$\left[\Gamma_{\mu_n} : \Gamma_{\mu_n}^0\right] \leqslant N_W.$$

To do this, consider the set $S_k$ of elements of $\Gamma_{\mu_n}$ that are products of at most $k$ generators in $\Gamma \cap V'$ and write $\varphi(k) = \left|S_k / \Gamma_{\mu_n}^0\right|$. Remark the following:

- For $k \geqslant 1$, $\varphi(k) \leqslant \varphi(k+1)$ since $S_k \subseteq S_{k+1}$.
- If $\varphi(k_0) = \varphi(k_0 + 1)$ for some $k_0 \geqslant 1$, then $\varphi(k) = \varphi(k_0)$ for all $k \geqslant k_0$.

  Indeed, if $\varphi(k_0) = \varphi(k_0 + 1)$, then any coset of the form $v_1 \cdots v_{k_0+1} \Gamma_{\mu_n}^0$ can be written as $v_1' \cdots v_{k_0}' \Gamma_{\mu_n}^0$, so any coset of the form $v_1 \cdots v_{k_0+2} \Gamma_{\mu_n}^0$ can be written as $v_1 v_2' \cdots v_{k_0+1}' \Gamma_{\mu_n}^0$ and therefore $\varphi(k_0 + 1) = \varphi(k_0 + 2)$; it follows by induction that $\varphi(k) = \varphi(k_0)$ for $k \geqslant k_0$.
- If $(\varphi(k))_{k \geqslant 1}$ is constant from rank $k_0$, then any $\Gamma_{\mu_n}^0$-coset in $\Gamma_{\mu_n}$ is represented by a product of at most $k_0$ generators, and there are at most $\varphi(k_0)$ such cosets. In particular, $\left[\Gamma_{\mu_n} : \Gamma_{\mu_n}^0\right] \leqslant \varphi(k_0)$.

The above arguments imply that it suffices to find some $k \leqslant N_W$ s.t. $\varphi(k) = \varphi(k+1)$. We may assume that $\Gamma_{\mu_n} \neq \Gamma_{\mu_n}^0$. In this case, we have:

- $\varphi(1) \geqslant 2$.

  Indeed, if $\varphi(1) = 1$, then $S_1 / \Gamma_{\mu_n}^0 = \left\{\Gamma_{\mu_n}^0\right\}$, so $\Gamma \cap V' \subseteq \Gamma_{\mu_n}^0$ and therefore $\Gamma_{\mu_n} = \Gamma_{\mu_n}^0$.

- $\varphi(N_W) \leqslant N_W$.

  Indeed, let $\gamma_1, \ldots, \gamma_k$ be representatives of distinct $\Gamma^0_{\mu_n}$-cosets in $\Gamma_{\mu_n}$, with each $\gamma_i$ a product of a most $N_W$ generators. Then we have
  $$\gamma_1, \ldots, \gamma_k \in V'^{N_W} \subseteq V \subseteq \bigcup_{i=1}^{N_W} g_i W.$$
  If $k > N_W$, then (by the pigeon-hole principle) there must exist $i \neq j$ s.t. $\gamma_i, \gamma_j \in g_\ell W$. Therefore,
  $$\gamma_j^{-1}\gamma_i \in W^2 \cap \Gamma_{\mu_n} \subseteq U_Z \cap \Gamma_{\mu_n} \subseteq \Gamma^0_{\mu_n},$$
  contradicting the fact that $\gamma_i \Gamma^0_{\mu_n} \neq \gamma_j \Gamma^0_{\mu_n}$.

- We have $2 \leqslant \varphi(1) \leqslant \varphi(2) \leqslant \cdots \leqslant \varphi(N_W) \leqslant N_W$. By the pigeon-hole principle, there must be some $k \leqslant N_W$ s.t. $\varphi(k) = \varphi(k+1)$.

This proves that $\varphi$ is constant from rank $N_W$, so $\left[\Gamma_{\mu_n} : \Gamma^0_{\mu_n}\right] \leqslant \varphi(N_W) \leqslant N_W$. □

**Remark 5.21.** (i) *In fact, we have proved the existence of a constant $N$ depending only on $n$ s.t. $\Gamma_{\mu_n}$ has a nilpotent subgroup of index at most $N$.*

(ii) *The Kazhdan-Margulis Lemma remains true with the same proof if* Isom$(\mathbb{H}^n)$ *is replaced by any Lie group.*

## 5.6 Elementary groups

**Definition 5.22** (Elementary group)**.** *A nontrivial discrete group $\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^n)$ is called* ***elementary*** *if it has a finite orbit in $\overline{\mathbb{H}}^n$.*

**Proposition 5.23.** *Let $\Gamma \leqslant \mathrm{Isom}(\mathbb{H}^n)$ be discrete.*

(i) *If $\Gamma$ is elementary and acts freely on $\mathbb{H}^n$, then we are in one of the following situations:*

  (a) *$\Gamma = \langle g \rangle$, where $g$ is hyperbolic,*

  (b) *$\Gamma = \langle g_i,\ i \in I \rangle$, where the $g_i$s are parabolic with a common fixed point in $\partial\mathbb{H}^n$.*

(ii) *If $\Gamma$ is virtually elementary, then $\Gamma$ is elementary.*

(iii) *If $\Gamma$ is discrete, acts freely on $\mathbb{H}^n$, and is virtually nilpotent, then $\Gamma$ is elementary.*

*Proof.* (i) First show that, if $\Gamma$ has a finite orbit in $\mathbb{H}^n$, then $\Gamma$ has a fixed point in $\mathbb{H}^n$ (this fixed point can be constructed by considering the barycentre of a finite orbit in the hyperboloid model); this is impossible because $\Gamma$ acts freely. Therefore, $\Gamma$ has a finite orbit $F$ in $\partial\mathbb{H}^n$. If $\Gamma$ has a unique global fixed point in $\partial\mathbb{H}^n$, prove that we are in the second situation. Otherwise, show that the union of all finite orbits of $\Gamma$ contains exactly two points in $\partial\mathbb{H}^n$, and these two points are the endpoints of the axis of a hyperbolic map generating $\Gamma$, so we are in the first situation.

(ii) Let $H \leqslant_{fi} \Gamma$ be a finite index elementary subgroup of $\Gamma$. By definition, there is a point $p \in \overline{\mathbb{H}}^n$ s.t. $H \cdot p$ is finite. Now $|\Gamma \cdot p| \leqslant [\Gamma : H] \cdot |H \cdot p|$, so $\Gamma \cdot p$ is finite.

(iii) Let $H \leqslant_{fi} \Gamma$ be a finite index nilpotent subgroup of $\Gamma$. Since $\Gamma$ acts freely, it is infinite and therefore $H \neq 1$. Therefore, by Proposition 5.16, $Z(H) \neq 1$. Now if $\gamma_1 \in Z(H) \smallsetminus \{1\}$ and $\gamma_2 \in H \smallsetminus \{1\}$, then the fact that $\gamma_1\gamma_2 = \gamma_2\gamma_1$ implies that $\gamma_1$ preserves the fixed points of $\gamma_2$. But $\gamma_2$ has one or two fixed points, and $\gamma_1$ cannot swap them (because it acts freely on $\mathbb{H}^n$), so it must fix them. By symmetry, $\gamma_2$ fixes the fixed points of $\gamma_1$, so $\gamma_1, \gamma_2$ have the same fixed points. Since $Z(H) \neq 1$, this implies that all elements of $H \smallsetminus \{1\}$ have the same fixed points, so $H$ is elementary. Therefore, $\Gamma$ is virtually elementary and hence elementary by (ii). □

**Theorem 5.24.** *For any $n \geqslant 2$, there exists $\mu_n > 0$ s.t. for any discrete subgroup $\Gamma \leqslant \operatorname{Isom}(\mathbb{H}^n)$ acting freely on $\mathbb{H}^n$, and for any $x \in \mathbb{H}^n$, the group*

$$\Gamma_{\mu_n}(x) = \langle \gamma \in \Gamma,\ d(x, \gamma x) < \mu_n \rangle$$

*is elementary.*

## 5.7 Injectivity radius

**Definition 5.25** (Injectivity radius)**.** *Let $M$ be a Riemannian manifold.*

- *The **injectivity radius at** $p$ is*

$$\operatorname{inj}_p(M) = \sup \Big\{ r > 0,\ \exp_p : B(0,r) \subseteq T_pM \to M \text{ is a diffeomorphism onto its image} \Big\}.$$

  *Note that $\operatorname{inj}_p(M) > 0$ by Proposition 4.11.*

- *The **injectivity radius of** $M$ is*

$$\operatorname{inj}(M) = \inf_{p \in M} \operatorname{inj}_p(M).$$

**Proposition 5.26.** *The map $p \mapsto \operatorname{inj}_p(M)$ is continuous.*
*In particular, if $M$ is compact, then $\operatorname{inj}(M) > 0$.*

**Lemma 5.27.** *Let $\gamma$ be a geodesic loop based at $p$ on $M$. Then*

$$\operatorname{inj}_p(M) \leqslant \frac{\ell(\gamma)}{2}.$$

*Proof.* We have $\gamma(t) = \exp_p(tu)$ for some $u \in T_pM$ with $\|u\| = \ell(\gamma)$. Since $\gamma(0) = \gamma(1)$, we have $\exp_p\left(\frac{u}{2}\right) = \exp_p\left(-\frac{u}{2}\right)$, so $\exp_p$ is not injective on $B\left(0, \frac{\ell(\gamma)}{2}\right)$. □

**Example 5.28.**
(i) $\operatorname{inj}(\mathbb{R}^n) = \operatorname{inj}(\mathbb{H}^n) = +\infty$.

(ii) $\operatorname{inj}(\mathbb{S}^n) = \pi$.

(iii) *If $T = \mathbb{H}^n / \langle f \rangle$ is a tube, where $f$ is a hyperbolic map with translation length $\ell$, then $\operatorname{inj}(T) = \ell$.*

(iv) *If $C$ is a cusp, then $\operatorname{inj}(C) = 0$.*

**Proposition 5.29.** *Let $M$ be a Riemannian manifold with $r = \operatorname{inj}(M) > 0$. Then any loop $c$ with $\ell(c) < 2r$ is contractible.*

*Proof.* Fix $\ell(c) < 2\rho < 2r$. First note that, if $c$ is a loop based at $p$ and not contained in $B_p(\rho) \subseteq M$, then $c$ has a subpath $c'$ contained in $B_p(\rho)$ and intersecting $\partial B_p(\rho)$ at two points $q_1, q_2$, therefore

$$\ell(c) \geqslant \ell(c') \geqslant d(q_1, p) + d(p, q_2) = 2\rho.$$

Hence, any loop $c$ based at $p$ of length $< 2\rho$ must be contained in $B_p(\rho)$. But $\rho < r = \operatorname{inj}(M) \leqslant \operatorname{inj}_p(M)$, so $B_p(\rho)$ is diffeomorphic to an open ball in $\mathbb{R}^n$; in particular, $B_p(\rho)$ is simply connected so $c$ is contractible. □

**Proposition 5.30.** *Let $M$ be a complete hyperbolic manifold and $x \in M$. Write $\Gamma = \pi_1 M \leqslant \operatorname{Isom}(\mathbb{H}^n)$ and fix a lift $\widetilde{x}$ of $x$ in $\mathbb{H}^n$. Then*

$$\operatorname{inj}_x(M) = \frac{1}{2} \inf_{\gamma \in \Gamma \smallsetminus \{\mathrm{id}\}} d(\widetilde{x}, \gamma \widetilde{x}).$$

*Proof.* ($\leqslant$) If $\gamma \in \Gamma \smallsetminus \{\mathrm{id}\}$, then the geodesic segment $[\widetilde{x}, \gamma\widetilde{x}]$ projects to a closed geodesic of length $d(\widetilde{x}, \gamma\widetilde{x})$, so $\mathrm{inj}_x(M) \leqslant \frac{1}{2} d(\widetilde{x}, \gamma\widetilde{x})$ by Lemma 5.27.

($\geqslant$) If $r < \frac{1}{2}\inf_{\gamma\in\Gamma\smallsetminus\{\mathrm{id}\}} d(\widetilde{x}, \gamma\widetilde{x})$, then the open balls $\{\gamma B_{\widetilde{x}}(r)\}_{\gamma\in\Gamma}$ in $\mathbb{H}^n$ are pairwise disjoint, so $B_x(r) \subseteq M$ is isometric to $B_{\widetilde{x}}(r)$ and therefore $\mathrm{inj}_x(M) \geqslant r$. □

**Definition 5.31** (Minimum displacement)**.** *Given $f \in \mathrm{Isom}(\mathbb{H}^n)$, the* ***minimum displacement*** *of $f$ is*

$$d(f) = \inf_{x\in\mathbb{H}^n} d(x, f(x)).$$

(i) *If $f$ is hyperbolic, then $d(f)$ is the translation length of $f$, and the points realising $d(f)$ are exactly the points of the axis of $f$.*

(ii) *If $f$ is parabolic, then $d(f) = 0$ and the infimum is not attained.*

(iii) *if $f$ is elliptic, then $d(f) = 0$, and the points realising $d(f)$ are exactly the fixed points of $f$.*

**Corollary 5.32.** *If $M$ is a compact hyperbolic manifold, then $\Gamma = \pi_1 M \leqslant \mathrm{Isom}(\mathbb{H}^n)$ does not contain any parabolic element.*

*Proof.* Since $M$ is compact, we have $\mathrm{inj}(M) > 0$. But

$$\mathrm{inj}(M) = \frac{1}{2} \inf_{\gamma\in\Gamma\smallsetminus\{\mathrm{id}\}} d(\gamma)$$

by Proposition 5.30, so all elements of $\Gamma \smallsetminus \{\mathrm{id}\}$ are hyperbolic. □

## 5.8 Thin-thick decompositions

**Definition 5.33** (Thin-thick decomposition)**.** *Let $M$ be a Riemannian manifold and $\varepsilon > 0$.*

- *The **$\varepsilon$-thin part** of $M$ if $M_{(0,\varepsilon]} = \left\{x \in M,\ \mathrm{inj}_x(M) \leqslant \frac{\varepsilon}{2}\right\}$.*
- *The **$\varepsilon$-thick part** of $M$ if $M_{[\varepsilon,\infty)} = \left\{x \in M,\ \mathrm{inj}_x(M) \geqslant \frac{\varepsilon}{2}\right\}$.*

*The **thin-thick decomposition** of $M$ is*

$$M = M_{(0,\mu_n]} \cup M_{[\mu_n,\infty)},$$

*where $\mu_n$ is the constant of the Kazhdan-Margulis Lemma (Lemma 5.20).*

**Proposition 5.34.** *Let $M$ be a complete orientable hyperbolic $n$-manifold with $n \leqslant 3$. Then the $\mu_n$-thin part of $M$ is a disjoint union of truncated cusps and tubes.*

*In fact, this result remains mostly true in higher dimensions, but the shapes of cusps become more complicated.*

*Proof.* $M_{(0,\mu_n]}$ is the image under the covering map $\mathbb{H}^n \to M$ of $S = \bigcup_{\gamma\in\Gamma} S_\gamma(\mu_n)$ with $\Gamma = \pi_1 M \leqslant \mathrm{Isom}(\mathbb{H}^n)$ and $S_\gamma(\mu_n) = \{x \in \mathbb{H}^n,\ d(x, \gamma x) \leqslant \mu_n\}$. Note that if $S_{\gamma_1}(x) \cap S_{\gamma_2}(x) \neq \varnothing$, then there exists $x \in \mathbb{H}^n$ s.t. $\gamma_1, \gamma_2 \in \Gamma_{\mu_n}$ with the notations of the Kazhdan-Margulis Lemma (Lemma 5.20). Therefore, the group $\langle\gamma_1, \gamma_2\rangle$ is elementary by Theorem 5.24. Hence, if $p$ is the image of $x$ under $\mathbb{H}^n \to M$, then the connected component of $p$ in $M$ is a tube in the hyperbolic case, or a cusp in the parabolic case. □

# 6 What next?

**Definition 6.1** (Curvature of a surface). *If $(\Sigma, g)$ is a Riemannian surface and $p \in \Sigma$, then the curvature of $\Sigma$ at $p$ is*

$$k_p = 3 \lim_{r \to 0} \left( \frac{2\pi r - \ell\left(C_p(r)\right)}{\pi r^3} \right),$$

*where $C_p(r)$ is the circle of centre $p$ and radius $r$ in $\Sigma$.*

**Example 6.2.** (i) *If $\Sigma = \mathbb{R}^2$, then $\ell\left(C_p(r)\right) = 2\pi r$ and $k_p = 0$ for all $p \in \Sigma$.*

(ii) *If $\Sigma = \mathbb{H}^2$, then $\ell\left(C_p(r)\right) = 2\pi \sinh r$ and $k_p = -1$ for all $p \in \Sigma$.*

(iii) *If $\Sigma = \mathbb{S}^2$, then $\ell\left(C_p(r)\right) = 2\pi \sin r$ and $k_p = +1$ for all $p \in \Sigma$.*

**Theorem 6.3** (Gauß-Bonnet). *Let $(\Sigma, g)$ be a Riemannian surface. Then*

$$\int_\Sigma k_p = 2\pi\chi(\Sigma).$$

*This formula relates the geometry of $\Sigma$ to its topology. Conversely, given a surface $\Sigma$ with appropriate topology, can we endow $\Sigma$ with a nice geometric structure?*

**Theorem 6.4.** *If $\Sigma$ is a closed surface with $\chi(\Sigma) < 0$ (i.e. with genus at least 2), then there exists a discrete subgroup $\Gamma \leqslant \mathrm{Isom}\left(\mathbb{H}^2\right)$ s.t. $\Sigma$ is homeomorphic to $\mathbb{H}^2/\Gamma$.*

*Proof.* Let $g = 1 - \frac{1}{2}\chi(\Sigma)$ be the genus of $\Sigma$. Construct $\Gamma$ using the Poincaré Theorem as a discrete group whose fundamental domain is a regular $4g$-gon with interior angles $\frac{\pi}{2g}$, with usual side-pairings. □

**Theorem 6.5** (Thurston's Hyperbolisation Theorem). *Let $M$ be a closed 3-manifold. Assume that $M$ is*

- *Irreducible: any 2-sphere bounds a 3-ball,*
- *Atoroidal: $\pi_1 M$ does not contain an embedded copy of $\mathbb{Z}^2$,*
- *Large enough: c.f. [11] for a complete definition.*

*Then $M$ has a complete hyperbolic metric of finite volume.*

*Proof.* See [11] or [7]. □

**Theorem 6.6** (Mostow's Rigidity Theorem). *Let $M_1, M_2$ be two $n$-dimensional ($n \geqslant 3$) compact connected oriented hyperbolic manifolds. Write $\Gamma_i = \pi_1\left(M_i\right) \leqslant \mathrm{Isom}\left(\mathbb{H}^n\right)$. If $\Gamma_1 \cong \Gamma_2$, then $M_1, M_2$ are isometric (and in fact, $\Gamma_1, \Gamma_2$ are conjugate in $\mathrm{Isom}\left(\mathbb{H}^n\right)$).*

*Proof.* See [9]. □

**Remark 6.7.** *Note that Mostow's Rigidity Theorem contrasts with the situation in dimension 2: the fundamental group of a hyperbolic surface may have many (non-conjugate) embeddings into $\mathrm{Isom}\left(\mathbb{H}^2\right)$, see for example Section 2.10.*

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