Algebraic Topology

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Michaelmas 2019
Part III course

Contents

Notation 0.1. *Unless otherwise specified, all spaces are topological and all maps are continuous.*

1 Homotopy

1.1 Homotopy of maps and homotopy equivalence

Definition 1.1 (Homotopy)**.** *Let $f_0, f_1 : X \to Y$ be two continuous maps between topological spaces. We say that f_0 is* homotopic *to f_1, and we write $f_0 \sim f_1$, if there exists a continuous map $F : X \times I \to Y$ (where $I = [0,1]$) such that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$. In other words, $(F(\cdot, t))_{t \in I}$ is a path from f_0 to f_1 in the space* $\mathrm{Map}(X,Y)$*.*

Notation 1.2. *If X is a topological space and c is an element of another topological space Y, we shall denote by c_X the constant map $X \to Y$ given by $x \mapsto c$.*

Example 1.3. (i) *The maps $\mathrm{id}_{\mathbb{R}^n}, 0_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ are homotopic* via $(x,t) \mapsto tx$.

(ii) *The maps $\mathrm{id}_{\mathbb{S}^1}, a : \mathbb{S}^1 \to \mathbb{S}^1$, where $a : z \mapsto -z$ is the antipodal map, are homotopic* via $(z,t) \mapsto e^{i\pi t} z$.

(iii) *However, the maps $\mathrm{id}_{\mathbb{S}^2}, a : \mathbb{S}^2 \to \mathbb{S}^2$ are not homotopic (we shall prove this fact later).*

(iv) *The maps $c_{\mathbb{S}^1}, j : \mathbb{S}^1 \to \mathbb{S}^2$, where $c = (0,0,1) \in \mathbb{S}^2$ and $j : (x,y) \mapsto (x,y,0)$, are homotopic* via $(x,y,t) \mapsto \left(tx, ty, \sqrt{1-t^2}\right)$.

(v) *Let $\mathbb{D}^n = \{v \in \mathbb{R}^n,\ \|v\| \leqslant 1\}$ and consider $\mathbb{S}^{n-1} \subseteq \mathbb{D}^n$. Then a map $f : \mathbb{S}^{n-1} \to Y$ extends to $\mathbb{D}^n$ if and only if f is homotopic to a constant map.*

Lemma 1.4. *Homotopy is an equivalence relation on* $\mathrm{Map}(X,Y)$*. Hence, we can define $[X,Y] = \mathrm{Map}(X,Y)/\sim$. The image of an element $f \in \mathrm{Map}(X,Y)$ in $[X,Y]$ will be denoted by $[f]$.*

Lemma 1.5. *If $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ with $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Corollary 1.6. *Any map $f : X \to \mathbb{R}^n$ is homotopic to $0_X : X \to \mathbb{R}^n$.*

Proof. We know by Example 1.3 that $\mathrm{id}_{\mathbb{R}^n} \sim 0_{\mathbb{R}^n}$, therefore $f = \mathrm{id}_{\mathbb{R}^n} \circ f \sim 0_{\mathbb{R}^n} \circ f = 0_X$. ☐

Definition 1.7 (Contractible space)**.** *Let X be a topological space. The following two assertions are equivalent:*

(i) *There exists $c \in X$ such that $\mathrm{id}_X \sim c_X$.*

(ii) *$[Z,X]$ has only one element for all spaces Z.*

If these conditions are satisfied, X is said to be contractible*.*

Proof. (i) ⇒ (ii) If $g \in \mathrm{Map}(Z,X)$, then $g = \mathrm{id}_X \circ g \sim c_X \circ g = c_Z$. (ii) ⇒ (i) The set $[X,X]$ has only one element. ☐

Definition 1.8 (Homotopy equivalence)**.** *Two spaces X and Y are said to be* homotopy equivalent *(which we denote by $X \sim Y$) if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.*

Example 1.9. *A space is contractible* iff *it is homotopy equivalent to a point.*

Proof. The map $f : \{*\} \to X$ being a homotopy equivalence with $c = f(*)$ means that $c_X \sim \mathrm{id}_X$, i.e. X is contractible. ☐

Lemma 1.10. *If $X_1 \sim X_2$ and $Y_1 \sim Y_2$, then there is a bijection between $[X_1, Y_1]$ and $[X_2, Y_2]$.*

1.2 Homotopy groups

Definition 1.11 (Map of pairs)**.** *We shall write $f : (X, A) \to (Y, B)$ to mean that f is a map $X \to Y$, $A \subseteq X$, $B \subseteq Y$ and $f(A) \subseteq B$.*

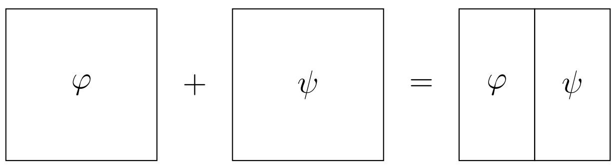
If $f_0, f_1 : (X, A) \to (Y, B)$, we say that $f_0 \sim f_1$ if there exists a continuous map $F : (X \times I, A \times I) \to (Y, B)$ with $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$. This defines the set $[(X, A), (Y, B)] = \mathrm{Map}\left((X, A), (Y, B)\right) / \sim$.

Definition 1.12 (Homotopy groups)**.** *Let X be a space, $p \in X$. We write $* = (-1, 0, \ldots, 0) \in \mathbb{S}^n$, and we define:*

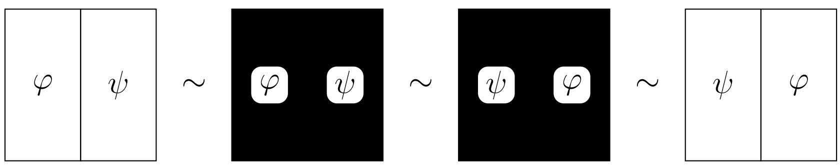
$$\pi_n(X, p) = \left[\left(\mathbb{S}^n, *\right), (X, p)\right] = \left[\left(\mathbb{D}^n, \mathbb{S}^{n-1}\right), (X, p)\right] = \left[\left(I^n, \partial I^n\right), (X, p)\right].$$

*$\pi_n(X, p)$ is called the n-*th homotopy group *of X at p.*

Proposition 1.13. (i) *If $n \geqslant 1$, then $\pi_n(X, p)$ is a group, where composition is defined by the following diagram (viewing $\pi_n(X, p)$ as $\left[\left(I^n, \partial I^n\right), (X, p)\right]$):*



Moreover, if $n \geqslant 2$, $\pi_n(X, p)$ is abelian, as shown by the following diagram, where black filling represents the constant map $x \mapsto p$:



(ii) *If we have a map $f : (X, p) \to (Y, q)$, then it induces maps $f_* : \pi_n(X, p) \to \pi_n(Y, q)$ defined by $f_*\left([\gamma]\right) = [f \circ \gamma]$.*

(iii) *The homotopy groups define functors* $\mathbf{Top}_*$ $\to$ **Gp** *from the category of pointed topological spaces to the category of groups.*

(iv) *The homotopy groups are* homotopy invariant*: if $f \sim g$, then $f_* = g_*$.*

Example 1.14. *Here are the first homotopy groups of $\mathbb{S}^1$ and $\mathbb{S}^2$:*

	π_1	π_2	π_3	π_4	π_5	$\cdots$
$\mathbb{S}^1$	$\mathbb{Z}$	0	0	0	0	$\cdots$
$\mathbb{S}^2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdots$

We see that the homotopy groups have an undesirable behaviour for large values of n.

2 Homology

Remark 2.1. *Our goal is to define functors H_n : **Top** $\to$ **AbGp** from the category of topological spaces to the category of abelian groups satisfying the following two conditions:*

(i) Homotopy invariance*: if $f \sim g$, then $f_* = g_*$.*

(ii) Dimension axiom*: $H_n(X) = 0$ if $n > \dim X$.*

2.1 Chain complexes

Definition 2.2 (Chain complex)**.** *Let R be a commutative ring. A* chain complex (C, d) *over R consists of:*

(i) *R-modules C_i for $i \in \mathbb{Z}$,*

(ii) *Homomorphisms $d_i : C_i \to C_{i-1}$,*

satisfying $d_i \circ d_{i+1} = 0$ for all i. We write:

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots$$

We shall denote $C_* = \bigoplus_{i \in \mathbb{Z}} C_i$.

Definition 2.3 (Simplex)**.** *The n-*dimensional simplex *is defined by:*

$$\Delta^n = \left\{ v = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1},\ \sum_{i=0}^{n} v_i = 1 \text{ and } \forall i \in \{0, \ldots, n\},\ v_i \geqslant 0 \right\}.$$

Note that $\Delta^{-1} = \varnothing$.

- *If $I = \{i_0 < i_1 < \cdots < i_k\} \subseteq \{0, \ldots, n\}$, the k-dimensional I-*face *of Δ^n is defined by $f_I = \{v \in \Delta^n,\ \forall i \notin I,\ v_i = 0\}$.*
- *Associated to the face f_I, there is a* face map *$F_I : \Delta^k \to f_I$ given by*

$$(F_I(w))_i = \begin{cases} 0 & \text{if } i \notin I \\ w_j & \text{if } i = i_j \end{cases}.$$

Definition 2.4 (Reduced chain complex of the simplex)**.** *The* reduced chain complex $\widetilde{S}(\Delta^n)$ over $\mathbb{Z}$ of the simplex Δ^n *is defined as follows: $\widetilde{S}_k(\Delta^n)$ is the free abelian group with basis $(f_I)_{|I|=k+1}$ and $d_k : \widetilde{S}_k(\Delta^n) \to \widetilde{S}_{k-1}(\Delta^n)$ is given by*

$$d_k(f_I) = \sum_{j=0}^{k} (-1)^j f_{I \setminus \{i_j\}},$$

where $I = \{i_0 < i_1 < \cdots < i_k\}$.

This is indeed a chain complex, i.e. $d_k \circ d_{k+1} = 0$.

Note that $\widetilde{S}_k(\Delta^n) = \mathbb{Z}$ if $k < 0$.

Proof. It suffices to prove that $d_k \circ d_{k+1}(f_I) = 0$ for $|I| = k+2$. But $d_k \circ d_{k+1}(f_I)$ is a sum of terms of the form $f_{I \setminus \{i_j, i_{j'}\}}$, with $j < j'$, and the coefficient of $f_{I \setminus \{i_j, i_{j'}\}}$ is $(-1)^j(-1)^{j'-1} + (-1)^{j'}(-1)^j = 0$. □

Definition 2.5 (Homology groups of a chain complex)**.** *If (C, d) is a chain complex, its i-*th homology group *is the R-module*

$$H_i(C) = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i+1}}.$$

If $x \in \operatorname{Ker} d_i$, we denote by $[x]$ the image of x in $H_i(C)$. We write $H_*(C) = \bigoplus_{i \in \mathbb{Z}} H_i(C)$.

Example 2.6. $H_*\left(\widetilde{S}\left(\Delta^2\right)\right) = 0$.

Definition 2.7 (Unreduced chain complex of the simplex)**.** *The* unreduced chain complex $S\left(\Delta^n\right)$ of the simplex Δ^n *is defined by*

$$S_k\left(\Delta^n\right) = \begin{cases} \widetilde{S}_k\left(\Delta^n\right) & \text{if } k \geqslant 0 \\ 0 & \text{if } k < 0 \end{cases}.$$

Example 2.8. $H_*\left(S\left(\Delta^2\right)\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}.$

Definition 2.9 (Chain map)**.** *If (C,d) and (C',d') are chain complexes over R, a* chain map f : $(C,d) \to (C',d')$ *is a sequence of homomorphisms $f_i : C_i \to C'_i$ such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+2} \arrow[r, "d_{i+2}"] \arrow[d, "f_{i+2}"'] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"'] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{i+2} \arrow[r, "d'_{i+2}"] & C'_{i+1} \arrow[r, "d'_{i+1}"] & C'_i \arrow[r, "d'_i"] & C'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

In other words, $d'f = fd$.

In that case, we have $f(\operatorname{Ker} d) \subseteq \operatorname{Ker} d'$ and $f(\operatorname{Im} d) \subseteq \operatorname{Im} d'$, so there is a well-defined map $f_* : H_*(C) \to H_*(C')$.

Example 2.10. *If f_I is a face of Δ^n with $I = \{i_0 < i_1 < \cdots < i_k\}$, then there is a chain map $\varphi_I : \widetilde{S}\left(\Delta^k\right) \to \widetilde{S}\left(\Delta^n\right)$ given by $\varphi_I\left(f_J\right) = f_{\varphi(J)}$, where $\varphi(j) = i_j$ for all $j \in \{0,\ldots,k\}$.*

Proposition 2.11. *Homology groups define functors $\mathbf{ChCmplx}_R \to \mathbf{Mod}_R$ from the category of chain complexes over R to the category of R-modules.*

2.2 The singular chain complex

Definition 2.12 (Singular chain complex)**.** *Let X be a topological space. A* singular k-simplex *in X is a map $\sigma : \Delta^k \to X$.*

The singular chain complex $C(X)$ *is defined as follows: $C_k(X)$ is the free abelian group generated by the set of singular k-simplices in X, for all $k \geqslant 0$. In other words, its elements are finite sums $\sum_{i=1}^r a_i\sigma_i$, where $a_i \in \mathbb{Z}$ and $\sigma_i : \Delta^k \to X$. For $k < 0$, $C_k(X) = 0$. The boundary map is defined by:*

$$d\sigma = \sum_{j=0}^k (-1)^j \sigma \circ F_{\{0,\ldots,k\}\setminus\{j\}},$$

where $F_{\{0,\ldots,k\}\setminus\{j\}}$ is the face map defined in Definition 2.3.

This is indeed a chain complex, i.e. $d^2 = 0$.

Proof. To show that $d^2 = 0$, consider for each $\sigma : \Delta^k \to X$ the homomorphism $\varphi_\sigma : S_*\left(\Delta^k\right) \to C_*\left(X\right)$ defined by $\varphi_\sigma\left(f_I\right) = \sigma \circ F_I$. By definition of d, we have $d \circ \varphi_\sigma = \varphi_\sigma \circ d$. Therefore

$$d^2(\sigma) = d^2\left(\sigma \circ F_{\{0,\ldots,k\}}\right) = d^2\left(\varphi_\sigma\left(f_{\{0,\ldots,k\}}\right)\right) = \varphi_\sigma\left(d^2\left(f_{\{0,\ldots,k\}}\right)\right) = 0,$$

since $d^2 = 0$ in $S\left(\Delta^k\right)$. $\square$

Definition 2.13 (Reduced singular chain complex)**.** *Let X be a topological space. We define the* reduced singular chain complex $\widetilde{C}(X)$ *as follows: for $k \geqslant 0$, $\widetilde{C}_k(X) = C_k(X)$, $\widetilde{C}_{-1}(X) = \langle\sigma_\varnothing\rangle \simeq \mathbb{Z}$ and $\widetilde{C}_k(X) = 0$ for $k < -1$. The boundary operator d is the same as in $C(X)$ for $k \neq 0$, and we set $d\sigma = \sigma_\varnothing \in \widetilde{C}_{-1}(X)$ for all $\sigma : \Delta^0 \to X$.*

Definition 2.14 (Singular homology groups)**.** *If X is a topological space, we define:*

(i) *The n-th* singular homology group $H_n(X) = H_n\left(C(X)\right)$,

(ii) *The n-th* reduced singular homology group $\widetilde{H}_n(X) = H_n\left(\widetilde{C}(X)\right)$.

Proposition 2.15. *If $f : X \to Y$ is a map of topological spaces, then it induces a chain map $f_\sharp : C(X) \to C(Y)$ given by $f_\sharp(\sigma) = f \circ \sigma$ for $\sigma : \Delta^k \to X$.*

This defines a functor $\mathbf{Top} \to \mathbf{ChCmplx}_{\mathbb{Z}}$.

Corollary 2.16. *Singular homology groups define functors* $\mathbf{Top} \to \mathbf{AbGp}$.

If $f : X \to Y$ is a map of topological spaces, we shall denote by $f_* : H_*(X) \to H_*(Y)$ the induced map.

Proposition 2.17. (i) *If X is path-connected, then $H_0(X) \simeq \mathbb{Z}$.*

(ii) $H_*\left(\{p\}\right) = \begin{cases} \mathbb{Z} & \textit{if } * = 0 \\ 0 & \textit{otherwise} \end{cases}$.

(iii) $\widetilde{H}_*\left(\{p\}\right) = 0$.

(iv) *Let $\pi_0(X)$ be the set of path-components of X. Then*

$$H_*(X) = \bigoplus_{P \in \pi_0(X)} H_*(P).$$

Proof. (i) We have $\operatorname{Ker} d_0 = C_0(X) = \bigoplus_{p \in X} \mathbb{Z}\sigma_p$, where $\sigma_p : \Delta^0 \to X$ is the constant map equal to p. Now $\operatorname{Im} d_1 = \langle \sigma_p - \sigma_q,\ p, q \in X \rangle$, so $H_0 = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \simeq \mathbb{Z}$.

(ii) For every $n \geqslant 0$, there is a unique map $\sigma_n : \Delta^n \to \{p\}$, and it satisfies:

$$d\sigma_n = \sum_{j=0}^{n} (-1)^j \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even and } n > 0 \\ 0 & \text{if } n \text{ is odd or } n = 0 \end{cases}.$$

Thus, $\operatorname{Ker} d = \langle \sigma_0, \sigma_1, \sigma_3, \sigma_5, \dots \rangle$ and $\operatorname{Im} d = \langle \sigma_1, \sigma_3, \sigma_5, \dots \rangle$, and $\operatorname{Ker} d / \operatorname{Im} d = \langle [\sigma_0] \rangle$.

(iii) The argument for (ii) remains valid, with the difference that $d\sigma_0 = \sigma_\varnothing$ and $d\sigma_\varnothing = 0$, so $\operatorname{Ker} d = \langle \sigma_\varnothing, \sigma_1, \sigma_3, \dots \rangle = \operatorname{Im} d$.

(iv) For $P \in \pi_0(X)$, denote by $\iota_P : P \to X$ the inclusion map. We have an injective map:

$$j = \sum_{P \in \pi_0(X)} (\iota_P)_\sharp : \bigoplus_{P \in \pi_0(X)} C_*(P) \longrightarrow C_*(X).$$

Since Δ^k is path connected, we know that any $\sigma : \Delta^k \to X$ must be in $\operatorname{Im} j$, so j is an isomorphism. This proves that $C_*(X) \simeq \bigoplus_{P \in \pi_0(X)} C_*(P)$, and we now conclude using the following remark. □

Remark 2.18. *If $(C^\alpha, d^\alpha)_{\alpha \in A}$ are chain complexes, then we have a new complex $(\bigoplus_{\alpha \in A} C^\alpha, \sum_{\alpha \in A} d^\alpha)$, and it satisfies:*

$$H_*\left(\bigoplus_{\alpha \in A} C^\alpha\right) = \bigoplus_{\alpha \in A} H_*\left(C_\alpha\right).$$

2.3 Homotopy invariance

Remark 2.19. *Our goal in this section is to show that, given two maps $g_0, g_1 : X \to Y$, $g_0 \sim g_1$ implies $g_{0*} = g_{1*} : H_*(X) \to H_*(Y)$.*

Definition 2.20 (Chain homotopy)**.** *We say that two chain maps $g_0, g_1 : (C, d) \to (C', d')$ are* chain homotopic, *and we write $g_0 \sim g_1$, if there exists a homomorphism $h : C_* \to C'_{*+1}$ (i.e. with $h(C_i) \subseteq C'_{i+1}$) s.t. $d'h + hd = g_1 - g_0$. The map h is called a* chain homotopy*.*

Lemma 2.21. *Chain homotopy is an equivalence relation.*

Proposition 2.22. *If $g_0, g_1 : (C, d) \to (C', d')$ are chain homotopic, then $g_{0*} = g_{1*} : H_*(C) \to H_*(C')$.*

Proof. Let $h : C_* \to C'_{*+1}$ be a chain homotopy between g_0 and g_1. Let $[x] \in H_*(C)$, i.e. $x \in \operatorname{Ker} d$. We have

$$(g_{1*} - g_{0*})[x] = [g_1(x) - g_0(x)] = [d'h(x) + hd(x)] = [d'h(x)] = 0. \qquad \square$$

Definition 2.23 (Chain homotopy equivalence)**.** *Chain complexes (C, d) and (C', d') are said to be* chain homotopy equivalent *(which we write $C \sim C'$) if there exist chain maps $f : C \to C'$ and $g : C' \to C$ such that $fg \sim \mathrm{id}_{C'}$ and $gf \sim \mathrm{id}_C$.*

Proposition 2.24. *If $C \sim C'$, then $H_*(C) \simeq H_*(C')$.*

Notation 2.25.

- *Define singular n-simplices $\iota_n, \iota'_n : \Delta^n \to \Delta^n \times [0,1]$ by $\iota_n(v) = (v, 0)$ and $\iota'_n(v) = (v, 1)$.*
- *Consider the chain maps $\varphi_{\iota_n}, \varphi_{\iota'_n} : S_*(\Delta^n) \to C_*(\Delta^n \times [0,1])$ given by $\varphi_{\iota_n}(f_I) = \iota_n \circ F_I$ and $\varphi_{\iota'_n}(f_I) = \iota'_n \circ F_I$.*
- *Given points $p_0, \dots, p_k \in \Delta^n \times [0,1]$, we define a singular k-simplex $[p_0, \dots, p_k] : \Delta^k \to \Delta^n \times [0,1]$ given by $v \mapsto \sum_{i=0}^k v_i p_i$. We thus have*

$$d[p_0, \dots, p_n] = \sum_{j=0}^{k} (-1)^j [p_0, \dots, \hat{p}_j, \dots, p_k].$$

- *Write $i = f_i \times 0 \in \Delta^n \times [0,1]$, $i' = f_i \times 1 \in \Delta^n \times [0,1]$.*

Proposition 2.26 (Universal Chain Homotopy)**.** *The maps $\varphi_{\iota_n}, \varphi_{\iota'_n} : S_*(\Delta^n) \to C_*(\Delta^n \times [0,1])$ are chain homotopic.*

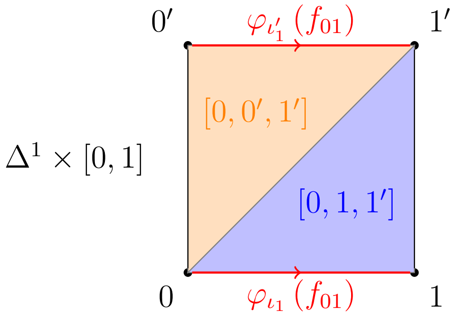


Figure 1: The Universal Chain Homotopy

Proof. Define $U_n : S_*(\Delta^n) \to C_{*+1}(\Delta^n \times [0,1])$ by

$$U_n(f_I) = \sum_{j=0}^{k} (-1)^j \left[i_0, \ldots, i_j, i'_j, \ldots, i'_k\right],$$

for $I = \{i_0 < i_1 < \cdots < i_k\}$. Then

$$U_n d(f_I) = \sum_{a<b} (-1)^{a+b-1} \left[i_0, \ldots, \hat{i}_a, \ldots, i_b, i'_b, \ldots, i'_k\right] + \sum_{a>b} (-1)^{a+b} \left[i_0, \ldots, i_b, i'_b, \ldots, \hat{i}'_a, \ldots, i'_k\right],$$

and likewise

$$\begin{aligned} dU_n(f_I) = &\sum_{a<b} (-1)^{a+b} \left[i_0, \ldots, \hat{i}_a, \ldots, i_b, i'_b, \ldots, i'_k\right] + \sum_{a>b} (-1)^{a+b+1} \left[i_0, \ldots, i_b, i'_b, \ldots, \hat{i}'_a, \ldots, i'_k\right] \\ &+ \sum_{b=0}^{k} (-1)^{b+b} [i_0, \ldots, i_{b-1}, i'_b, \ldots, i'_k] + \sum_{b=1}^{k+1} (-1)^{b-1+b} [i_0, \ldots, i_{b-1}, i'_b, \ldots, i'_k]. \end{aligned}$$

Therefore

$$(U_n d + dU_n)(f_I) = [i'_0, \ldots, i'_k] - [i_0, \ldots, i_k] = \varphi_{\iota'_n}(f_I) - \varphi_{\iota_n}(f_I).$$

□

Lemma 2.27. *Write $\overline{F}_I = F_I \times \mathrm{id}_{[0,1]} : \Delta^k \times [0,1] \to \Delta^n \times [0,1]$. Then the following diagram commutes:*

$$\begin{tikzcd} S_*\left(\Delta^k\right) \arrow[r, "F_{I\sharp}"] \arrow[d, "U_k"'] & S_*(\Delta^n) \arrow[d, "U_n"'] \\ C_{*+1}\left(\Delta^k \times [0,1]\right) \arrow[r, "\overline{F}_{I\sharp}"] & C_{*+1}(\Delta^n \times [0,1]) \end{tikzcd}$$

Theorem 2.28. *Let $g_0, g_1 : X \to Y$. If $g_0 \sim g_1$, then $g_{0\sharp} \sim g_{1\sharp}$.*

Proof. Let $G : X \times [0,1] \to Y$ be a homotopy from g_0 to g_1. Given a singular n-simplex $\sigma : \Delta^n \to X$, define

$$G_\sigma : (v,t) \in \Delta^n \times [0,1] \longmapsto G(\sigma(v), t) \in Y.$$

Note that

$$G_{\sigma \circ F_I} = G_\sigma \circ \overline{F}_I.$$

Now define $h : C_*(X) \to C_{*+1}(Y)$ by

$$h(\sigma) = G_{\sigma\sharp} U_n(f_{0,\ldots,n}).$$

Thus

$$dh(\sigma) = dG_{\sigma\sharp} U_n(f_{0,\ldots,n}) = G_{\sigma\sharp} dU_n(f_{0,\ldots,n}).$$

Writing $\hat{j} = \{0, \ldots, n\} \setminus \{j\}$, we also have

$$\begin{aligned} hd(\sigma) &= h\left(\sum_{j=0}^{n} (-1)^j \sigma \circ F_{\hat{j}}\right) = \sum_{j=0}^{n} (-1)^j G_{\sigma \circ F_{\hat{j}}\sharp} U_{n-1}(f_{0,\ldots,n-1}) \\ &= \sum_{j=0}^{n} (-1)^j \left(G_\sigma \circ \overline{F}_{\hat{j}}\right)_\sharp U_{n-1}(f_{0,\ldots,n-1}) = \sum_{j=0}^{n} (-1)^j G_{\sigma\sharp} U_n F_{\hat{j}\sharp}(f_{0,\ldots,n-1}) \\ &= G_{\sigma\sharp} U_n \left(\sum_{j=0}^{n} (-1)^j F_{\hat{j}\sharp}(f_{0,\ldots,n-1})\right) = G_{\sigma\sharp} U_n d(f_{0,\ldots,n}). \end{aligned}$$

Using Proposition 2.26, we therefore obtain

$$(dh + hd)(\sigma) = G_{\sigma\sharp}(dU_n + U_n d)(f_{0,\ldots,n}) = G_{\sigma\sharp}\left(\varphi_{\iota'_n} - \varphi_{\iota_n}\right)(f_{0,\ldots,n}) = g_{1\sharp}(\sigma) - g_{0\sharp}(\sigma).$$

□

Corollary 2.29. *Let $g_0, g_1 : X \to Y$. If $g_0 \sim g_1$, then $g_{0*} = g_{1*} : H_*(X) \to H_*(Y)$.*

Corollary 2.30. *If $X \sim Y$, then $H_*(X) \simeq H_*(Y)$.*

Proof. If $X \sim Y$, then there exist maps $f : X \to Y$ and $g : Y \to X$ with $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. Therefore:

$$f_*g_* = (fg)* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_*(Y)},$$

and likewise $g_*f_* = \mathrm{id}_{H_*(X)}$. □

Corollary 2.31. *If a space X is contractible, then $H_*(X) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$.*

2.4 Exact sequences and the Snake Lemma

Definition 2.32 (Exact sequence of modules)**.** *Consider a sequence of R-modules and homomorphisms:*

$$\cdots \to A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots . \tag{$*$}$$

- *We say that the sequence $(*)$ is exact at A_i if $\operatorname{Ker} f_i = \operatorname{Im} f_{i+1}$.*
- *We say that the sequence $(*)$ is exact if it is exact at A_i for all i.*

Note that the sequence $(*)$ is exact if and only if the sequence

$$0 \to \operatorname{Coker} f_{i+2} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} \operatorname{Ker} f_{i-1} \to 0$$

is exact for all i.

Definition 2.33 (Exact sequence of chain complexes)**.** *Saying that a short sequence of chain complexes*

$$0 \to A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \to 0$$

is exact means that:

(i) *A, B, C are chain complexes and ι, π are chain maps.*

(ii) *For all i, the sequence $0 \mapsto A_i \xrightarrow{\iota} B_i \xrightarrow{\pi} C_i \to 0$ is exact.*

Lemma 2.34 (Snake Lemma)**.** *Let $0 \to A_* \xrightarrow{\iota} B_* \xrightarrow{\pi} C_* \to 0$ be a short exact sequence of chain complexes. Then there is a map $\partial : H_*(C) \to H_{*-1}(A)$ s.t. we have a long exact sequence of homology:*

$$\begin{tikzcd}
{} \arrow[r] & H_*(A) \arrow[r, "\iota_*"] & H_*(B) \arrow[r, "\pi_*"] & H_*(C) \arrow[dll, rounded corners, "\partial"] \\
& H_{*-1}(A) \arrow[r, "\iota_*"] & H_{*-1}(B) \arrow[r, "\pi_*"] & H_{*-1}(C) \arrow[r] & {}
\end{tikzcd}$$

Proof. The map ∂ is defined as follows: if $[c] \in H_n(C)$, then there exists $b \in B_n$ s.t. $\pi b = c$, and we have $\pi db = d\pi b = dc = 0$, so there exists $a \in A_{n-1}$ s.t. $\iota a = db$, and we have $\iota da = d\iota a = d^2 b = 0$, so $da = 0$ and we set $\partial[c] = [a]$.

This is well-defined because $[c] = 0$ means that $c = dc'$ for some $c' \in C_{n+1}$, so $c' = \pi b'$ for some $b' \in B_{n-1}$ by surjectivity of π. Continuing the construction as above yields the existence of an $a' \in A_{n-1}$ s.t. $\iota a' = db'$, and therefore $a = da'$ so $[a] = 0$.

The sequence is exact at $H_{n-1}(A)$ because

$$[a] \in \operatorname{Ker} \iota_* \iff \exists b \in B_n,\ \iota a = db \iff \exists b \in B_n,\ [a] = \partial[\pi b] \iff [a] \in \operatorname{Im} \partial.$$

The sequence is exact at $H_n(C)$ because

$$\begin{aligned}[c] \in \operatorname{Ker} \partial &\iff \exists a' \in A_{n+1},\ \exists b \in B_n,\ \iota d a' = db \text{ and } c = \pi b \\ &\iff \exists a' \in A_{n+1},\ \exists b \in B_n,\ [c] = \pi_* [b - \iota a'] \iff [c] \in \operatorname{Im} \pi_*.\end{aligned}$$

The exactness at $H_n(B)$ is clear from the exactness of $0 \to A_* \to B_* \to C_* \to 0$. □

Corollary 2.35. *Let $X \neq \varnothing$ be a topological space. Then*

$$H_*(X) = \begin{cases} \widetilde{H}_*(X) \oplus \mathbb{Z} & \text{if } * = 0 \\ \widetilde{H}_*(X) & \text{otherwise} \end{cases}.$$

Proof. Define a chain complex K by

$$K_* = \begin{cases} \langle \sigma_\varnothing \rangle & \text{if } * = -1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus $H_*(K) = \begin{cases} \langle \sigma_\varnothing \rangle & \text{if } * = -1 \\ 0 & \text{otherwise} \end{cases}$. Moreover, we have a short exact sequence of chain complexes

$$0 \to K_* \to \widetilde{C}_*(X) \to C_*(X) \to 0.$$

Applying the Snake Lemma yields, for $* \neq 0$:

$$0 = H_*(K) \to \widetilde{H}_*(X) \to H_*(X) \to H_{*-1}(K) = 0,$$

which implies that $\widetilde{H}_*(X) \simeq H_*(X)$ for $* \neq 0$. Now, for $* = 0$:

$$0 = H_0(K) \to \widetilde{H}_0(X) \to H_0(X) \xrightarrow{\partial} H_{-1}(K) = \langle \sigma_\varnothing \rangle \to \widetilde{H}_{-1}(X) \to H_{-1}(X) = 0.$$

Let us compute ∂: for $p \in X$, denote by $\sigma_p : \Delta^0 \to X$ the 0-simplex given by $f_0 \mapsto p$. In $\widetilde{H}_0(X)$, we have $d\sigma_p = \sigma_\varnothing$, so $\partial [\sigma_p] = [\sigma_\varnothing]$. Therefore, ∂ is surjective and we have a short exact sequence

$$0 \to \widetilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0.$$

This implies that $H_0(X) \simeq \widetilde{H}_0(X) \oplus \mathbb{Z}$. □

2.5 Homology of a pair

Definition 2.36 (Subcomplexes and quotient complexes)**.** *Let (C, d) be a chain complex. We say that A is a subcomplex of C if*

(i) $A_* = \bigoplus_{i \in \mathbb{Z}} A_i$ *with A_i a submodule of C_i for all i,*

(ii) $d(A_i) \subseteq A_{i-1}$.

If so, (A, d) is a chain complex.

We set $Q_i = C_i / A_i$. Since $d(A_i) \subseteq A_{i-1}$, the map $d : C_i \to C_{i-1}$ induces $d_Q : Q_i \to Q_{i-1}$ with $d_Q^2 = 0$. We call (Q, d_Q) the quotient complex*.*

We have a short exact sequence

$$0 \to A_* \xrightarrow{\iota} C_* \xrightarrow{\pi} Q_* \to 0.$$

Example 2.37. *Let $A \subseteq X$ be an inclusion of spaces. If $\sigma : \Delta^k \to X$ has $\sigma\left(\Delta^k\right) \subseteq A$, then $d\sigma \in C_*(A)$. In other words, $C(A)$ is a subcomplex of $C(X)$.*

Definition 2.38 (Homology of a pair)**.** *If $A \subseteq X$, we define*

$$C_*(X,A) = C_*(X)/C_*(A),$$

and $H_*(X,A) = H_*(C(X,A))$. The group $H_*(X,A)$ is called the homology of the pair (X,A)*. By the Snake Lemma, the short exact sequence $0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$ induces* the long exact sequence of the pair (X,A)*:*

$$\cdots \to H_*(A) \xrightarrow{\iota_*} H_*(X) \xrightarrow{\pi_*} H_*(X,A) \xrightarrow{\partial} H_{*-1}(A) \to \cdots.$$

Example 2.39. *Consider the pair $(\mathbb{D}^1, \mathbb{S}^0)$. We have*

$$H_*\left(\mathbb{S}^0\right) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H_*\left(\mathbb{D}^1\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Writing the long exact homology sequence of the pair $(\mathbb{D}^1, \mathbb{S}^0)$ yields $H_*(\mathbb{D}^1, \mathbb{S}^0) = \begin{cases} \mathbb{Z} & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$.

Proposition 2.40. *Consider a* map of pairs *$f : (X,A) \to (Y,B)$, i.e. a map $f : X \to Y$ s.t. $f(A) \subseteq B$. If $\sigma : \Delta^k \to A$ is a k-simplex, then $f_\sharp(\sigma) = f \circ \sigma : \Delta^k \to B$, which shows that $f_\sharp(C_*(A)) \subseteq C_*(B)$. Hence $f_\sharp$ descends to a map $f_\sharp^{(q)} : C_*(X,A) \to C_*(Y,B)$, which induces a map $f_* : H_*(X,A) \to H_*(Y,B)$.*

Lemma 2.41. *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "\iota"] \arrow[d, "f"'] & B_* \arrow[r, "\pi"] \arrow[d, "f"'] & C_* \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & A'_* \arrow[r, "\iota'"] & B'_* \arrow[r, "\pi'"] & C'_* \arrow[r] & 0 \end{tikzcd}$$

is a commutative diagram of chain complexes with exact rows. Then we have a commutative diagram of long exact sequences:

$$\begin{tikzcd} \cdots \arrow[r] & H_*(A) \arrow[r, "\iota_*"] \arrow[d, "f_*"'] & H_*(B) \arrow[r, "\pi_*"] \arrow[d, "f_*"'] & H_*(C) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"'] & \cdots \\ \cdots \arrow[r] & H_*(A') \arrow[r, "\iota'_*"] & H_*(B') \arrow[r, "\pi'_*"] & H_*(C') \arrow[r, "\partial'"] & H_{*-1}(A') \arrow[r] & \cdots \end{tikzcd}$$

In other words, there is a functor from the category of short exact sequences of chain complexes over R to the category of long exact sequences of R-modules.

Proof. We only check that the rightmost square commutes. If $[c] \in H_n(C)$, pick $b \in B_n$ and $a \in A_{n-1}$ s.t. $\pi b = c$ and $\iota a = db$. Then $\partial[c] = a$. Set $a' = fa$, $b' = fb$, $c' = fc$. We now have $\pi b' = c'$ and $\iota a' = db'$, therefore $\partial'[c'] = [a']$, i.e. $\partial' f_*[c] = f_*\partial[c]$. □

Corollary 2.42. *If $(X,A) \to (Y,B)$ is a map of pairs, then there is a commutative diagram of long exact sequences:*

$$\begin{tikzcd} \cdots \arrow[r] & H_*(A) \arrow[r, "\iota_*"] \arrow[d, "f_*"'] & H_*(X) \arrow[r, "\pi_*"] \arrow[d, "f_*"'] & H_*(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{*-1}(A) \arrow[r] \arrow[d, "f_*"'] & \cdots \\ \cdots \arrow[r] & H_*(B) \arrow[r, "\iota_*"] & H_*(Y) \arrow[r, "\pi_*"] & H_*(Y,B) \arrow[r, "\partial"] & H_{*-1}(B) \arrow[r] & \cdots \end{tikzcd}$$

Proposition 2.43. *Let $g_0, g_1 : (X, A) \to (Y, B)$. If g_0 and g_1 are homotopic as maps of pairs, then $g_{0*} = g_{1*} : H_*(X, A) \to H_*(Y, B)$.*

Proof. The maps $g_{0\sharp}, g_{1\sharp} : C_*(X) \to C_*(Y)$ are chain homotopic via $h(\sigma) = G_{\sigma\sharp}U_n\left(f_{0,\dots,n}\right)$, where G is a homotopy of maps of pairs from g_0 to g_1. We have $G\left(A \times [0,1]\right) \subseteq B$, which implies that $h\left(C_*(A)\right) \subseteq C_{*+1}(B)$, so h descends to a map $h^{(q)} : C_*(X, A) \to C_{*+1}(Y, B)$ with $dh^{(q)} + h^{(q)}d = g_{1\sharp}^{(q)} - g_{0\sharp}^{(q)}$. Hence $g_{1\sharp}^{(q)} \sim g_{0\sharp}^{(q)}$ and $g_{1*} = g_{0*}$. □

Remark 2.44. *We could define the* reduced homology of a pair (X, A) *by* $\widetilde{C}_*(X, A) = \widetilde{C}_*(X)/\widetilde{C}_*(A)$ *and* $\widetilde{H}_*(X, A) = H_*\left(\widetilde{C}(X, A)\right)$. *Again, we will have the* long exact sequence of a pair.

Proposition 2.45. (i) *For any pair (X, A) with $A \neq \varnothing$, we have $\widetilde{H}_*(X, A) \simeq H_*(X, A)$.*

(ii) *If $p \in X$, then $\widetilde{H}_*(X) \simeq H_*(X, p)$.*

(iii) *$H_*\left(\mathbb{D}^n, \mathbb{S}^{n-1}\right) \simeq \widetilde{H}_{*-1}\left(\mathbb{S}^{n-1}\right)$.*

Proof. (i) We have $\widetilde{C}_*(X) = C_*(X) \oplus \langle \sigma_\varnothing \rangle$ and $\widetilde{C}_*(A) = C_*(A) \oplus \langle \sigma_\varnothing \rangle$. Therefore $\widetilde{C}_*(X, A) \simeq C_*(X, A)$.

(ii) The long exact (reduced) homology sequence of (X, p) is written as

$$\cdots \to \widetilde{H}_*\left(\{p\}\right) \to \widetilde{H}_*(X) \to H_*(X, p) \to \widetilde{H}_{*-1}\left(\{p\}\right) \to \cdots.$$

Since $\widetilde{H}_*\left(\{p\}\right) = 0$, it follows that $\widetilde{H}_*(X) \simeq H_*(X, p)$.

(iii) Same proof as (ii), using the fact that $\mathbb{D}^n$ is contractible and so $\widetilde{H}_*\left(\mathbb{D}^n\right) = 0$. □

2.6 Collapsing a pair

Definition 2.46 (Deformation retract)**.** *We say that a subset A of a space U is a* deformation retract *of U if there exists $\pi : (U, A) \to (A, A)$ with $i \circ \pi \sim \mathrm{id}_{(U,A)}$ as maps of pairs (where $i : (A, A) \to (U, A)$ is the inclusion).*

Example 2.47. *$\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{D}^n \backslash \{0\}$.*

Definition 2.48 (Good pair)**.** *The pair (X, A) is said to be* good *if*

(i) *A is closed in X,*

(ii) *There exists an open subset U of X s.t. $A \subseteq U$ and A is a deformation retract of U.*

Example 2.49. (i) *$(\mathbb{D}^n, \mathbb{S}^{n-1})$ is good with $U = \mathbb{D}^n \backslash \{0\}$.*

(ii) *$(\mathbb{D}^n, \mathbb{D}^n \backslash \{0\})$ is not good because $\mathbb{D}^n \backslash \{0\}$ is not closed in $\mathbb{D}^n$.*

(iii) *If $A = \left\{\frac{1}{n}, \ , n \in \mathbb{Z}\backslash\{0\}\right\} \cup \{0\} \subseteq \mathbb{R}$, then A is closed in $\mathbb{R}$ but $(\mathbb{R}, A)$ is not good.*

(iv) *If K is a compact submanifold of a smooth manifold M, then (M, K) is good.*

(v) *If L is a subcomplex of a simplicial complex Λ, then (Λ, L) is good.*

Theorem 2.50 (Collapsing a pair)**.** *Let (X, A) be a good pair. Then the quotient map $\pi : (X, A) \to (X/A, \{*_A\})$ induces an isomorphism $\pi_* : H_*(X, A) \xrightarrow{\simeq} H_*\left(X/A, \{*_A\}\right)$. In particular*

$$H_*(X, A) \simeq \widetilde{H}_*(X/A).$$

Proof. See Theorem 2.69. □

Example 2.51. $\widetilde{H}_*\left(\mathbb{S}^n\right) = \begin{cases} \mathbb{Z} & \textit{if } * = n \\ 0 & \textit{otherwise} \end{cases}.$

Proof. By induction on n. For $n=0$, note that $\mathbb{S}^0 = \{-1\} \cup \{+1\}$. Using the fact that $\widetilde{H}_*(\{pt\}) = 0$, the result follows. For the induction step, note that $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a good pair, and $\mathbb{D}^n/\mathbb{S}^{n-1} \simeq \mathbb{S}^n$, so by Theorem 2.50, $H_*(\mathbb{D}^n, \mathbb{S}^{n-1}) \simeq \widetilde{H}_*(\mathbb{S}^n)$. Moreover, Proposition 2.45 shows that $H_*(\mathbb{D}^n, \mathbb{S}^{n-1}) \simeq \widetilde{H}_{*-1}(\mathbb{S}^{n-1})$, from which we deduce that

$$\widetilde{H}_*(\mathbb{S}^n) \simeq \widetilde{H}_{*-1}\left(\mathbb{S}^{n-1}\right).$$

The result follows. ☐

Corollary 2.52. (i) $\mathbb{S}^n$ *is not contractible.*

(ii) $\mathbb{S}^m \sim \mathbb{S}^n \Longrightarrow m = n$.

(iii) *The map* $\mathrm{id} : \mathbb{S}^n \to \mathbb{S}^n$ *does not extend to a map* $\mathbb{D}^{n+1} \to \mathbb{S}^n$.

(iv) *The group* $\pi_n(\mathbb{S}^n, *)$ *is nontrivial.*

Example 2.53. $H_*(\mathbb{T}^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0,2 \\ \mathbb{Z}^2 & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}.$

Proof. First step: we compute $H_*(\mathbb{S}^2, \mathbb{S}^0)$. Writing the long exact sequence of $(\mathbb{S}^2, \mathbb{S}^0)$ yields:

$$\underbrace{\widetilde{H}_2\left(\mathbb{S}^0\right)}_{=0} \to \underbrace{\widetilde{H}_2\left(\mathbb{S}^2\right)}_{=\mathbb{Z}} \to H_2\left(\mathbb{S}^2, \mathbb{S}^0\right) \to \underbrace{\widetilde{H}_1\left(\mathbb{S}^0\right)}_{=0} \to \underbrace{\widetilde{H}_1\left(\mathbb{S}^2\right)}_{=0} \to H_1\left(\mathbb{S}^2, \mathbb{S}^0\right) \to \underbrace{\widetilde{H}_0\left(\mathbb{S}^0\right)}_{=\mathbb{Z}} \to \underbrace{\widetilde{H}_0\left(\mathbb{S}^2\right)}_{=0}.$$

It follows that $H_*(\mathbb{S}^2, \mathbb{S}^0) = \begin{cases} \mathbb{Z} & \text{if } * = 1,2 \\ 0 & \text{otherwise} \end{cases}.$

Second step: Let $B = \mathbb{S}^1 \times 1 \subseteq \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$. Note that $\mathbb{T}^2/B \simeq \mathbb{S}^2/\mathbb{S}^0$, so $H_*(\mathbb{T}^2, B) \simeq H_*(\mathbb{S}^2, \mathbb{S}^0)$ by Theorem 2.50. The long exact sequence of $(\mathbb{T}^2, B)$ is

$$\underbrace{\widetilde{H}_2(B)}_{=0} \to \widetilde{H}_2\left(\mathbb{T}^2\right) \to \underbrace{H_2\left(\mathbb{T}^2, B\right)}_{=\mathbb{Z}} \to \underbrace{\widetilde{H}_1(B)}_{=\mathbb{Z}} \xrightarrow{\iota_*} \widetilde{H}_1\left(\mathbb{T}^2\right) \to \underbrace{H_1\left(\mathbb{T}^2, B\right)}_{=\mathbb{Z}} \to \underbrace{\widetilde{H}_0(B)}_{=0}.$$

We claim that ι_* is injective: to prove it, consider $\pi : (x,y) \in \mathbb{S}^1 \times \mathbb{S}^1 \longmapsto x \in \mathbb{S}^1$, then $\pi \circ \iota = \mathrm{id}_{\mathbb{S}^1}$, so $\pi_* \circ \iota_* = \mathrm{id}_{\widetilde{H}_*(\mathbb{S}^1)}$ and ι_* is injective. We now split the above long exact sequence into short exact sequences:

$$0 \to \widetilde{H}_2\left(\mathbb{T}^2\right) \to \underbrace{H_2\left(\mathbb{T}^2, B\right)}_{=\mathbb{Z}} \to \underbrace{\operatorname{Ker} \iota_*}_{=0} \to 0,$$

which gives $\widetilde{H}_2(\mathbb{T}^2) = \mathbb{Z}$. Likewise, we have

$$0 \to \underbrace{\widetilde{H}_1(B)}_{=\mathbb{Z}} \to \widetilde{H}_1\left(\mathbb{T}^2\right) \to \underbrace{H_1\left(\mathbb{T}^2, B\right)}_{=\mathbb{Z}} \to 0,$$

and therefore $\widetilde{H}_1(\mathbb{T}^2) = \mathbb{Z}^2$. ☐

2.7 Subdivide, excise and collapse!

Notation 2.54. *Let* $\mathcal{U} = \{U_\alpha,\ \alpha \in A\}$ *be an open cover of* X. *If a singular* k*-simplex* $\sigma : \Delta^k \to X$ *is s.t. there exists* $U \in \mathcal{U}$ *with* $\sigma\left(\Delta^k\right) \subseteq U$, *then we write* $\sigma \lhd \mathcal{U}$. *We define* $C_k^{\mathcal{U}}(X)$ *to be the submodule of* $C_k(X)$ *generated by the singular* k*-simplices* σ *with* $\sigma \lhd \mathcal{U}$

If $\sigma \lhd \mathcal{U}$, *note that* $(\sigma \circ F_I) \lhd \mathcal{U}$ *for all* I, *and therefore* $d\sigma \in C_{k-1}^{\mathcal{U}}(X)$. *Hence,* $C^{\mathcal{U}}(X)$ *is a subcomplex of* $C(X)$.

Theorem 2.55 (Subdivision Theorem)**.** *If $i : C^{\mathcal{U}}_*(X) \to C_*(X)$ is the inclusion, then the induced map $i_* : H^{\mathcal{U}}_*(X) \to H_*(X)$ is an isomorphism.*

Proposition 2.56 (Mayer-Vietoris Sequence)**.** *Suppose $U_1, U_2 \subseteq X$ are two open subsets s.t. $U_1 \cup U_2 = X$. We have the following diagram of inclusion maps:*

$$\begin{tikzcd}
 & U_1 \arrow[dr, "j_1"] & \\
U_1 \cap U_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\
 & U_2 \arrow[ur, "j_2"'] &
\end{tikzcd}$$

There is a long exact sequence:

$$\cdots \to H_*\left(U_1 \cap U_2\right) \xrightarrow{i_{1*} \oplus i_{2*}} H_*\left(U_1\right) \oplus H_*\left(U_2\right) \xrightarrow{j_{1*} - j_{2*}} H_*(X) \xrightarrow{\partial} H_{*-1}\left(U_1 \cap U_2\right) \to \cdots .$$

Proof. There is a short exact sequence of chain complexes:

$$0 \to C_*\left(U_1 \cap U_2\right) \xrightarrow{i_{1\sharp} \oplus i_{2\sharp}} C_*\left(U_1\right) \oplus C_*\left(U_2\right) \xrightarrow{j_{1\sharp} - j_{2\sharp}} C^{\mathcal{U}}_*(X) \to 0.$$

Taking the long exact homology sequence given by the Snake Lemma (Lemma 2.34) and using the fact that $H^{\mathcal{U}}_*(X) \simeq H_*(X)$ by the Subdivision Theorem (Theorem 2.55) yields the result. $\square$

Remark 2.57. *The Mayer-Vietoris Sequence can also be written with reduced homology groups.*

Example 2.58. $H_*\left(\mathbb{S}^n\right) \simeq H_{*-1}\left(\mathbb{S}^{n-1}\right)$.

Proof. Take $U_1 = \mathbb{S}^n \backslash \{p\} \simeq \mathbb{R}^n \sim \{p\}$, $U_2 = \mathbb{S}^n \backslash \{q\} \simeq \mathbb{R}^n \sim \{q\}$, note that $U_1 \cap U_2 \simeq \mathbb{R}^n \backslash \{0\} \sim \mathbb{S}^{n-1}$ and write the Mayer-Vietoris sequence of (X, U_1, U_2). $\square$

Lemma 2.59 (Five Lemma)**.** *Consider a commutative diagram with exact rows as below:*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Corollary 2.60. *Suppose $A \subseteq X$ and $\mathcal{U}$ is an open cover of X. Define an open cover $\mathcal{U}_A = \{U \cap A,\ U \in \mathcal{U}\}$ of A. Then $C^{\mathcal{U}_A}_*(A)$ is a subcomplex of $C^{\mathcal{U}}_*(X)$, so we can define $C^{\mathcal{U}}_*(X, A) = C^{\mathcal{U}}_*(X)/C^{\mathcal{U}_A}_*(A)$, and we have*

$$H^{\mathcal{U}}_*(X, A) \simeq H_*(X, A).$$

Proof. We have a map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & C^{\mathcal{U}_A}_*(A) \arrow[r] \arrow[d, "i"'] & C^{\mathcal{U}}_*(X) \arrow[r] \arrow[d, "i"'] & C^{\mathcal{U}}_*(X, A) \arrow[r] \arrow[d, "i"'] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

By the Snake Lemma (Lemma 2.34), it induces a map of long exact sequences:

$$\begin{tikzcd}
H^{\mathcal{U}_A}_*(A) \arrow[r] \arrow[d] & H^{\mathcal{U}}_*(X) \arrow[r] \arrow[d] & H^{\mathcal{U}}_*(X, A) \arrow[r] \arrow[d, red] & H^{\mathcal{U}_A}_{*-1}(A) \arrow[r] \arrow[d] & H^{\mathcal{U}}_{*-1}(X) \arrow[d] \\
H_*(A) \arrow[r] & H_*(X) \arrow[r] & H_*(X, A) \arrow[r] & H_{*-1}(A) \arrow[r] & H_{*-1}(X)
\end{tikzcd}$$

The black vertical arrows are isomorphisms by the Subdivision Theorem (Theorem 2.55); it follows that the red arrow is also an isomorphism by the Five Lemma (Lemma 2.59). $\square$

Theorem 2.61 (Excision Theorem). *Let $B \subseteq A \subseteq X$ s.t. $\overline{B} \subseteq \mathring{A}$. Then the inclusion map $j : (X\backslash B, A\backslash B) \to (X, A)$ induces an isomorphism*

$$H_* (X\backslash B, A\backslash B) \simeq H_*(X, A).$$

Proof. The set $\mathcal{U} = \left\{X\backslash\overline{B}, \mathring{A}\right\}$ is an open cover of X by assumption. Note that

$$C_*^{\mathcal{U}}(X) = C_*^{\mathcal{U}_{X\backslash B}}(X\backslash B) \oplus \left\langle \sigma, \operatorname{Im}\sigma \subseteq \mathring{A} \right\rangle,$$

and similarly

$$C_*^{\mathcal{U}_A}(A) = C_*^{\mathcal{U}_{A\backslash B}}(A\backslash B) \oplus \left\langle \sigma, \operatorname{Im}\sigma \subseteq \mathring{A} \right\rangle.$$

Therefore, $C_*^{\mathcal{U}}(X, A) \simeq C_*^{\mathcal{U}_{X\backslash B}}(X\backslash B, A\backslash B)$. Now, we have the following commutative diagram:

$$\begin{tikzcd}
C_*^{\mathcal{U}_{X\backslash B}}(X\backslash B, A\backslash B) \arrow[r, "j_\sharp^{\mathcal{U}}"] \arrow[d, "i"'] & C_*^{\mathcal{U}}(X, A) \arrow[d, "i'"'] \\
C_*(X\backslash B, A\backslash B) \arrow[r, "j_\sharp"] & C_*(X, A)
\end{tikzcd}$$

We have just seen that $j_\sharp^{\mathcal{U}}$ is an isomorphism (and therefore, so is $j_*^{\mathcal{U}}$), and we know that i_* and i'_* are isomorphisms by Corollary 2.60. Therefore, j_* is an isomorphism. $\square$

Example 2.62. *If U is an open subset of $\mathbb{R}^n$ and $p \in U$, then*

$$H_* (U, U\backslash\{p\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$$

Proof. First step: compute $H_* (\mathbb{R}^n, \mathbb{R}^n\backslash\{p\})$ by noting that $\mathbb{R}^n\backslash\{p\} \sim \mathbb{S}^{n-1}$ and by writing the long exact sequence of $(\mathbb{R}^n, \mathbb{R}^n\backslash\{p\})$. *Second step*: set $C = \mathbb{R}^n\backslash U$; as C is closed in $\mathbb{R}^n$, we have $\overline{C} \subseteq \mathbb{R}^n\backslash\{p\}$ and therefore, by the Excision Theorem (Theorem 2.61), we obtain $H_* (U, U\backslash\{p\}) \simeq H_* (\mathbb{R}^n, \mathbb{R}^n\backslash\{p\})$. $\square$

Corollary 2.63. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be two nonempty open subsets. If U and V are homeomorphic, then $m = n$.*

Proof. A homeomorphism $f : U \to V$ induces an isomorphism $H_* (U, U\backslash\{p\}) \to H_* (V, V\backslash \{f(p)\})$. $\square$

2.8 Deformation retracts and collapsing a pair

Definition 2.64 (Deformation retract). *Suppose $A \subseteq U$ and let $i : A \to U$ be the inclusion. If $\pi : U \to A$, we have maps of pairs $(U, A) \xrightarrow{\widetilde{\pi}} (A, A) \xrightarrow{\widetilde{i}} (U, A)$.*

We say that $\pi : U \to A$ is a deformation retract *if $\widetilde{i} \circ \widetilde{\pi} \sim \mathrm{id}_{(U,A)}$ as maps of pairs. This implies that $A \sim U$ (because $i \circ \pi \sim \mathrm{id}_U$ and $\pi \circ i \sim \mathrm{id}_A$).*

Lemma 2.65. *If $\pi : U \to A$ is a deformation retract, so is $\pi' : U/A \to A/A$.*

Lemma 2.66. *Suppose $B \subseteq A \subseteq X$. Then there is a long exact sequence*

$$\cdots \to H_*(A, B) \xrightarrow{j_*} H_*(X, B) \xrightarrow{i_*} H_*(X, A) \xrightarrow{\partial} H_{*-A}(A, B) \to \cdots.$$

Proof. Apply the Snake Lemma (Lemma 2.34) to the following short exact sequence:

$$0 \to \frac{C_*(A)}{C_*(B)} \xrightarrow{i_\sharp} \frac{C_*(X)}{C_*(B)} \xrightarrow{j_\sharp} \frac{C_*(X)}{C_*(A)} \to 0. \qquad \square$$

Lemma 2.67. *Suppose $A \subseteq U \subseteq X$ and A is a deformation retract of U. Then the map $i_* : H_*(X, A) \to H_*(X, U)$ induced by inclusion is an isomorphism.*

Proof. Note that $i : A \to U$ is a homotopy equivalence, so $i_* : H_*(A) \to H_*(U)$ is an isomorphism. Splitting the long exact sequence of (U, A) gives

$$0 \to \underbrace{\operatorname{Coker} i_*}_{=0} \to H_*(U, A) \to \underbrace{\operatorname{Ker} i_*}_{=0} \to 0,$$

which implies that $H_*(U, A) = 0$. Writing the long exact sequence of the triple (X, U, A) as in Lemma 2.66 shows that $i_* : H_*(X, A) \to H_*(X, U)$ is an isomorphism. $\square$

Definition 2.68 (Good pair)**.** *The pair (X, A) is said to be* good *if*

(i) *A is closed in X,*

(ii) *There exists an open subset U of X s.t. $A \subseteq U$ and A is a deformation retract of U.*

Theorem 2.69 (Collapsing a pair)**.** *Let (X, A) be a good pair. Then the quotient map $\pi : (X, A) \to (X/A, \{*_A\})$ induces an isomorphism $\pi_* : H_*(X, A) \xrightarrow{\simeq} H_*(X/A, \{*_A\})$. In particular*

$$H_*(X, A) \simeq \widetilde{H}_*(X/A).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
H_*(X, A) \arrow[r, "i_*"] \arrow[d, "\pi_*"'] & H_*(X, U) \arrow[d, "\pi_{2*}"'] & H_*(X \backslash A, U \backslash A) \arrow[l, "j_*"'] \arrow[d, "\pi_{3*}"'] \\
H_*(X/A, \{*_A\}) \arrow[r, "i'_*"] & H_*(X/A, U/A) & H_*((X/A) \setminus \{*_A\}, (U/A) \setminus \{*_A\}) \arrow[l, "j'_*"']
\end{tikzcd}$$

Note that π_3 is a homeomorphism, so π_{3*} is an isomorphism. Moreover, A is closed, U is open, $\overline{A} \subseteq \mathring{U}$, so j_* and j'_* are isomorphisms by the Excision Theorem (Theorem 2.61). Since the right-hand square commutes, π_{2*} is an isomorphism. Now i_* and i'_* are isomorphisms by Lemmas 2.65 and 2.67. Since the left-hand square also commutes, π_* is an isomorphism as wanted. $\square$

2.9 Maps of the sphere

Notation 2.70. *We want to make a consistent choice of generators for $\widetilde{H}_n(\mathbb{S}^n) \simeq \mathbb{Z}$. We start by defining $[S^0] = \sigma_{+1} - \sigma_{-1}$, a generator of $\widetilde{H}_0(\mathbb{S}^0)$ with $\mathbb{S}^0 = \{\pm 1\}$, and then we define a generator $[S^n]$ of $\widetilde{H}_n(\mathbb{S}^n)$ by induction in such a way that the following diagram of isomorphisms carries $[S^{n-1}]$ to $[S^n]$:*

$$\begin{tikzcd}
\widetilde{H}_n(\mathbb{S}^n) & H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[l, "p_*"'] \arrow[r, "f_*"] \arrow[d, "\partial"'] & H_n(I^n, \partial I^n) \\
& \widetilde{H}_{n-1}(\mathbb{S}^{n-1}) &
\end{tikzcd}$$

We shall also write $[D^n, S^{n-1}]$ and $[I^n, \partial I^n]$ for the corresponding generators of $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ and of $H_n(I^n, \partial I^n)$.

Definition 2.71 (Degree)**.** *Let $f : \mathbb{S}^n \to \mathbb{S}^n$. Then there exists $\kappa \in \mathbb{Z}$ s.t. $f_*[S^n] = \kappa [S^n]$. The integer* $\deg f = \kappa$ *is called the* degree *of f.*

Proposition 2.72. (i) $\deg(f \circ g) = (\deg f)(\deg g)$.

(ii) $\deg(\mathrm{id}_{\mathbb{S}^n}) = 1$.

(iii) $f \sim g \Longrightarrow \deg f = \deg g$.

(iv) *If* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is constant, then* $\deg f = 0$.

(v) *If* $f : \mathbb{S}^n \to \mathbb{S}^n$ *is a homeomorphism, then* $\deg f \in \{\pm 1\}$.

Proposition 2.73. *If* $\rho : \mathbb{S}^n \to \mathbb{S}^n$ *is a reflection in a hyperplane, then* $\deg \rho = -1$.

Proof. See Proposition 2.81. □

Corollary 2.74. *If* $a : \mathbb{S}^n \to \mathbb{S}^n$ *is the antipodal map, then* $\deg a = (-1)^{n+1}$.
In particular, if n *is even, then* $a \not\sim \mathrm{id}_{\mathbb{S}^n}$.

Proof. Note that $a = \rho_1 \circ \cdots \circ \rho_{n+1}$, where $\rho_i(v) = (v_1, \ldots, -v_i, \ldots, v_n)$, and use Proposition 2.73. □

2.10 The Hurewicz homomorphism

Definition 2.75 (Hurewicz homomorphism)**.** *The* Hurewicz homomorphism ψ *is the map* $\psi : \pi_n(X, p) \to \widetilde{H}_n(X)$*, defined by* $[\alpha] \in \pi_n(X, p) \subseteq [\mathbb{S}^n, X] \longmapsto \alpha_* [S^n] \in \widetilde{H}_n(X)$*. Note that* ψ *is well-defined because* $\alpha \sim \beta \Longrightarrow \alpha_* = \beta_*$.
We are now going to prove that ψ *is a group homomorphism.*

Definition 2.76 (Wedge product)**.** *Let* $(X_\alpha, p_\alpha)_{\alpha \in A}$ *be a family of pointed spaces. Their* wedge product *is defined by*

$$\bigvee_{\alpha \in A} (X_\alpha, p_\alpha) = \left(\coprod_{\alpha \in A} X_\alpha \right) / \left(\coprod_{\alpha \in A} \{p_\alpha\} \right).$$

Given maps $f_\alpha : (X_\alpha, p_\alpha) \to (Y, q)$*, we define* $\bigvee_{\alpha \in A} f_\alpha : \bigvee_{\alpha \in A} (X_\alpha, p_\alpha) \to Y$ *by* $\left(\bigvee_{\alpha \in A} f_\alpha\right)(x) = f_\alpha(x)$ *if* $x \in X_\alpha$*. This makes sense because* $f_\alpha(p_\alpha) = q$ *for all* $\alpha \in A$.
If the spaces X_α *are homogeneous (i.e. s.t. the group of homeomorphisms acts transitively), then* $\bigvee_{\alpha \in A} X_\alpha$ *does not depend on the choice of points* p_α*, and we shall drop them from the notation.*

Lemma 2.77. *Let* $(X_\alpha, p_\alpha)_{\alpha \in A}$ *be a family of pointed spaces s.t.* $(X_\alpha, p_\alpha)_{\alpha \in A}$ *is a good pair for all* $\alpha \in A$*. Denote by* $\iota_\alpha : X_\alpha \to \bigvee_{\alpha \in A} (X_\alpha, p_\alpha)$ *the inclusion and by* $\pi_\alpha : \bigvee_{\alpha \in A} (X_\alpha, p_\alpha) \to X_\alpha$ *the projection (with* $\pi_\alpha : x \notin X_\alpha \longmapsto p_\alpha$*). Then there are isomorphisms*

$$\bigoplus_{\alpha \in A} \widetilde{H}_*(X_\alpha) \simeq \widetilde{H}_* \left(\bigvee_{\alpha \in A} (X_\alpha, p_\alpha) \right),$$

given by $\sum_{\alpha \in A} \iota_{\alpha *}$ *and* $\bigoplus_{\alpha \in A} \pi_{\alpha *}$.

Proof. By collapsing pairs, we obtain isomorphisms

$$\bigoplus_{\alpha \in A} \widetilde{H}_*(X_\alpha) \simeq \bigoplus_{\alpha \in A} H_*(X_\alpha, p_\alpha) \simeq H_* \left(\coprod_{\alpha \in A} X_\alpha, \coprod_{\alpha \in A} p_\alpha \right) \simeq \widetilde{H}_* \left(\bigvee_{\alpha \in A} (X_\alpha, p_\alpha) \right).$$ □

Corollary 2.78. *Let* $(X_\alpha, p_\alpha)_{\alpha \in A}$ *be a family of pointed spaces s.t.* $(X_\alpha, p_\alpha)_{\alpha \in A}$ *is a good pair for all* $\alpha \in A$*, and let* $f_\alpha : (X_\alpha, p_\alpha) \to (Y, q)$ *be maps. Then we have the following commutative diagram:*

$$\begin{tikzcd} \widetilde{H}_*\left(\bigvee_\alpha (X_\alpha, p_\alpha)\right) \arrow[r, "(\bigvee_\alpha f_\alpha)_*"] \arrow[d, "\simeq"'] & \widetilde{H}_*(Y) \\ \bigoplus_\alpha \widetilde{H}_*(X_\alpha) \arrow[ur, "\sum_\alpha f_{\alpha *}"'] & \end{tikzcd}$$

Proposition 2.79. *The Hurewicz homomorphism ψ is indeed a group homomorphism.*

Proof. Let $\alpha, \beta : (\mathbb{S}^n, *) \to (X, p)$. The group law in $\pi_n(\mathbb{S}^n, *)$ can be understood *via* the composite $\mathbb{S}^n \xrightarrow{\pi} \mathbb{S}^n / C \simeq \mathbb{S}^n_a \vee \mathbb{S}^n_b$, where C is the equator of $\mathbb{S}^n$. We have

$$\begin{aligned}\psi[\alpha+\beta] &= (\alpha+\beta)_* [S^n] = (\alpha \vee \beta)_* \pi_* [S_n] = \alpha_* p_{a*} \pi_* [S^n] + \beta_* p_{b*} \pi_* [S^n] \\ &= \alpha_* [S^n] + \beta_* [S^n] = \psi[\alpha] + \psi[\beta],\end{aligned}$$

because $p_a \pi \sim \mathrm{id}_{\mathbb{S}^n}$ and similarly for b. □

Corollary 2.80. *The Hurewicz homomorphism of the sphere $\psi : \pi_n(\mathbb{S}^n, *) \to \widetilde{H}_n(\mathbb{S}^n)$ is surjective because $\psi(\mathrm{id}_{\mathbb{S}^n}) = [S^n]$.*

Proposition 2.81. *If $\rho : \mathbb{S}^n \to \mathbb{S}^n$ is a reflection in a hyperplane, then $\deg \rho = -1$.*

Proof. Consider $R : (I^n, \partial I^n) \to (I^n, \partial I^n)$ given by $x \mapsto (1 - x_1, x_2, \dots, x_n)$. Then $\alpha + \alpha \circ R = 0$ for all $\alpha \in \pi_n(\mathbb{S}^n, *)$. Applying the Hurewicz homomorphism, we obtain $\alpha_* [S^n] = -\alpha_* R_* [S^n]$ for all α, and in particular $R_* [S^n] = -[S^n]$ so $\deg R = -1$. Now there exists $f : (I^n, \partial I^n) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ with $f \circ R = \rho_1 \circ f$, where $\rho_1(x) = (-x_1, x_2, \dots, x_n)$; therefore $\rho_{1*}[D^n, S^{n-1}] = -[D^n, S^{n-1}]$. Finally, $\rho_{1*}[S^n] = \rho_{1*}\partial [D^n, S^{n-1}] = \partial \rho_{1*} [D^n, S^{n-1}] = -[S^n]$, so $\deg \rho_1 = -1$. Since any two reflections are homotopic, it follows that $\deg \rho = -1$ for all reflections ρ. □

Example 2.82. *In general, the Hurewicz homomorphism is neither injective nor surjective.*

- *$\pi_n(\mathbb{S}^2, *)$ is non trivial for many $n > 2$ but $\widetilde{H}_n(\mathbb{S}^2) = 0$ for $n > 2$, so $\psi : \pi_n(\mathbb{S}^2, *) \to \widetilde{H}_n(\mathbb{S}^2)$ cannot be injective.*

- *If $\alpha : \mathbb{S}^2 \to \mathbb{T}^2$, then α lifts to $\widetilde{\alpha} : \mathbb{S}^2 \to \mathbb{R}^2$, so $\alpha_* [S^2] = p_* \widetilde{\alpha}_* [S^2] = 0$ since $\widetilde{H}_2(\mathbb{R}^2) = 0$. Therefore, $\psi : \pi_n(\mathbb{T}^2, *) \to \widetilde{H}_2(\mathbb{T}^2)$ is not surjective.*

Theorem 2.83 (Hurewicz)**.** *Let X be a path-connected space.*

(i) *The group $H_1(X)$ is isomorphic to the abelianisation of $\pi_1(X, *)$.*

(ii) *If $\pi_k(X, *) = 0$ for all $1 \leqslant k \leqslant n$, then $\psi : \pi_{n+1}(X, *) \to H_{n+1}(X)$ is an isomorphism and $H_k(X) = 0$ for all $1 \leqslant k \leqslant n$.*

Corollary 2.84. *Let X be a path-connected space. If $\pi_1(X, *) = 0$ and $H_k(X) = 0$ for all $1 \leqslant k \leqslant n$, then $\pi_k(X, *) = 0$ for all $1 \leqslant k \leqslant n$ and $\pi_{n+1}(X, *) \simeq H_{n+1}(X)$.*

Example 2.85. *$\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ and $\pi_k(\mathbb{S}^n) = 0$ for $1 \leqslant k < n$.*

2.11 Local degree of a map of the sphere

Notation 2.86. *If $p \in \mathbb{S}^n$, the space $\mathbb{S}^n \backslash \{p\} \simeq \mathbb{R}^n$ is contractible and we have an isomorphism $\pi_* : \widetilde{H}_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{p\})$. We define $[S^n, S^n \backslash \{p\}] = \pi_* [S^n]$.*

Likewise, if $U \subseteq \mathbb{S}^n$ is open and $p \in U$, we have an isomorphism $\iota_* : H_n(U, U \backslash \{p\}) \to H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{p\})$ by excision. We therefore define $[U, U \backslash \{p\}] = \iota_*^{-1} [S^n, S^n \backslash \{p\}]$.

Definition 2.87 (Local degree)**.** *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ and $q \in \mathbb{S}^n$ be such that $f^{-1}(f(q)) = \{q_1, \dots, q_N\}$ is finite. For each $1 \leqslant i \leqslant N$, pick an open subset $U_i \subseteq \mathbb{S}^n$ containing q_i s.t. $U_i \cap U_j = \varnothing$ for $i \neq j$. Then f can be seen as a map $f : (U, U \backslash \{q\}) \to (\mathbb{S}^n, \mathbb{S}^n \backslash \{f(q)\})$, so the induced map satisfies*

$$f_* [U, U \backslash \{q\}] = \kappa [S^n, S^n \backslash \{f(q)\}],$$

for some $\kappa \in \mathbb{Z}$.

The local degree *of f at q is defined by* $\deg_q f = \kappa$, *assuming that $f^{-1}(f(q))$ is finite.*

Theorem 2.88. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ and let $p \in \mathbb{S}^n$ s.t. $f^{-1}(p)$ is finite. Then*

$$\deg f = \sum_{q \in f^{-1}(p)} \deg_q f.$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & \widetilde{H}_n(\mathbb{S}^n) \arrow[r, "f_*"] \arrow[d] \arrow[dr] \arrow[dl, bend right, "\gamma"'] \arrow[d, bend right=60, "\beta'"'] \arrow[d, phantom, shift left=25, "\alpha = \beta"] & \widetilde{H}_n(\mathbb{S}^n) \arrow[d, "\pi_*", "\simeq"'] \\
 & H_n(\mathbb{S}^n, \mathbb{S}^n \backslash f^{-1}(p)) \arrow[r] & H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{p\}) \\
\bigoplus_i H_n(\mathbb{S}^n, \mathbb{S}^n \backslash \{q_i\}) & \bigoplus_i H_n(U_i, U_i \backslash \{q_i\}) \arrow[l, "j_*"', "\simeq"] \arrow[u, "\iota_*"', "\simeq"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Let $\beta = \delta\beta'$ and $\alpha = \pi_* f_*$. We have

$$\alpha [S^n] = (\deg f) [S^n, S^n \backslash \{p\}] = \beta [S^n].$$

Note that

$$\beta' [S^n] = j_*^{-1} \gamma [S^n] = j_*^{-1} \bigoplus_i [S^n, S^n \backslash \{q_i\}] = \bigoplus_i [U_i, U_i \backslash \{q_i\}]$$

This implies that

$$\beta [S^n] = \delta \beta' [S^n] = \sum_i \left(\deg_{q_i} f \right) [S^n, S^n \backslash \{p\}],$$

which proves the result. ☐

2.12 Finite cell complexes

Definition 2.89 (Glueing along a map)**.** *Let $A \subseteq X$ and $B \subseteq Y$ and consider a map $f : B \to A$. We define*

$$X \cup_f Y = (X \amalg Y) / \sim,$$

where $\sim$ is the equivalence relation given by $b \sim f(b)$ for all $b \in B$.

If $(Y, B) = \left(\mathbb{D}^k, \mathbb{S}^{k-1}\right)$, we say that $X \cup_f \mathbb{D}^k$ is obtained by attaching a k-cell *to X.*

Definition 2.90 (Finite cell complex)**.** *An n-dimensional* finite cell complex *consists of*

(i) *A space X,*

(ii) *Closed subspaces $\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$, with each X_k called the k-*skeleton *of X,*

(iii) *Such that X_k is obtained by attaching finitely many k-cells to X_{k-1}.*

In other words, there is a finite set A_k and maps $\left(\iota_\alpha : \mathbb{D}^k \to X_k\right)_{\alpha \in A_k}$ s.t. $\iota_\alpha \left(\mathbb{S}^{k-1}\right) \subseteq X_{k-1}$ and we have an isomorphism

$$\left(\coprod_{\alpha \in A_k} \iota_\alpha \right) : \coprod_{\alpha \in A_k} \mathring{\mathbb{D}}^k \xrightarrow{\simeq} X_k \backslash X_{k-1}.$$

Example 2.91. (i) *$\mathbb{S}^k$ is a cell complex formed of one 0-cell and one k-cell.*

(ii) *$\vee^n \mathbb{S}^k$ is a cell complex formed of one 0-cell and n k-cells.*

(iii) $\mathbb{S}^1$ *is a cell complex formed of two* 0*-cells and two* 1*-cells.*

(iv) $\mathbb{T}^2$ *is a cell complex formed of one* 0*-cell, two* 1*-cells and one* 2*-cell.*

Definition 2.92 (Complex projective space)**.** *The n-dimensional* complex projective space *is defined by*

$$\begin{aligned}\mathbb{CP}^n &= \left(\mathbb{C}^{n+1}\backslash\{0\}\right)/\mathbb{C}^* \\ &= \underbrace{\left\{z \in \mathbb{C}^{n+1},\ |z| = 1\right\}}_{\mathbb{S}^{2n+1}}/\mathbb{S}^1.\end{aligned}$$

The projection map $\mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$ is called the Hopf map*. Given $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}\backslash\{0\}$, we write $[z_0 : \cdots : z_n]$ for its image in $\mathbb{CP}^n$.*

Proposition 2.93. *$\mathbb{CP}^n$ is obtained by attaching a $2n$-cell to $\mathbb{CP}^{n-1}$.*

By induction, it follows that $\mathbb{CP}^n$ is a finite cell complex with one 0*-cell, one* 2*-cell, …, one $2n$-cell.*

Proof. Consider the embedding $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ given by $[z_0 : \cdots : z_{n-1}] \mapsto [z_0 : \cdots : z_{n-1} : 0]$. Consider also the map

$$\iota : (z_0, \ldots, z_{n-1}) \in \mathbb{D}^{2n} \subseteq \mathbb{C}^n \longmapsto \left[z_0 : \cdots : z_{n-1} : 1 - \|z\|^2\right] \in \mathbb{CP}^n.$$

We see that $\iota\left(\mathbb{S}^{2n-1}\right) = \mathbb{CP}^{n-1}$, and we have an isomorphism

$$\iota_{|\mathring{\mathbb{D}}^{2n}} : \mathring{\mathbb{D}}^{2n} \xrightarrow{\simeq} \mathbb{CP}^n\backslash\mathbb{CP}^{n-1}.$$ □

Corollary 2.94. $\mathbb{CP}^1 \simeq \mathbb{S}^2$.

Proposition 2.95. $H_*\left(\mathbb{CP}^n\right) = \begin{cases}\mathbb{Z} & \text{if } * = 0, 2, \ldots, 2n \\ 0 & \text{otherwise}\end{cases}.$

Proof. We proceed by induction on n. Since $\mathbb{CP}^0$ is a point, the result is clear. For $n \geqslant 0$, $\left(\mathbb{CP}^n, \mathbb{CP}^{n-1}\right)$ is a good pair, so

$$H_*\left(\mathbb{CP}^n, \mathbb{CP}^{n-1}\right) \simeq \widetilde{H}_*\left(\mathbb{CP}^n/\mathbb{CP}^{n-1}\right) \simeq \widetilde{H}_*\left(\mathbb{S}^{2n}\right) = \begin{cases}\mathbb{Z} & \text{if } * = 2n \\ 0 & \text{otherwise}\end{cases}.$$

Writing the long exact sequence of $\left(\mathbb{CP}^n, \mathbb{CP}^{n-1}\right)$ and using the fact that $H_{2n}\left(\mathbb{CP}^n, \mathbb{CP}^{n-1}\right) \xrightarrow{\partial} H_{2n-1}\left(\mathbb{CP}^{n-1}\right)$ is zero by induction, we obtain:

$$H_*\left(\mathbb{CP}^n\right) \simeq H_*\left(\mathbb{CP}^{n-1}\right) \oplus \widetilde{H}_*\left(\mathbb{S}^{2n}\right).$$ □

Definition 2.96 (Real projective space)**.** *The n-dimensional* real projective space *is defined by*

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1}\backslash\{0\}\right)/\mathbb{R}^* = \mathbb{S}^n/\sim,$$

where $\sim$ is the antipodal equivalence relation.

$\mathbb{RP}^n$ is a finite cell complex with one 0*-cell, one* 1*-cell, …, one n-cell.*

Remark 2.97. *The argument we used to compute the homology of $\mathbb{CP}^n$ in Proposition 2.95 won't work for $\mathbb{RP}^n$ as is. To make it work, we introduce the notion of cellular chain complexes.*

2.13 Cellular homology

Definition 2.98 (Cellular chain complex)**.** *Let X be an n-dimensional finite cell complex with k-skeleton X_k. Define the* cellular chain complex *$C^{\text{cell}}(X)$ by*

$$C_k^{\text{cell}}(X) = H_k\left(X_k, X_{k-1}\right),$$

and $d_k^{\text{cell}} : C_k^{\text{cell}}(X) \to C_{k-1}^{\text{cell}}(X)$ is the boundary map in the long exact sequence of the triple (X_k, X_{k-1}, X_{k-2}).

Lemma 2.99. *Let X be a finite cell complex. Then*

$$d_k^{\text{cell}} = \pi_{k-1}\partial_k,$$

where $\partial_k : H_k\left(X_k, X_{k-1}\right) \to H_{k-1}\left(X_{k-1}\right)$ is the boundary map of the pair (X_k, X_{k-1}) and $\pi_{k-1} : H_{k-1}\left(X_{k-1}\right) \to H_{k-1}\left(X_{k-1}, X_{k-2}\right)$ is the map induced by the projection.

Proof. Let $[c] \in H_k\left(X_k, X_{k-1}\right)$, $c \in C_k\left(X_k\right)$, $dc \in C_{k-1}\left(X_{k-1}\right)$. Then $\partial_k[c] = [dc] \in H_{k-1}\left(X_{k-1}\right)$, and $d_k^{\text{cell}}[c] = [dc] \in H_{k-1}\left(X_{k-1}, X_{k-2}\right)$, which shows that $\pi_{k-1}\partial_k[c] = d_k^{\text{cell}}(c)$. □

Corollary 2.100. *The cellular chain complex is a chain complex, i.e. $\left(d^{\text{cell}}\right)^2 = 0$.*

Proof. By Lemma 2.99, we have $d_k^{\text{cell}} d_{k+1}^{\text{cell}} = \pi_{k-1}\partial_k\pi_k\partial_{k+1}$. Now, writing the long exact sequence of (X_k, X_{k-1}), we have

$$\cdots \to H_k\left(X_k\right) \xrightarrow{\pi_k} H_k\left(X_k, X_{k-1}\right) \xrightarrow{\partial_k} H_{k-1}\left(X_{k-1}\right) \to \cdots,$$

so $\partial_k\pi_k = 0$. □

Remark 2.101. *Suppose given maps $\left(\iota_\alpha : \mathbb{D}^k \to X_k\right)_{\alpha \in A_k}$ as in Definition 2.90. Since the pair (X_k, X_{k-1}) is good, it follows that*

$$C_k^{\text{cell}}(X) = H_k\left(X_k, X_{k-1}\right) \simeq \widetilde{H}_k\left(X_k/X_{k-1}\right) \simeq \widetilde{H}_k\left(\bigvee_{\alpha \in A_k} \mathbb{S}_\alpha^k\right) \simeq \bigoplus_{\alpha \in A_k} \mathbb{Z}e_\alpha^k,$$

where $e_\alpha^k = \iota_{\alpha *}\left[D^k, S^{k-1}\right] \in H_k\left(X_k, X_{k-1}\right)$. To determine the boundary map, note that $d_k^{\text{cell}} e_\alpha^k = \pi_{k-1}\partial_k e^k\alpha = \pi_{k-1}\iota_{\alpha *}\left[S^{k-1}\right]$, and therefore

$$de_\alpha^k = \sum_{\beta \in A_{k-1}} n_{\alpha\beta} e_\beta^{k-1},$$

where $n_{\alpha\beta}$ is the degree of the composite

$$\mathbb{S}^{k-1} \xrightarrow{\iota_\alpha} X_{k-1} \xrightarrow{\pi_{k-1}} X_{k-1}/X_{k-2} \simeq \bigvee_{\beta \in A_{k-1}} \mathbb{S}_\beta^{k-1} \to \mathbb{S}_\beta^{k-1}.$$

Example 2.102. *Since $\mathbb{RP}^n$ is a cell complex with one 0-cell, one 1-cell, ..., one n-cell, we have $C_k^{\text{cell}}\left(\mathbb{RP}^n\right) = \mathbb{Z}e^k$ if $0 \leqslant k \leqslant n$, 0 otherwise. To compute de^k, we consider the composite f of $\mathbb{S}^{k-1} \xrightarrow{\pi} \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \simeq \mathbb{S}^{k-1}$. We note that the preimage of $p \in \mathbb{S}^{k-1}$ will consist of a pair $\{q, aq\} \subseteq \mathbb{S}^{k-1}$, where $a : \mathbb{S}^{k-1} \to \mathbb{S}^{k-1}$ is the antipodal map. It follows that* $\deg f = \deg_q f + \deg_{aq} f = \deg_q f\,(1 + \deg a)$*. But f is a homeomorphism near q, so $\deg_q f = 1$, which implies that*

$$de^k = \left(1 + (-1)^k\right) e^{k-1}.$$

This determines entirely the chain complex $C^{\text{cell}}\left(\mathbb{RP}^n\right)$. Our aim is now to show that the homology of $C^{\text{cell}}(X)$ is isomorphic to the singular homology of X.

Lemma 2.103. *If X is a finite cell complex with one 0-cell and all other cells of dimension at least m, then $\widetilde{H}_*(X_k) = 0$ unless $m \leqslant * \leqslant k$.*

Proof. We use induction on k. If $k < m$, then $X_k = X_0 = \{p\}$, so $\widetilde{H}_*(X_k) = 0$. If $k = m$, then $X_k = X_m \simeq \vee^r \mathbb{S}^m$, so $\widetilde{H}_*(X_k) = 0$ unless $* = m$. Now suppose the statement holds for X_{k-1}, i.e. $\widetilde{H}_*(X_{k-1}) = 0$ unless $m \leqslant * \leqslant k-1$. Moreover, we have $H_*(X_k, X_{k-1}) \simeq \widetilde{H}_*(X_k/X_{k-1}) \simeq \widetilde{H}_*\left(\vee^s \mathbb{S}^k\right) = 0$ unless $* = k$. The long exact sequence of (X_k, X_{k-1}) is

$$\cdots \to \widetilde{H}_*(X_{k-1}) \to \widetilde{H}_*(X_k) \to H_*(X_k, X_{k-1}) \to \cdots .$$

Since $\widetilde{H}_*(X_{k-1})$ and $H_*(X_k, X_{k-1})$ are both zero unless $m \leqslant * \leqslant k$, it follows that $\widetilde{H}_*(X_k) = 0$ unless $m \leqslant * \leqslant k$. □

Corollary 2.104. *If X is a finite cell complex, then $H_k(X) \simeq H_k(X_{k+1})$.*

Proof. Write the long exact sequence of (X, X_{k+1}):

$$\cdots \to H_{k+1}(X, X_{k+1}) \to H_k(X_{k+1}) \xrightarrow{j_*} H_k(X) \to H_k(X, X_{k+1}) \to \cdots .$$

But $H_*(X, X_{k+1}) \simeq \widetilde{H}_*(X/X_{k+1})$ and X/X_{k+1} has all cells of dimension at least $k+2$ (except for one 0-cell), so $H_{k+1}(X, X_{k+1}) = H_k(X, X_{k+1}) = 0$ by Lemma 2.103. This implies that j_* is an isomorphism. □

Theorem 2.105. *Let X be a finite cell complex. Then*

$$H_*(X) \simeq H_*^{\text{cell}}(X),$$

where $H_*^{\text{cell}}(X) = H_*\left(C^{\text{cell}}(X)\right)$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & H_k(X_{k+1}, X_k) & \\
H_k(X_{k-1}) \arrow[dr] & & H_k(X_{k+1}) \arrow[ur] & & \\
 & H_k(X_k) \arrow[ur, "i"] \arrow[dr, "\pi_k"] & & & \\
H_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "d_{k+1}"] & & H_k(X_k, X_{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial_k"'] & & H_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & H_{k-1}(X_{k-1}) \arrow[ur, "\pi_{k-1}"'] & \\
 & & H_{k-1}(X_{k-2}) \arrow[ur] & &
\end{tikzcd}$$

The diagonal lines are exact and the horizontal line is the chain complex $C^{\text{cell}}(X)$. Note that the blue groups are zero by Lemma 2.103. This implies that π_{k-1} and π_k are injective, and i is surjective. Therefore

$$\operatorname{Ker} d_k = \operatorname{Ker}(\pi_{k-1} \circ \partial_k) = \partial_k^{-1}(\operatorname{Ker} \pi_{k-1}) = \operatorname{Ker} \partial_k = \operatorname{Im} \pi_k \simeq H_k(X_k),$$

and this isomorphism $H_k(X_k) \xrightarrow{\pi_k} \operatorname{Ker} d_k$ maps $\operatorname{Im} \partial_{k+1}$ to $\operatorname{Im} d_{k+1}$ because $\pi_k \circ \partial_{k+1} = d_{k+1}$. Therefore

$$H_k^{\text{cell}}(X) = \operatorname{Ker} d_k / \operatorname{Im} d_{k+1} \simeq H_k(X_k) / \operatorname{Im} \partial_{k+1} = \operatorname{Coker} \partial_{k+1} \simeq \operatorname{Im} i = H_k(X_{k+1}).$$

But Corollary 2.104 implies that $H_k(X_{k+1}) \simeq H_k(X)$; the result follows. □

Corollary 2.106 (Dimension Axiom)**.** *If X is a finite cell complex of dimension n, then $H_*(X) = 0$ for $* > n$.*

Corollary 2.107. *If X is a finite cell complex, then $H_*(X)$ is a finitely generated abelian group:*

$$H_*(X) = \mathbb{Z}^N \oplus T,$$

where T is a torsion group.

Corollary 2.108. (i) $H_*\left(\mathbb{RP}^{2n}\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * = 1, 3, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases}.$

(ii) $H_*\left(\mathbb{RP}^{2n+1}\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2n+1 \\ \mathbb{Z}/2 & \text{if } * = 1, 3, \ldots, 2n-1 \\ 0 & \text{otherwise} \end{cases}.$

Theorem 2.109 (Whitehead)**.** *If X and Y are connected finite cell complexes and $f : X \to Y$ is a map such that the induced maps $f_* : \pi_i(X) \xrightarrow{\simeq} \pi_i(Y)$ are isomorphisms for all $i \geqslant 1$, then f is a homotopy equivalence.*

Corollary 2.110. *If X and Y are simply connected finite cell complexes and $f : X \to Y$ is a map such that the induced map $f_* : H_*(X) \xrightarrow{\simeq} H_*(Y)$ is an isomorphism, then f is a homotopy equivalence.*

Corollary 2.111. *Suppose X is a simply connected finite cell complex with trivial homology. Then X is contractible.*

3 Cohomology and products

3.1 Homology with coefficients

Remark 3.1. *If (C, d) is a chain complex over R and M is an R-module, then $(C \otimes M, d \otimes \mathrm{id}_M)$ is a chain complex over R.*

Example 3.2. $C_*^{\text{cell}}\left(\mathbb{RP}^2\right) = \left(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}\right)$ *and* $C_*^{\text{cell}}\left(\mathbb{RP}^2\right) \otimes \mathbb{Z}/2 = \left(\mathbb{Z}/2 \xrightarrow{\times 0} \mathbb{Z}/2 \xrightarrow{\times 0} \mathbb{Z}/2\right)$.

Note that $H_*\left(C^{\text{cell}}\left(\mathbb{RP}^2\right) \otimes \mathbb{Z}/2\right) \not\simeq H_*\left(C^{\text{cell}}\left(\mathbb{RP}^2\right)\right) \otimes \mathbb{Z}/2$.

Definition 3.3 (Singular homology with coefficients)**.** *Let G be an abelian group (i.e. a $\mathbb{Z}$-module) and let X be a topological space. We define:*

- *The* singular chain complex of X with coefficients in G *by* $C_*(X; G) = C_*(X) \otimes_{\mathbb{Z}} G$.
- *The* singular homology of X with coefficients in G *by* $H_*(X; G) = H_*(C(X; G))$.

We define similarly $C_*^{\text{cell}}(X; G)$ and $H_*^{\text{cell}}(X; G)$ if X is a finite cell complex, $C_*(X, A; G)$ and $H_*(X, A; G)$ if (X, A) is a pair (in that case, $C_*(X, A; G) = C_*(X; G)/C_*(A; G)$).

Remark 3.4. *If R is a ring, then $C(X; R)$ is a chain complex over R.*

Proposition 3.5. *A map $f : X \to Y$ induces a chain map $f_\sharp \otimes \mathrm{id}_G : C_*(X; G) \to C_*(Y; G)$ and therefore a map $f_* : H_*(X; G) \to H_*(Y; G)$.*

This defines a (covariant) functor **Top** $\to$ **AbGp**.

Proposition 3.6. *Given an element $g \in G$, there is a chain map $C_*(X) \to C_*(X; G)$ given by $x \mapsto x \otimes g$, and which induces a map $H_*(X) \to H_*(X; G)$. For any map $f : X \to Y$, we have a commutative square:*

$$\begin{array}{ccc} H_*(X) & \xrightarrow{f_*} & H_*(Y) \\ \cdot\otimes g \downarrow & & \cdot\otimes g \downarrow \\ H_*(X;G) & \xrightarrow{f_*} & H_*(Y;G) \end{array}$$

Definition 3.7 (Reduced singular homology with coefficients)**.** *If X is a space and G is an abelian group, we define*

$$\widetilde{H}_*(X;G) = \operatorname{Ker}\left(H_*(X;G) \xrightarrow{f_*} H_*(\{p\};G)\right),$$

with $f : X \to \{p\}$.

Theorem 3.8. *If X is a finite cell complex, then $H_*(X;G) \simeq H_*^{\text{cell}}(X;G)$.*

Proof. The proof is done in several steps:

(i) Show that $H_*(-;G)$ defines a functor **Pair** $\to$ **AbGp** from the category of pairs of spaces to the category of abelian groups.

(ii) If $f \sim g$, show that $f_* = g_*$.

(iii) If $f : (X, A) \to (Y, B)$ is a map of pairs, show that there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow H_*(A;G) & \xrightarrow{\iota_*} & H_*(X;G) & \xrightarrow{\pi_*} & H_*(X,A;G) & \xrightarrow{\partial} & H_{*-1}(A;G) \to \cdots \\ f_* \downarrow & & f_* \downarrow & & f_* \downarrow & & f_* \downarrow \\ \cdots \longrightarrow H_*(B;G) & \xrightarrow{\iota_*} & H_*(Y;G) & \xrightarrow{\pi_*} & H_*(Y,B;G) & \xrightarrow{\partial} & H_{*-1}(B;G) \to \cdots \end{array}$$

(iv) If $\overline{B} \subseteq \mathring{A}$, show that we have the Excision Property: $j_* : H_*(X\backslash B, A\backslash B; G) \xrightarrow{\simeq} H_*(X,A;G)$ is an isomorphism.

Properties (i) – (iv) mean that $H_*(-;G)$ is a *generalised homology theory.* Then:

(v) Show that $H_*(\{p\};G) = \begin{cases} G & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}.$

(vi) Show that $\widetilde{H}_*(\mathbb{S}^n;G) \simeq \widetilde{H}_*(\mathbb{D}^n, \mathbb{S}^{n-1};G) = \begin{cases} G & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$

(vii) If $f : \mathbb{S}^n \to \mathbb{S}^n$, we have a commutative square:

$$\begin{array}{ccc} \widetilde{H}_n(\mathbb{S}^n) & \xrightarrow{f_*} & \widetilde{H}_n(\mathbb{S}^n) \\ \cdot\otimes g \downarrow & & \cdot\otimes g \downarrow \\ \widetilde{H}_n(\mathbb{S}^n;G) & \xrightarrow{f_*} & \widetilde{H}_n(\mathbb{S}^n;G) \end{array}$$

It follows that $f_* : \widetilde{H}_n(\mathbb{S}^n;G) \to \widetilde{H}_n(\mathbb{S}^n;G)$ is given by multiplication by $\deg f$.

Then complete the proof as in Theorem 2.105. □

Example 3.9. $H_*(\mathbb{RP}^n;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \textit{if } * = 0,1,\ldots,n \\ 0 & \textit{otherwise} \end{cases}.$

3.2 Cohomology

Definition 3.10 (Cochain complex)**.** *Let R be a commutative ring. A* cochain complex (C, d) *over R consists of R-modules C^i for $i \in \mathbb{Z}$, and homomorphisms $d^i : C^i \to C^{i+1}$, satisfying $d^{i+1} \circ d^i = 0$ for all i. We write:*

$$\cdots \leftarrow C^{i+1} \xleftarrow{d^i} C^i \xleftarrow{d^{i-1}} C^{i-1} \leftarrow \cdots$$

We shall denote $C^* = \bigoplus_{i \in \mathbb{Z}} C^i$.

The cohomology *of (C, d) is defined by $H^k(C) = \frac{\operatorname{Ker} d^k}{\operatorname{Im} d^{k-1}}$.*

Remark 3.11. *If (C, d) is a chain complex over R and M is an R-module, then $(\operatorname{Hom}(C, M), d)$ is a cochain complex.*

Definition 3.12 (Singular cohomology)**.** *Let G be an abelian group (i.e. a $\mathbb{Z}$-module) and let X be a topological space. We define:*

- *The* singular cochain complex of X with coefficients in G *by $C^*(X; G) = \operatorname{Hom}(C^*(X), G)$.*
- *The* singular cohomology of X with coefficients in G *by $H^*(X; G) = H^*(C(X; G))$.*

We define similarly $C^*_{\text{cell}}(X; G)$ and $H^*_{\text{cell}}(X; G)$ if X is a finite cell complex.

Proposition 3.13. *A map $f : X \to Y$ induces a cochain map $f^\sharp : C^*(Y; G) \to C^*(X; G)$ and therefore a map $f^* : H^*(Y; G) \to H^*(X; G)$.*

This defines a contravariant functor **Top** $\to$ **AbGp**.

Theorem 3.14. *If X is a finite cell complex, then $H^*(X; G) \simeq H^*_{\text{cell}}(X; G)$.*

Example 3.15. $C^{\text{cell}}_*\left(\mathbb{RP}^2\right) = \left(\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}\right)$ *and* $C^*_{\text{cell}}\left(\mathbb{RP}^2; \mathbb{Z}\right) = \left(\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 0} \mathbb{Z}\right)$.

Note that $H^*\left(\mathbb{RP}^2; \mathbb{Z}\right) \not\simeq \operatorname{Hom}\left(H_*\left(\mathbb{RP}^2\right), \mathbb{Z}\right)$.

Example 3.16. *If M is a smooth manifold, then any differential form $\omega \in \Omega^k(M)$ defines a $\mathbb{R}$-cochain on smooth simplices $\sigma : \Delta^k \to M$ by*

$$\omega(\sigma) = \int_{\Delta^k} \sigma_*(\omega).$$

If $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative, then $\mathrm{d}\omega(\sigma) = \omega(d\sigma)$ by Stokes' Formula. In other words, the above defines a cochain map.

Theorem 3.17 (De Rham)**.** *If M is a smooth manifold, then $H^*(\Omega(M), \mathrm{d}) \simeq H^*(M; \mathbb{R})$.*

Definition 3.18 (Cohomology of pairs)**.** *If (X, A) is a pair, we define*

$$C^*(X, A; G) = \{a \in C^*(X; G) = \operatorname{Hom}(C_*(X), G),\ \operatorname{Ker} a \supseteq C_*(A)\}.$$

We have a short exact sequence $0 \to C^*(X, A; G) \to C^*(X; G) \to C^*(A; G) \to 0$, which gives the long exact sequence of a pair*:*

$$\cdots \to H^*(X, A; G) \to H^*(X; G) \to H^*(A; G) \to H^{*+1}(X, A; G) \to \cdots.$$

Lemma 3.19. *There is a bilinear pairing $\langle \cdot, \cdot \rangle : C^k(X; G) \times C_k(X) \longrightarrow G$ defined by $\langle a, x \rangle = a(x)$. It descends to a pairing*

$$\langle \cdot, \cdot \rangle : H^*(X; G) \times H_*(X) \longrightarrow G.$$

Proof. First note that $\langle da, x \rangle = \langle a, dx \rangle$ and if $f : X \to Y$, then $\left\langle f^\sharp a, x \right\rangle = \langle a, f_\sharp x \rangle$. We must now check that $\langle a + db, x + dy \rangle = \langle a, x \rangle$ when $da = 0$ and $dx = 0$. Indeed

$$\langle a + db, x + dy \rangle = \langle a, x \rangle + \langle b, dx \rangle + \langle da, y \rangle + \left\langle b, d^2 y \right\rangle = \langle a, x \rangle. \qquad \square$$

3.3 Free chain complexes over a PID

Definition 3.20 (Short injective chain complex)**.** *A chain complex (C, d) over a ring R is said to be* short injective *if*

(i) $C_* = 0$ *for* $* \neq k, k+1$,

(ii) C_k, C_{k+1} *are free over* R,

(iii) $d_{k+1} : C_{k+1} \to C_k$ *is injective.*

In other words, (C, d) *has the form*

$$0 \to C_{k+1} \hookrightarrow C_k \to 0.$$

In particular, $H_*(C) = \begin{cases} C_k/C_{k+1} & \text{if } * = k \\ 0 & \text{otherwise} \end{cases}.$

Lemma 3.21. *If (C, d) is short injective and $d_{k+1} : C_{k+1} \to C_k$ is invertible, then (C, d) is contractible.*

Proof. Set $h = d_{k+1}^{-1} : C_k \to C_{k+1}$ and check that $dh + hd = \mathrm{id}_{C^*}$, which proves that (C, d) is contractible. □

Proposition 3.22. *A few facts from commutative algebra:*

(i) $\mathbb{Z}$, $k[t]$ *and* $k[t, t^{-1}]$ *are all principal ideal domains (PIDs), where k is a field.*

(ii) *If R is a PID, M is free over R and $N \subseteq M$, then N is free over R.*

(iii) *If $0 \to A \to B \to C \to 0$ is exact and C is free, then the sequence splits, i.e. $B \simeq A \oplus C$.*

Theorem 3.23. *If (C, d) is a free chain complex over a PID R, then it is isomorphic to a direct sum of short injective complexes.*

Proof. Let $Z_k = \mathrm{Ker}\left(C_k \xrightarrow{d_k} C_{k-1}\right) \subseteq C_k$ and $B_{k-1} = \mathrm{Im}\left(C_k \xrightarrow{d_k} C_{k-1}\right) \subseteq C_{k-1}$. Then $Z_k, B_k \subseteq C_k$, and C_k is free, so Z_k and B_k are free by Proposition 3.22. Moreover, we have a short exact sequence

$$0 \to Z_k \to C_k \to B_{k-1} \to 0.$$

Since B_{k-1} is free, Proposition 3.22 implies that $C_k \simeq Z_k \oplus B_{k-1}$. Moreover, $d_k(Z_k) = 0$ and $d_k(B_{k-1}) \subseteq Z_{k-1}$ since $d^2 = 0$ (note that this is a different object from $d_k(B_k) = 0$). In other words,

$$C_* = \bigoplus_{k \in \mathbb{Z}} \left(B_{k-1} \overset{d_k}{\hookrightarrow} Z_{k-1}\right).$$ □

Theorem 3.24 (Smith Normal Form)**.** *If R is a PID and $f : R^n \hookrightarrow R^m$ is injective, then there are bases $(e_i)_{1 \leqslant i \leqslant n}$ of R^n and $\left(e'_j\right)_{1 \leqslant j \leqslant m}$ of R^m such that $f(e_i) = a_i e'_i$ for $1 \leqslant i \leqslant n$, with $a_i \in R \backslash \{0\}$.*

Corollary 3.25. *If (C, d) is a free, finitely generated complex over a PID R, then it is isomorphic to a direct sum of complexes of the following forms:*

(i) $0 \to R \to 0$,

(ii) $0 \to R \xrightarrow{\times a} R \to 0$.

Proof. Apply Theorem 3.24 to each short injective summand of C_* in the decomposition given by Theorem 3.23. □

Corollary 3.26. *If (C, d) is a finitely generated complex over a field k, then it is homotopic to the complex $(H(C), 0)$.*

Proof. Since k is a field, note that (C, d) is free. Now apply Corollary 3.25 and note that complexes of type (ii) are contractible by Lemma 3.21 since any $a \in k \backslash \{0\}$ is invertible. □

3.4 The Universal Coefficient Theorems

Notation 3.27. *Suppose R is a PID and (C,d) is a free finitely generated chain complex over R. By the Structure Theorem for finitely generated modules over PIDs, we can write*

$$H_*(C) = F_* \oplus T_*,$$

where F_* is free and T_* is torsion. In the decomposition given by Corollary 3.25, summands of type (i) *account for F_* and summands of type* (ii) *account for T_*.*

Proposition 3.28. *Let (C,d) be a free, finitely generated chain complex over a PID R. Then*

(i) $H_k\left(C \otimes R/b\right) \simeq (F_k \otimes R/b) \oplus (T_k \otimes R/b) \oplus (T_{k-1} \otimes R/b)$.

(ii) $H^k\left(\operatorname{Hom}(C,R)\right) \simeq F_k \oplus T_{k-1}$.

(iii) $H^k\left(\operatorname{Hom}(C,R/a)\right) \simeq \operatorname{Hom}\left(F_k, R/a\right) \oplus \operatorname{Hom}\left(T_k, R/a\right) \oplus \operatorname{Hom}\left(T_{k-1}, R/a\right)$.

The general 'metatheorem' underlying this proposition is the fact that the groups $H_*(X;G)$ and $H^*(X;G)$ are determined by $H_*(X)$.

Proof. Check this for each summand in the decomposition given by Corollary 3.25. □

Remark 3.29. *We have only proved Proposition 3.28 for free, finitely generated chain complexes. Hence, this will only apply to the computation of cellular homology of finite cell complexes. But the result actually remains true for all free chain complexes, so it can be applied to the computation of singular homology in general.*

Example 3.30. *Suppose X is a topological space s.t.*

$$\widetilde{H}_*(X) = \begin{cases} \mathbb{Z}/4 & \text{if } * = 3 \\ \mathbb{Z} & \text{if } * = 2 \\ \mathbb{Z}/2 & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\widetilde{H}^*(X) = \begin{cases} \mathbb{Z}/4 & \text{if } * = 4 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \widetilde{H}_*\left(X;\mathbb{Z}/4\right) = \begin{cases} \mathbb{Z}/4 & \text{if } * = 3,4 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } * = 2 \\ \mathbb{Z}/2 & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Definition 3.31 (Free resolution)**.** *If M is an R-module, a* free resolution *of M is a free chain complex (C,d) with $C_k = 0$ for $k < 0$ and*

$$H_*(C) = \begin{cases} M & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Example 3.32. (i) *If M is free, then $0 \to M \to 0$ is a free resolution of M.*

(ii) *If R is a PID and $a \neq 0$, then $0 \to R \xrightarrow{\times a} R \to 0$ is a free resolution of R/a.*

(iii) *If $0 \to C_1 \hookrightarrow C_0 \to 0$ is short injective, then it is a free resolution of $H_*(C) = H_0(C)$.*

(iv) *If $R = \mathbb{C}[x,y]$ and $M = R/(x,y)$, then $R \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -y & x \end{pmatrix}} R \to 0$ is a free resolution of M.*

Definition 3.33 (Tor and Ext). *Let M, N be R-modules. We define*

$$\mathrm{Tor}_*^R(M,N) = H_*\left(C \otimes N\right) \qquad \text{and} \qquad \mathrm{Ext}_R^*(M,N) = H^*\left(\mathrm{Hom}(C,N)\right),$$

where C is a free resolution of M. This definition does not depend on the choice of C.

Example 3.34. (i) *If M is free, then*

$$\mathrm{Tor}_*(M,N) = \begin{cases} M \otimes N & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathrm{Ext}^*(M,N) = \begin{cases} \mathrm{Hom}(M,N) & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}.$$

(ii) *If R is a PID, $a, b \neq 0$, then*

$$\mathrm{Tor}_*\left(R/a, R/b\right) = \begin{cases} R/\gcd(a,b) & \text{if } * = 0,1 \\ 0 & \text{otherwise} \end{cases}.$$

(iii) *If $R = \mathbb{C}[x,y]$ and $M = R/(x,y)$, then*

$$\mathrm{Tor}_*(M,M) = H_*\left(M \xrightarrow{0} M^2 \xrightarrow{0} M \to 0\right) = \begin{cases} M & \text{if } * = 0,2 \\ M^2 & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 3.35. *If C is a free chain complex over a PID R, then*

$$H_k(C \otimes N) = \mathrm{Tor}_0\left(H_k(C), N\right) \oplus \mathrm{Tor}_1\left(H_{k-1}(C), N\right) \simeq \left(H_k(C) \otimes N\right) \oplus \mathrm{Tor}_1\left(H_{k-1}(C), N\right),$$

and

$$H^k\left(\mathrm{Hom}(C,N)\right) = \mathrm{Ext}^0\left(H_k(C), N\right) \oplus \mathrm{Ext}^1\left(H_{k-1}(C), N\right) \simeq \mathrm{Hom}\left(H_k(C), N\right) \oplus \mathrm{Ext}^1\left(H_{k-1}(C), N\right).$$

Proof. Since C is free, Theorem 3.23 implies that it suffices to check the result for a short injective complex. □

Corollary 3.36. *If X is a space such that $H_*(X)$ is free abelian, then*

$$H_*(X;G) = H_*(X) \otimes G \qquad \text{and} \qquad H^*(X;G) = \mathrm{Hom}\left(H_*(X), G\right).$$

Corollary 3.37. *If X is a space such that $H_*(X)$ is free abelian, then $H^*(X)$ is the dual of $H_*(X)$, and for any map $f : X \to Y$, the induced map $f^* : H^*(Y) \to H^*(X)$ is dual to $f_* : H_*(X) \to H_*(Y)$.*

Proof. This follows from the pairing formula $\langle f^*a, x\rangle = \langle a, f_*x\rangle$. □

3.5 Products

Notation 3.38. *If C is a chain complex and $x \in C_i$, we write $|x| = i$.*

Definition 3.39 (Tensor product of chain complexes). *If C and C' are chain complexes over R, then $C \otimes C'$ is the chain complex defined by*

$$\left(C \otimes C'\right)_k = \bigoplus_{i+j=k} \left(C_i \otimes C'_j\right),$$

and $d\left(y \otimes y'\right) = dy \otimes y' + (-1)^{|y|} y \otimes d'y'$.

Proposition 3.40. *If Y and Y' are finite cell complexes and A_i (resp. A'_i) is the set of i-cells of Y (resp. Y'), then $Z = Y \times Y'$ is a finite cell complex and the set of k-cells of Z is in bijection with*

$$\left\{(\alpha, \alpha'),\ \alpha \in A_i,\ \alpha' \in A'_j,\ i + j = k\right\}.$$

Proof. Let $Z_k = \bigcup_{i+j=k} Y_i \times Y'_j$. If $\alpha \in A_i$ and $\alpha' \in A'_j$, we have $\iota_\alpha : \mathbb{D}^i \to Y_i$ and $\iota_{\alpha'} : \mathbb{D}^j \to Y'_j$, from which we obtain

$$\iota_\alpha \times \iota_{\alpha'} : \underbrace{\mathbb{D}^i \times \mathbb{D}^j}_{\simeq \mathbb{D}^{i+j}} \longrightarrow Y_i \times Y'_j \subseteq Z_k. \qquad \square$$

Theorem 3.41. *If Y and Y' are finite cell complexes, then*

$$C_*^{\text{cell}}\left(Y \times Y'\right) = C_*^{\text{cell}}(Y) \otimes C_*^{\text{cell}}\left(Y'\right).$$

Proof. We have an obvious correspondence at the level of chain groups given by Proposition 3.40; we need to check that it preserves the boundary map. $\square$

Example 3.42. *We wish to compute $H_*\left(\mathbb{RP}^2 \times \mathbb{RP}^2\right) \simeq H_*\left(C^{\text{cell}}\left(\mathbb{RP}^2\right) \otimes C^{\text{cell}}\left(\mathbb{RP}^2\right)\right)$. We represent the tensor product in a grid, as below:*

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "\times 2"'] & & \mathbb{Z} \arrow[d, "\times 2"'] & \mathbb{Z} \arrow[l, "\times 0"] \arrow[d, "\times(-2)"'] & \mathbb{Z} \arrow[l, "\times 2"] \arrow[d, "\times 2"'] \\
\mathbb{Z} \arrow[d, "\times 0"'] & & \mathbb{Z} \arrow[d, "\times 0"'] & \mathbb{Z} \arrow[l, "\times 0"] \arrow[d, "\times 0"'] & \mathbb{Z} \arrow[l, "\times 2"] \arrow[d, "\times 0"'] \\
\mathbb{Z} & & \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 0"] & \mathbb{Z} \arrow[l, "\times 2"] \\
 & & & & \\
 & & \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 0"] & \mathbb{Z} \arrow[l, "\times 2"]
\end{tikzcd}$$

Each diagonal line corresponds to one value of k in the complex $\left(C^{\text{cell}}\left(\mathbb{RP}^2\right) \otimes C^{\text{cell}}\left(\mathbb{RP}^2\right)\right)_k$. We obtain

$$H_*\left(\mathbb{RP}^2 \times \mathbb{RP}^2\right) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ (\mathbb{Z}/2)^2 & \text{if } * = 1 \\ \mathbb{Z}/2 & \text{if } * = 2, 3 \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 3.43 (Künneth Formula)**.** *If C, C' are free finitely generated complexes over a PID R, then*

$$H_*\left(C \otimes C'\right) \simeq \left(H_*(C) \otimes H_*\left(C'\right)\right) \oplus \operatorname{Tor}_1\left(H_*(C), H_*\left(C'\right)\right).$$

More precisely,

$$H_k\left(C \otimes C'\right) \simeq \left(\bigoplus_{i+j=k} H_i(C) \otimes H_j\left(C'\right)\right) \oplus \left(\bigoplus_{i+j=k-1} \operatorname{Tor}_1\left(H_i(C), H_j\left(C'\right)\right)\right).$$

In particular $H_*(X \times Y)$ is determined by $H_*(X)$ and $H_*(Y)$ for finite cell complexes X and Y.

Proof. By distributivity of the tensor product and Theorem 3.25, it suffices to check the result for chain complexes of types (i) and (ii). $\square$

Remark 3.44. *The Künneth Formula (Theorem 3.43) remains valid even if C and C' are not finitely generated.*

Corollary 3.45. *Suppose X and Y are finite cell complexes. If $H_*(X)$ is free over $\mathbb{Z}$, then*

$$H_*(X \times Y) \simeq H_*(X) \otimes H_*(Y).$$

This actually remains true for all topological spaces.

Proof. If M is free then $\mathrm{Tor}_1(M,N)=0$. $\square$

Corollary 3.46. *Suppose X and Y are finite cell complexes. If k is a field, then*

$$H_*\left(X\times Y;k\right)\simeq H_*(X;k)\otimes H_*(Y;k).$$

This actually remains true for all topological spaces.

Proof. Note that

$$\begin{aligned}C_*^{\mathrm{cell}}\left(X\times Y;k\right)&=\left(C_*^{\mathrm{cell}}(X)\otimes_{\mathbb{Z}}C_*^{\mathrm{cell}}(Y)\right)\otimes_{\mathbb{Z}}k\\&=\left(C_*^{\mathrm{cell}}(X)\otimes_{\mathbb{Z}}k\right)\otimes_k\left(C_*^{\mathrm{cell}}(Y)\otimes_{\mathbb{Z}}k\right)\\&=C_*^{\mathrm{cell}}\left(X;k\right)\otimes_k C_*^{\mathrm{cell}}\left(Y;k\right),\end{aligned}$$

and use Corollary 3.45 together with the fact that any module over k is free. $\square$

Example 3.47. $H_*\left(\mathbb{RP}^2\times\mathbb{RP}^2;\mathbb{Z}/2\right)=\begin{cases}\mathbb{Z}/2 & \text{if } *=0,4\\ (\mathbb{Z}/2)^2 & \text{if } *=1,3\\ (\mathbb{Z}/2)^3 & \text{if } *=2\\ 0 & \text{otherwise}\end{cases}.$

Definition 3.48 (Poincaré polynomial)**.** *If X is a space, we define the* Poincaré polynomial *of X over a field k by*

$$\mathcal{P}_k(X)=\sum_{i\geqslant 0}\left(\dim_k H_i\left(X;k\right)\right)t^i\in\mathbb{Z}[t].$$

Thus

$$\mathcal{P}_k(X\times Y)=\mathcal{P}_k(X)\times\mathcal{P}_k(Y).$$

Remark 3.49. *If $H_*(X)$ is free, then we have isomorphisms*

$$H_*(X;G)\simeq H_*(X)\otimes G\qquad\text{and}\qquad H^*(X;G)\simeq\mathrm{Hom}\left(H_*(X);G\right),$$

which are realised by natural maps. We would like to also have a natural map $H_*(X)\otimes H_*(Y)\xrightarrow{\simeq}H_*\left(X\times Y\right)$. Such a map exists, but it is painful to construct. This is why we introduce the cup product.

3.6 The cup product

Definition 3.50 (Cup product)**.** *If $\alpha\in C^k(X;R)$ and $\beta\in C^\ell(X;R)$, we define the* cup product *$\alpha\cup\beta\in C^{k+\ell}(X;R)$ of α and β by*

$$\left(\alpha\cup\beta\right)(\sigma)=\alpha\left(\sigma\circ F_{0,\dots,k}\right)\beta\left(\sigma\circ F_{k,\dots,k+\ell}\right)$$

for all $\sigma:\Delta^{k+\ell}\to X$.

Lemma 3.51. *$d(\alpha\cup\beta)=d\alpha\cup\beta+(-1)^{|\alpha|}\alpha\cup d\beta$.*

Proof. For every $\sigma : \Delta^{k+\ell+1} \to X$, we have

$$
\begin{aligned}
d(\alpha \cup \beta)(\sigma) = (\alpha \cup \beta)(d\sigma) = \sum_{j=0}^{k+\ell+1} (-1)^j (\alpha \cup \beta)\left(\sigma \circ F_{0,\dots,\hat{j},\dots,k+\ell+1}\right) \\
&= \sum_{j=0}^{k} (-1)^j \alpha\left(\sigma \circ F_{0,\dots,\hat{j},\dots,k+1}\right) \beta\left(\sigma \circ F_{k+1,\dots,k+\ell+1}\right) \\
&\qquad + \sum_{j=k+1}^{k+\ell+1} (-1)^j \alpha\left(\sigma \circ F_{0,\dots,k}\right) \beta\left(\sigma \circ F_{k,\dots,\hat{j},\dots,k+\ell+1}\right) \\
&= \sum_{j=0}^{k+1} (-1)^j \alpha\left(\sigma \circ F_{0,\dots,\hat{j},\dots,k+1}\right) \beta\left(\sigma \circ F_{k+1,\dots,k+\ell+1}\right) \\
&\qquad + \sum_{j=k}^{k+\ell+1} (-1)^j \alpha\left(\sigma \circ F_{0,\dots,k}\right) \beta\left(\sigma \circ F_{k,\dots,\hat{j},\dots,k+\ell+1}\right) \\
&= (d\alpha \cup \beta)(\sigma) + (-1)^k (\alpha \cup d\beta)(\sigma).
\end{aligned}
$$

□

Corollary 3.52. *The map* $\cup : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$ *descends to a map* $\cup : H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$ *given by*

$$
[\alpha] \cup [\beta] = [\alpha \cup \beta].
$$

Proof. Note that, if $d\alpha = d\beta = 0$, then, by Lemma 3.51,

$$
(\alpha + d\alpha') \cup (\beta + d\beta') = \alpha \cup \beta + d\alpha' \cup \beta + \alpha \cup d\beta' + d\alpha' \cup d\beta' = \alpha \cup \beta + d\left(\alpha' \cup \beta + \alpha \cup \beta' + \alpha' \cup d\beta'\right).
$$

□

Proposition 3.53. $H^*(X;R)$ *equipped with the cup product* $\cup$ *is a ring. Moreover, if* $f : X \to Y$ *is a map, then* $f^* : H^*(Y;R) \to H^*(X;R)$ *is a ring homomorphism.*

Proof. Define $1 \in \mathcal{C}^0(X;R)$ by $1(\sigma) = 1 \in R$ for all $\sigma : \Delta^0 \to R$. Then $(d1)(\tau) = 1(d\tau) = 1(\tau \circ F_1 - \tau \circ F_0) = 0$ for all $\tau : \Delta^1 \to R$, so $d1 = 0$ and we can define $1 = [1] \in H^0(X;R)$. We must check the ring axioms for $H^*(X;R)$. All of them are actually true at the level of cochains, for instance associativity:

$$
((\alpha \cup \beta) \cup \gamma)(\sigma) = \alpha\left(\sigma \circ F_{0,\dots,k}\right) \beta\left(\sigma \circ F_{k,\dots,k+\ell}\right) \gamma\left(\sigma \circ F_{k+\ell+1,\dots,k+\ell+m}\right) = (\alpha \cup (\beta \cup \gamma))(\sigma).
$$

Now given $f : X \to Y$, we have

$$
f^\sharp(\alpha \cup \beta)(\sigma) = (\alpha \cup \beta)(f_\sharp \sigma) = (\alpha \cup \beta)(f \circ \sigma) = \left(f^\sharp \alpha \cup f^\sharp \beta\right)(\sigma),
$$

and therefore $f^*([\alpha] \cup [\beta]) = f^*[\alpha] \cup f^*[\beta]$. □

Remark 3.54. *Let* M *be a smooth manifold. Then Theorem 3.17 provides a (group) isomorphism*

$$
H^*(\Omega(M), \mathrm{d}) \simeq H^*(M; \mathbb{R}).
$$

This is actually a ring isomorphism when $H^*(\Omega(M), \mathrm{d})$ *is equipped with* $\wedge$ *and* $H^*(M;\mathbb{R})$ *is equipped with* $\cup$.

Proposition 3.55. *If* $a, b \in H^*(X)$, *then*

$$
a \cup b = (-1)^{|a||b|} b \cup a.
$$

We say that $\cup$ *is* graded-commutative.

Proof. Consider the map

$$\rho : (v_0, \dots, v_k) \in \Delta^k \longmapsto (v_k, v_{k-1}, \dots, v_0) \in \Delta^k.$$

It induces a map $r_\sharp : C_*(X) \to C_*(X)$ defined by $r_\sharp(\sigma) = \varepsilon(|\sigma|)\,\sigma \circ \rho$ where $\varepsilon(k) = (-1)^{\frac{1}{2}k(k+1)}$. This is a chain map because

$$dr_\sharp(\sigma) = \varepsilon(k) \sum_{j=0}^{k} (-1)^j \sigma \circ \rho \circ F_{\hat{j}} = \varepsilon(k-1) \sum_{j=0}^{k} (-1)^k (-1)^j \sigma \circ F_{\widehat{k-j}} \circ \rho = \varepsilon(k-1) d\sigma \circ \rho = r_\sharp d(\sigma).$$

Moreover, we show that $r_\sharp$ is chain homotopic to $\mathrm{id}_{C_*(X)}$. Dualizing, we obtain $r^\sharp : C^*(X) \to C^*(X)$ with $r^\sharp \sim \mathrm{id}_{C^*(X)}$, therefore $\left[r^\sharp \alpha\right] = [\alpha]$ for all α. Now we have

$$r^\sharp(\alpha \cup \beta) = \underbrace{\frac{\varepsilon(|\alpha| + |\beta|)}{\varepsilon(|\alpha|)\,\varepsilon(|\beta|)}}_{=(1)^{|\alpha||\beta|}} r^\sharp(\beta) \cup r^\sharp(\alpha),$$

from which it follows that

$$[\alpha] \cup [\beta] = \left[r^\sharp(\alpha \cup \beta)\right] = (-1)^{|\alpha||\beta|} \left[r^\sharp(\beta) \cup r^\sharp(\alpha)\right] = (-1)^{|\alpha||\beta|} [\beta] \cup [\alpha]. \qquad \square$$

Remark 3.56. *For the rest of the section, we shall work over $R = \mathbb{Z}$, but the results will remain valid over any ring.*

Lemma 3.57. *Let (X, A) be a pair. Then the cup product defines a map $\cup : C^k(X, A) \times C^\ell(X) \to C^{k+\ell}(X, A)$, and this descends to a map*

$$\cup : H^k(X, A) \times H^\ell(X) \to H^{k+\ell}(X, A).$$

Moreover, for any $\beta \in H^*(X)$, the following square commutes:

$$\begin{tikzcd} H^*(X,A) \arrow[r] \arrow[d, "\cdot \cup \beta"'] & H^*(X) \arrow[d, "\cdot \cup \beta"'] \\ H^*(X,A) \arrow[r] & H^*(X) \end{tikzcd}$$

Example 3.58. (i) *If X is path-connected, then $H^0(X) = \langle 1 \rangle \simeq \mathbb{Z}$, where 1 is the neutral element for $\cup$.*

(ii) *$H^*(X \amalg Y) \simeq H^*(X) \times H^*(Y)$ as rings.*

(iii) *$H^*(\mathbb{S}^n) \simeq \mathbb{Z}[a]/(a^2)$ if $n > 0$.*

Proof. (i) Note that $H_0(X) \simeq \mathbb{Z}$ so $H^0(X) \simeq \mathbb{Z}$ by the Universal Coefficient Theorem (Corollary 3.37). Moreover, if $p \in X$, then $\langle 1, [\sigma_p] \rangle = 1$, which implies that 1 generates $H^0(X)$ (otherwise it would be a multiple of something, and so would $\langle 1, [\sigma_p] \rangle$).

(ii) There is an isomorphism $\left(\iota_X^\sharp \times \iota_Y^\sharp\right) : C^*(X \amalg Y) \to C^*(X) \times C^*(Y) = C^*(X) \oplus C^*(Y)$, where $\iota_X : X \hookrightarrow X \amalg Y$, and this isomorphism induces the claimed (group) isomorphism, which is also a ring homomorphism because ι_X^* and ι_Y^* are.

(iii) As a group,

$$H^*(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & \text{if } * = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

Let a be a generator of $H^n(\mathbb{S}^n)$. Then $H^*(\mathbb{S}^n) = \langle 1, a \rangle$, and we have the relations $1 \cup 1 = 1$, $1 \cup a = a \cup 1 = a$ and $a \cup a = 0$ since $H^{2n}(\mathbb{S}^n) = 0$. $\square$

3.7 The exterior product

Definition 3.59 (Exterior product)**.** *Let X and Y be two spaces, let $a \in H^k(X)$ and $b \in H^\ell(Y)$. The* exterior product *of a and b is defined by*

$$a \times b = \pi_1^*(a) \cup \pi_2^*(b) \in H^{k+\ell}(X \times Y),$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections.

Definition 3.60 (Generalised cohomology theory)**.** *A* generalised cohomology theory *is a contravariant functor $h^* : \mathbf{Pair} \to \mathbf{GrdMod}_{\mathbb{Z}}$ from the category of pairs of spaces to the category of graded $\mathbb{Z}$-modules, satisfying the following three axioms:*

(i) Homotopy invariance*: if $f \sim g$, then $f^* = g^*$.*

(ii) Functorial long exact sequence of a pair*: pairs have long exact cohomology sequences and this is functorial, i.e. the following diagram commutes for every $f : (X, A) \to (Y, B)$:*

$$\begin{tikzcd}
\cdots \arrow[r] & h^*(X,A) \arrow[r] & h^*(X) \arrow[r] & h^*(A) \arrow[r] & h^{*+1}(X,A) \arrow[r] & \cdots \\
\cdots \arrow[r] & h^*(Y,B) \arrow[r] \arrow[u, "f^*"] & h^*(Y) \arrow[r] \arrow[u, "f^*"] & h^*(B) \arrow[r] \arrow[u, "f^*"] & h^{*+1}(Y,B) \arrow[r] \arrow[u, "f^*"] & \cdots
\end{tikzcd}$$

(iii) Excision*: if $\overline{B} \subseteq \mathring{A}$, then the map*

$$h^*(X, A) \xrightarrow{\cong} h^*(X \backslash B, A \backslash B)$$

induced by the inclusion $(X \backslash B, A \backslash B) \to (X, A)$ is an isomorphism.

If h^* is a generalised cohomology theory, then it will also satisfy the following condition:

(iv) Collapsing a pair*: if (X, A) is a good pair, then we have an isomorphism*

$$\pi^* : h^*(X/A, \{*_A\}) \xrightarrow{\cong} h^*(X, A),$$

induced by the projection $\pi : (X, A) \to (X/A, \{*_A\})$.

Proposition 3.61. *Let $f : (X, A) \to (Y, B)$ be a map of pairs. If $f_* : H_*(X, A) \xrightarrow{\cong} H_*(Y, B)$ is an isomorphism, then $f^* : H^*(Y, B) \xrightarrow{\cong} H^*(X, A)$ is also an isomorphism.*

Remark 3.62. *There are contravariant functors $H^*, \overline{h}^*, \underline{h}^* : \mathbf{Pair} \to \mathbf{GrdMod}_{\mathbb{Z}}$, defined by*

$$\overline{h}^* \left((X_1, A_1) \xrightarrow{f} (X_2, A_2) \right) = H^*(X_2 \times Y, A_2 \times Y) \xrightarrow{(f \times \mathrm{id}_Y)^*} H^*(X_1 \times Y, A_1 \times Y),$$

and

$$\underline{h}^* \left((X_1, A_1) \xrightarrow{f} (X_2, A_2) \right) = H^*(X_2, A_2) \otimes H^*(Y) \xrightarrow{f^* \otimes \mathrm{id}_{H^*(Y)}} H^*(X_1, A_1) \otimes H^*(Y).$$

All three of these functors satisfy the axioms for a generalised cohomology theory.

Proof. The homotopy invariance is clear in each case. For the functorial long exact sequence of a pair, we already know the result for H^*. For $\overline{h}^*$, use the long exact sequence of $(X \times Y, A \times Y)$. For $\underline{h}^*$, use the fact that $H^*(Y)$ is free, so we can tensor long exact sequences. For excision, apply Proposition 3.61 to obtain the result for H^*, then use excision for $(X \times Y, A \times Y)$ to obtain it for $\overline{h}^*$, and tensor by $\mathrm{id}_{H^*(Y)}$ for $\underline{h}^*$. $\square$

Lemma 3.63. *Using the notations of Remark 3.62, we have a natural transformation* $\Phi : \underline{h}^* \to \overline{h}^*$ *defined by*

$$\Phi_{(X,A)} : a \otimes b \in \underline{h}^*(X, A) \longmapsto a \times b = \pi_1^*(a) \cup \pi_2^*(b) \in \overline{h}^*(X, A).$$

In other words, the following diagrams commute:

$$\begin{array}{ccc} (X,A) \\ f \downarrow \\ (X',A') \end{array} \qquad \begin{array}{ccc} \underline{h}^*(X,A) & \xrightarrow{\Phi_{(X,A)}} & \overline{h}^*(X,A) \\ \underline{f}^* \uparrow & & \overline{f}^* \uparrow \\ \underline{h}^*(X',A') & \xrightarrow{\Phi_{(X',A')}} & \overline{h}^*(X',A') \end{array} \qquad \begin{array}{ccc} \underline{h}^*(X,A) & \xrightarrow{\Phi_{(X,A)}} & \overline{h}^*(X,A) \\ \delta \downarrow & & \delta \downarrow \\ \underline{h}^{*+1}(A) & \xrightarrow{\Phi_A} & \overline{h}^{*+1}(A) \end{array}$$

Proof. We prove that the first square commute (we write $F = f \times \mathrm{id}_Y : X \times Y \to X' \times Y$ and note that $\pi_1' \circ F = f \circ \pi_1$ and $\pi_2' \circ F = \pi_2$):

$$\begin{aligned} \overline{f}^* \Phi_{(X',A')}(a \otimes b) &= \overline{f}^* \left({\pi_1'}^*(a) \cup {\pi_2'}^*(b) \right) = F^* \left({\pi_1'}^*(a) \right) \cup F^* \left({\pi_2'}^*(b) \right) \\ &= \left(\pi_1' \circ F \right)^* (a) \cup \left(\pi_2' \circ F \right)^* (b) = \pi_1^* f^*(a) \cup \pi_2^*(b) \\ &= f^*(a) \times b = \Phi_{(X,A)} \underline{f}^*(a \otimes b). \end{aligned}$$

□

Theorem 3.64. *If X is homotopic to a finite cell complex and Y is such that $H^*(Y)$ is free over $R = \mathbb{Z}$, then the map*

$$\Phi : H^*(X) \otimes H^*(Y) \xrightarrow{\simeq} H^*(X \times Y)$$

induced by the bilinear map $\times : H^*(X) \times H^*(Y) \to H^*(X \times Y)$ is an isomorphism.

This actually remains true for any topological spaces X and Y such that $H^*(Y)$ is free.

Proof. Given a pair (X, A), we denote by $\mathcal{P}(X, A)$ the statement that

$$\Phi_{(X,A)} : \underline{h}^*(X, A) \longrightarrow \overline{h}^*(X, A)$$

is an isomorphism.

(a) $\mathcal{P}(\mathbb{D}^0)$ and $\mathcal{P}(\mathbb{S}^0)$ hold. We use the facts that

$$\underline{h}^*\left(\mathbb{D}^0\right) \simeq \mathbb{Z} \otimes H^*(Y) \simeq H^*(Y) \simeq \overline{h}^*\left(\mathbb{D}^0\right)$$

and

$$\underline{h}^*\left(\mathbb{S}^0\right) \simeq \mathbb{Z}^2 \otimes H^*(Y) \simeq H^*(Y \amalg Y) \simeq \overline{h}^*\left(\mathbb{S}^0\right)$$

and we check that these isomorphisms are induced by Φ.

(b) If $X \sim X'$, then $\mathcal{P}(X) \Leftrightarrow \mathcal{P}(X')$. To prove this, write the naturality square of Φ associated to the homotopy equivalence $f : X \to X'$, and use the fact that both $\underline{f}^*$ and $\overline{f}^*$ are isomorphisms.

(c) If two of $\mathcal{P}(A), \mathcal{P}(X), \mathcal{P}(X, A)$ hold, then so does the third. To prove it, note that we have a commutative map of long exact sequences:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \underline{h}^*(X,A) & \longrightarrow & \underline{h}^*(X) & \longrightarrow & \underline{h}^*(A) & \longrightarrow & \underline{h}^{*+1}(X,A) & \longrightarrow \cdots \\ & \Phi_{(X,A)} \downarrow & & \Phi_X \downarrow & & \Phi_A \downarrow & & \Phi_{(X,A)} \downarrow & \\ \cdots \longrightarrow & \overline{h}^*(X,A) & \longrightarrow & \overline{h}^*(X) & \longrightarrow & \overline{h}^*(A) & \longrightarrow & \overline{h}^{*+1}(X,A) & \longrightarrow \cdots \end{array}$$

The result now follows from the Five Lemma (Lemma 2.59).

(d) If (X, A) is a good pair, then $\mathcal{P}(X, A) \Leftrightarrow \mathcal{P}(X/A)$. Indeed, by collapsing a pair, we see that $\mathcal{P}(X, A) \Leftrightarrow \mathcal{P}(X/A, \{*_A\})$, but $\mathcal{P}(\{*_A\})$ holds by (a), so $\mathcal{P}(X/A, \{*_A\}) \Leftrightarrow \mathcal{P}(X/A)$ by (c).

(e) $\mathcal{P}(\mathbb{D}^n, \mathbb{S}^{n-1})$ and $\mathcal{P}(\mathbb{S}^n)$ hold for all n. We prove this by induction on n. For $n = 0$, this is (a). Assuming the result is true for n, we have $\mathcal{P}(\mathbb{D}^n, \mathbb{S}^{n-1}) \Leftrightarrow \mathcal{P}(\mathbb{D}^n/\mathbb{S}^{n-1}) = \mathcal{P}(\mathbb{S}^n)$ by (d), and $\mathcal{P}(\mathbb{D}^{n+1})$ holds by (a) and (b) because $\mathbb{D}^{n+1} \sim \mathbb{D}^0$, so $\mathcal{P}(\mathbb{D}^{n+1}, \mathbb{S}^n)$ holds by (c).

(f) $\mathcal{P}(X) \Rightarrow \mathcal{P}\left(X \cup_f \mathbb{D}^k\right)$, where $f : \mathbb{S}^{k-1} \to X$. Prove this by considering the pair $\left(X \cup_f \mathbb{D}^k, X\right)$ and by noting that $\left(X \cup_f \mathbb{D}^k\right)/X \simeq \mathbb{S}^k$. Thus, if $\mathcal{P}(X)$ holds, we know that $\mathcal{P}\left(\mathbb{S}^k\right)$ holds by (e), so $\mathcal{P}\left(X \cup_f \mathbb{D}^k\right)$ holds by (c) and (d).

(g) $\mathcal{P}(X)$ holds if X is a finite cell complex.

(h) $\mathcal{P}(X)$ holds if X is homotopic to a finite cell complex. ☐

3.8 Computations of cohomology rings

Example 3.65. (i) $(a_1 \times b_1) \cup (a_2 \times b_2) = (-1)^{|b_1||a_2|} (a_1 \cup a_2) \times (b_1 \cup b_2)$.

(ii) $H^*(\mathbb{T}^2) \simeq \langle a, b,\ ab = -ba \text{ and } a^2 = b^2 = 0\rangle$ *as a ring (in other words,* $H^*(\mathbb{T}^2)$ *is the exterior algebra* $\Lambda^*(a, b)$ *on two generators).*

(iii) $H^*(\mathbb{T}^n) \simeq \langle a_1, \ldots, a_n,\ a_i a_j = -a_j a_i \text{ and } a_i^2 = 0\rangle$.

(iv) $H^*(\mathbb{S}^2 \times \mathbb{S}^2) \simeq \mathbb{Z}[a, b] / (a^2, b^2)$.

(v) *If* X *and* Y *are path-connected, then* $H^*(X \vee Y)$ *is a subring of* $H^*(X \amalg Y) = H^*(X) \times H^*(Y)$.

(vi) $H^k(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^4) \simeq H^*(\mathbb{S}^2 \times \mathbb{S}^2)$ *as groups, but not as rings. It follows that* $\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^4 \not\simeq \mathbb{S}^2 \times \mathbb{S}^2$.

(vii) *If* Σ_2 *is the genus 2 surface, then* $H^*(\Sigma_2) = \langle a_1, b_1, a_2, b_2, c\rangle$*, with* $|a_i| = |b_i| = 1$*,* $|c| = 2$ *and* $a_i \cup b_j = \delta_{ij} c$*,* $a_i \cup a_j = b_i \cup b_j = 0$.

More generally, $H^*(\Sigma^g) = \left\langle c, (a_i)_{1 \leqslant i \leqslant g}, (b_i)_{1 \leqslant i \leqslant g}\right\rangle$ *with* $|a_i| = |b_i| = 1$*,* $|c| = 2$ *and* $a_i \cup b_j = \delta_{ij} c$*,* $a_i \cup a_j = b_i \cup b_j = 0$.

Proof. (i) Note that

$$\begin{aligned}(a_1 \times b_1) \cup (a_2 \times b_2) &= \pi_1^*(a_1) \cup \pi_2^*(b_1) \cup \pi_1^*(a_2) \cup \pi_2^*(b_2) \\ &= (-1)^{|b_1||a_2|} \pi_1^*(a_1) \cup \pi_1^*(a_2) \cup \pi_2^*(b_1) \cup \pi_2^*(b_2) \\ &= (1)^{|b_1||a_2|} \pi_1^*(a_1 \cup a_2) \cup \pi_2^*(b_1 \cup b_2) \\ &= (-1)^{|b_1||a_2|} (a_1 \cup a_2) \times (b_2 \cup b_2).\end{aligned}$$

(ii) and (iii) We know that $H^*(\mathbb{S}^1) \simeq \mathbb{Z}[c]/(c^2)$ as a ring, with $|c| = 1$ (c.f. Example 3.58.(iii)). By Theorem 3.64,

$$H^*\left(\mathbb{T}^2\right) = H^*\left(\mathbb{S}^1 \times \mathbb{S}^1\right) \simeq \left\langle 1 \times 1, \underbrace{c \times 1}_{a}, \underbrace{1 \times c}_{b}, c \times c\right\rangle,$$

as a group. Moreover, $a \cup b = (c \times 1) \cup (1 \times c) = (c \cup 1) \times (1 \cup c) = c \times c$ and $b \cup a = -a \cup b$. Likewise, $a \cup a = b \cup b = 0$, from which the result follows.

(iv) Use the fact that $H^*(\mathbb{S}^2) \simeq \mathbb{Z}[c] / (c^2)$ with $|c| = 2$ and proceed as for (ii).

(v) We have $H^k(X \vee Y) \simeq H^k(X \amalg Y) \simeq \left\{(a, b),\ a \in H^k(X),\ b \in H^k(Y)\right\}$ as groups for $k > 0$. Since X, Y are path-connected, $H^0(X \vee Y) \simeq \langle 1\rangle$, and the result follows from the fact that

$$(a_1, b_1) \cup (a_2, b_2) = (a_1 \cup a_2, b_1 \cup b_2).$$

(vi) We have $H^*(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^4) \simeq H^*(\mathbb{S}^2) \times H^*(\mathbb{S}^2) \times H^*(\mathbb{S}^4)$. This is isomorphic to $H^*(\mathbb{S}^2 \times \mathbb{S}^2)$ as groups, but we have for example $H^2(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^4) = \langle \alpha, \beta\rangle$, with $\alpha = (c, 0, 0)$ and $\beta = (0, c, 0)$. Therefore $\alpha^2 = \beta^2 = 0$, and $\alpha\beta = (c, 0, 0) \cup (0, c, 0) = 0$.

(vii) Let A be a circle separating the two holes of Σ_2. We have a projection map $\pi : \Sigma_2 \to \Sigma_2/A \simeq \mathbb{T}_1^2 \vee \mathbb{T}_2^2$. We first compute homology, obtaining that $\pi_* : H_1(\Sigma_2) \to H_1(\mathbb{T}^2)^2$ is an isomorphism, and $\pi_* : H_2(\Sigma_2) \to H_2(\mathbb{T}^2)^2$ is given by the matrix $\begin{pmatrix}1\\1\end{pmatrix}$. Since $H_*(\Sigma_2)$ is free over $\mathbb{Z}$, Corollary 3.37 implies that π^* is dual to π_*. The result follows. ☐

4 Vector bundles and manifolds

4.1 Vector bundles

Definition 4.1 (Vector bundle). *An* n-dimensional real vector bundle *over a space B is a map $\pi : E \to B$ such that*

(i) *$\pi^{-1}(b)$ is an n-dimensional real vector space for all $b \in B$,*

(ii) *There is an open cover $(U_\alpha)_{\alpha \in A}$ of B and homeomorphisms $f_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ such that the square*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "f_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{R}^n \arrow[d, "\pi_1"] \\ U_\alpha \arrow[r, "\mathrm{id}_{U_\alpha}"] & U_\alpha \end{tikzcd}$$

commutes for all $\alpha \in A$, and the maps $\pi_2 \circ f_{\alpha|\pi^{-1}(b)} : \pi^{-1}(b) \to \mathbb{R}^n$ are linear isomorphisms.

The space B is the base *of the vector bundle, E is the* total space*, the sets $\pi^{-1}(b)$ are the* fibres *and the maps f_α are* local trivialisations*.*

Remark 4.2. *There is an analogous definition of complex vector bundles (replace $\mathbb{R}$ by $\mathbb{C}$).*

Definition 4.3 (Morphisms of vector bundles). *A* morphism of vector bundles *between $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ is a commuting square*

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"'] \\ B \arrow[r, "f_B"] & B' \end{tikzcd}$$

This implies that we have linear maps

$$f_{E|\pi^{-1}(b)} : \pi^{-1}(b) \longrightarrow (\pi')^{-1}(f(b)).$$

There is a category of vector bundles and morphisms of vector bundles.

Definition 4.4 (Subbundle). *We say that a bundle $E \xrightarrow{\pi} B$ is a* subbundle *of $E' \xrightarrow{\pi'} B$ if there is an injective morphism $f : E \hookrightarrow E'$ making the following square commute:*

$$\begin{tikzcd} E \arrow[r, tail, "f"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"'] \\ B \arrow[r, "\mathrm{id}_B"] & B \end{tikzcd}$$

Remark 4.5. *Let $E \xrightarrow{\pi} B$ be a vector bundle. Consider the maps $f_\alpha \circ f_\beta^{-1}$; there are functions $f_{\alpha\beta} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that are linear in the second coordinate, such that*

$$f_\alpha \circ f_\beta^{-1} : (b, \vec{v}) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longmapsto (b, f_{\alpha\beta}(b, \vec{v})) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^n.$$

In other words, we can write $f_{\alpha\beta}(b, \vec{v}) = g_{\alpha\beta}(b)\vec{v}$, with $g_{\alpha\beta}(b) \in GL_n(\mathbb{R})$. This defines maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R}),$$

called transition functions*.*

Lemma 4.6. *Let $E \xrightarrow{\pi} B$ be a vector bundle. Then the transition functions $(g_{\alpha\beta})_{\alpha,\beta\in A}$ satisfy*

(i) $g_{\alpha\alpha}(b) = I_n$,

(ii) $g_{\beta\alpha}(b) = (g_{\alpha\beta}(b))^{-1}$,

(iii) $g_{\alpha\beta}(b)g_{\beta\gamma}(b) = g_{\alpha\gamma}(b)$.

Proposition 4.7. *Suppose $(U_\alpha)_{\alpha\in A}$ is an open cover of a space B and there are maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ satisfying conditions* (i)–(iii) *of Lemma 4.6. Then there is a vector bundle $E \xrightarrow{\pi} B$ with transition functions $g_{\alpha\beta}$. Moreover, any two such bundles are isomorphic.*

Proof. Construct E by

$$E = \coprod_{\alpha\in A} (U_\alpha \times \mathbb{R}^n) / \sim,$$

where $\sim$ is defined by $(b, \vec{v}) \sim (b, g_{\alpha\beta}(b)\vec{v})$ for all $b \in U_\alpha \cap U_\beta$. Conditions (i)–(iii) imply that $\sim$ is indeed an equivalence relation. □

Example 4.8. *$B \times \mathbb{R}^n \xrightarrow{\pi_1} B$ is the n-dimensional* trivial bundle *over B.*

Definition 4.9 (Section)**.** *A* section *of $E \xrightarrow{\pi} B$ is a map $B \xrightarrow{s} E$ such that $\pi \circ s = \mathrm{id}_B$.*

For instance, we have a section $b \in B \longmapsto 0_{\pi^{-1}(b)} \in \pi^{-1}(b) \subseteq E$ called the zero section.

A section $s : B \to E$ is called nonvanishing *if $s(b) \neq 0_{\pi^{-1}(b)}$ for all $b \in B$.*

Proposition 4.10. *A vector bundle $E \xrightarrow{\pi} B$ is isomorphic to the trivial bundle* iff *there are sections $s_1, \dots, s_n : B \to E$ such that $(s_i(b))_{1\leqslant i\leqslant n}$ is a basis of $\pi^{-1}(b)$ for all $b \in B$.*

Proof. If $s_1, \dots, s_n$ are such sections, define

$$f : (b, \vec{v}) \in B \times \mathbb{R}^n \longmapsto \sum_{i=1}^{n} v_i s_i(b) \in \pi^{-1}(b).$$

This defines an isomorphism, and the converse is easy. □

4.2 Examples of vector bundles

Example 4.11. (i) *The* Möbius bundle *is*

$$M = [0, 1] \times \mathbb{R} / \sim,$$

where $\sim$ is defined by $(0, x) \sim (1, -x)$, with projection $M \xrightarrow{\pi} ([0, 1]/\sim) \simeq \mathbb{S}^1$.

This is a line bundle *over $\mathbb{S}^1$.*

Note that, if $s : \mathbb{S}^1 \to M$ is a section, then $s(t) = (t, f(t)) \in [0, 1] \times \mathbb{R}$, where $f(t)$ satisfies $f(0) = -f(1)$. It follows that $f(t_0) = 0$ for some $t_0 \in [0, 1]$, and therefore $(s(t_0))$ cannot be a basis of $\pi^{-1}(t_0)$, so $M \to \mathbb{S}^1$ is not trivial.

(ii) *The* tautological bundle *is*

$$\mathcal{T}_{\mathbb{RP}^n} = \left\{([x], \vec{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1},\ \vec{v} \in \mathbb{R}x\right\},$$

with natural projection $\mathcal{T}_{\mathbb{RP}^n} \to \mathbb{RP}^n$.

We have local trivialisations given by $U_i = \{[x] \in \mathbb{RP}^n,\ x_i \neq 0\}$ and $f_i([x], \vec{v}) = ([x], v_i)$. The associated transition functions are

$$g_{ij}([x]) = \frac{x_i}{x_j} \in \mathbb{R}^*.$$

Note that $\mathcal{T}_{\mathbb{RP}^1}$ is the Möbius bundle.

(iii) *The* complex tautological bundle *is*

$$\mathcal{T}_{\mathbb{CP}^n} = \left\{([z], \vec{v}) \in \mathbb{CP}^n \times \mathbb{C}^{n+1},\ \vec{v} \in \mathbb{C}z\right\},$$

with natural projection $\mathcal{T}_{\mathbb{CP}^n} \to \mathbb{CP}^n$.

The map $\pi_2 : \mathcal{T}_{\mathbb{CP}^n} \to \mathbb{C}^{n+1}$ *given by* $([z], \vec{v}) \mapsto \vec{v}$ *is called the* blowup map *in algebraic geometry. If* $\vec{v} \neq 0$ *then* $\pi_2^{-1}(\vec{v}) = \{([\vec{v}], \vec{v})\}$*; if* $\vec{v} = 0$ *then* $\pi_2^{-1}(0) = \mathbb{CP}^n \times \{0\}$.

(iv) *The* tangent bundle of the sphere *is*

$$T\mathbb{S}^n = \left\{(\vec{x}, \vec{v}) \in \mathbb{S}^n \times \mathbb{R}^{n+1},\ \vec{x} \cdot \vec{v} = 0\right\},$$

with natural projection $T\mathbb{S}^n \to \mathbb{S}^n$.

We have local trivialisations given by $U_i = \{\vec{x} \in \mathbb{S}^n\ x_i \neq 0\}$ *and* $f_i(\vec{x}, \vec{v}) = (\vec{x}, \pi_{\hat{i}}(\vec{v}))$*, where* $\pi_{\hat{i}} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *is the map omitting the* i*-th coordinate.*

Since $T\mathbb{S}^{2n}$ *has no nonvanishing section (for such a section could be used to construct a homotopy between* $\mathrm{id}_{\mathbb{S}^{2n}}$ *and the antipodal map, contradicting Corollary 2.74), it follows that* $T\mathbb{S}^{2n}$ *is not trivial. However,* $T\mathbb{S}^1$ *is trivial. In general, it can be proved that* $T\mathbb{S}^n$ *is trivial* iff $n \in \{1, 3, 7\}$.

Definition 4.12 (Product of vector bundles). *Let* $E \xrightarrow{\pi} B$ *and* $E' \xrightarrow{\pi'} B'$ *be vector bundles. Their* product *is the vector bundle*

$$E \times E' \xrightarrow{\pi \times \pi'} B \times B'.$$

At the level of fibres, $(\pi \times \pi')^{-1}(b, b') = \pi^{-1}(b) \times (\pi')^{-1}(b')$.

Definition 4.13 (Pullback of a vector bundle). *Let* $E \xrightarrow{\pi} B$ *be a vector bundle and* $X \xrightarrow{f} B$ *be a map. The* pullback *of* π *along* f *is defined by*

$$f^*(E) = \{(x, \vec{v}) \in X \times E,\ f(x) = \pi(\vec{v})\},$$

with natural projection $\pi' : f^*(E) \to X$.

At the level of fibres, $(\pi')^{-1}(x) \simeq \pi^{-1}(f(x))$.

If E *is trivial on* U_α *with transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$*, then* $f^*(E)$ *is trivial on* $f^{-1}(U_\alpha)$ *with transition functions* $g_{\alpha\beta} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to GL_n(\mathbb{R})$.

Lemma 4.14. *Let* $E \xrightarrow{\pi} B$ *be a vector bundle and let* $X \xrightarrow{g} Y \xrightarrow{f} B$ *be maps. Then*

(i) $(\mathrm{id}_B)^* E \simeq E$,

(ii) $(f \circ g)^* E \simeq g^*(f^*(E))$.

Definition 4.15 (Whitney sum of vector bundles). *Let* $E \xrightarrow{\pi} B$ *and* $E' \xrightarrow{\pi'} B$ *be two vector bundles over* B*. Then their* Whitney sum *is defined by*

$$E \oplus E' = \Delta^*(E \times E'),$$

where $\Delta : B \to B \times B$ *is the diagonal map. There is a natural projection* $\pi_\oplus : E \oplus E' \to B$.

At the level of fibres, $(\pi_\oplus)^{-1}(b) \simeq \pi^{-1}(b) \oplus (\pi')^{-1}(b)$.

4.3 Partitions of unity

Notation 4.16. *If $\varphi : B \to \mathbb{R}$, we write*

$$\operatorname{Supp}\varphi = \overline{\{b \in B,\ \varphi(b) \neq 0\}} \subseteq B.$$

Definition 4.17 (Partition of unity)**.** *If $\mathcal{U} = \{U_\alpha,\ \alpha \in A\}$ is an open cover of a space B, a* partition of unity *subordinate to $\mathcal{U}$ is a collection of functions $(\varphi_i : B \to [0,1])_{i\in\mathbb{N}}$ such that*

(i) *For all $i \in \mathbb{N}$, $\operatorname{Supp}\varphi_i \subseteq U$ for some $U \in \mathcal{U}$,*

(ii) *For any $b \in B$, $\varphi_i(b) = 0$ for all but finitely many i,*

(iii) *For any $b \in B$, $\sum_{i\in\mathbb{N}} \varphi_i(b) = 1$.*

We say that B admits partitions of unity if whenever $\mathcal{U}$ is an open cover of B, there is a partition of unity subordinate to $\mathcal{U}$.

Example 4.18. *Compact Hausdorff spaces, metrisable spaces, manifolds, all admit partitions of unity.*

In general, a space B admits partitions of unity iff *B is paracompact Hausdorff.*

Notation 4.19. *Let $E \xrightarrow{\pi} B$ be a vector bundle and let $B' \subseteq B$. We define the* restriction *of E to B' by*

$$E_{|B'} = \iota^*(E),$$

where $\iota : B' \to B$ is the inclusion map.

Lemma 4.20. *Let $E \xrightarrow{\pi} B \times [0,1]$ be a vector bundle. If $E_{|B\times[0,\frac{1}{2}]}$ and $E_{|B\times[\frac{1}{2},1]}$ are both trivial, then so is E.*

Lemma 4.21. *Let $E \xrightarrow{\pi} B \times [0,1]$ be a vector bundle. Then any $b \in B$ has an open neighbourhood $U_b \subseteq B$ such that $E_{|U_b\times[0,1]}$ is trivial.*

Proof. Since E is locally trivial, given $b \in B$ and $s \in [0,1]$, there exists an open neighbourhood $U_{b,s} \subseteq B$ of b and an open neighbourhood $I_s \subseteq [0,1]$ of s such that $E_{|U_{b,s}\times I_s}$ is trivial. Since $[0,1]$ is compact, we can find $0 = t_0 < s_1 < t_1 < s_2 < \cdots < t_n = 1$ such that $E_{|U_{b,s_i}\times[t_{i-1},t_i]}$ is trivial. Now let $U_b = \bigcap_{i=1}^{s} U_{b,s_i}$ and apply Lemma 4.20. $\square$

Proposition 4.22. *Let $E \xrightarrow{\pi} B \times [0,1]$ be a vector bundle. If B admits partitions of unity, then $E_{|B\times 0} \simeq E_{|B\times 1}$.*

Proof. Pick an open cover $\mathcal{U} = \{U_b,\ b \in B\}$ of B as in Lemma 4.21. Let $(\varphi_i)_{i\in\mathbb{N}}$ be a partition of unity subordinate to $\mathcal{U}$, with $\operatorname{Supp}\varphi_i \subseteq U_{b_i}$ for some $b_i \in B$. Let $\psi_n = \sum_{i=1}^{n} \varphi_i$ and $p_n : b \in B \longmapsto (b, \psi_n(b)) \in B \times [0,1]$. Define

$$E_n = p_n^*(E) = \{(b, \vec{v}) \in B \times E,\ \pi(\vec{v}) = (b, \psi_n(b))\}.$$

Let $f_i : (\pi')^{-1}(U_{b_i} \times [0,1]) \to U_{b_i} \times [0,1] \times \mathbb{R}^n$ be a local trivialisation of E_n. There is an isomorphism $\beta_n : E_{n-1} \xrightarrow{\simeq} E_n$ given by

$$\beta_n(b, \vec{v}) = \begin{cases} (b, \vec{v}) & \text{if } b \notin U_{b_n} \\ f_i^{-1}(b, \psi_n(b), \vec{v}') & \text{if } b \in U_{b_n} \end{cases},$$

where $f_i(b, \vec{v}) = (b, \psi_{n-1}(b), \vec{v}')$. Now if

$$\beta = \lim_{n\to+\infty} (\beta_n \circ \cdots \circ \beta_2 \circ \beta_1),$$

then $\beta : E_{|B\times 0} \xrightarrow{\simeq} E_{|B\times 1}$. $\square$

Theorem 4.23. *Let $E \overset{\pi}{\to} B$ be a vector bundle, $f_0, f_1 : X \to B$ be two homotopic maps. If X admits partitions of unity, then*

$$f_0^*(E) \simeq f_1^*(E).$$

Proof. Let $f_\bullet : X \times [0,1] \to B$ be a homotopy from f_0 to f_1. Then

$$f_0^*(E) \simeq f_\bullet^*(E)_{|B\times 0} \simeq f_\bullet^*(E)_{|B\times 1} \simeq f_1^*(E). \qquad \square$$

Corollary 4.24. *If $E \overset{\pi}{\to} B$ is a vector bundle where B is contractible and admits partitions of unity, then E is trivial.*

Proof. Let $c_{b_0} : b \in B \longmapsto b_0 \in B$, so that $\mathrm{id}_B \sim c_{b_0}$ because B is contractible. It follows that

$$E \simeq \mathrm{id}_B^*(E) \simeq c_{b_0}^*(E) \simeq B \times \pi^{-1}(b_0). \qquad \square$$

4.4 The Thom isomorphism

Notation 4.25. *Let $E \overset{\pi}{\to} B$ be an n-dimensional (real) vector bundle. For $b \in B$, we denote by $E_b = \pi^{-1}(b)$ the fibre at b, and $\iota_b : E_b \hookrightarrow E$ the inclusion map. We also write $s_0 : B \to E$ for the zero section (i.e. $s_0(b) = \vec{0} \in E_b$ for all b), and we write $E^\sharp = E \backslash \operatorname{Im} s_0$ and $E_b^\sharp = E_b \backslash \left\{\vec{0}\right\} \simeq \mathbb{R}^n \backslash \{0\}$.*

Remark 4.26. *For all $b \in B$, we have*

$$H_*\left(E_b, E_b^\sharp\right) = H_*\left(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}\right) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{cases},$$

so by Corollary 3.36, for any ring R,

$$H^*\left(E_b, E_b^\sharp; R\right) = \begin{cases} R & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$$

Definition 4.27 (Thom class)**.** *An element $u \in H^n\left(E, E^\sharp; R\right)$ is said to be an R-*Thom class *(or an R-*orientation*) for E if $\iota_b^*(u)$ generates $H^n\left(E_b, E_b^\sharp; R\right) \simeq R$ for all $b \in B$.*

Notation 4.28. *From now on, we shall always work with R-coefficients and omit them from the notations.*

Example 4.29. *Assume that $E = B \times \mathbb{R}^n$ is the trivial bundle over B. By Theorem 3.64, since $H^*\left(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}\right)$ is free over R, we have*

$$H^*\left(E, E^\sharp\right) \simeq H^*\left(B \times \mathbb{R}^n, B \times \left(\mathbb{R}^n\backslash\{0\}\right)\right) \simeq H^*(B) \otimes H^*\left(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}\right).$$

Therefore, if c is a generator of $H^n\left(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}\right)$, then we have an isomorphism

$$H^k(B) \overset{\simeq}{\to} H^{n+k}\left(E, E^\sharp\right)$$

given by $a \mapsto (a \times c)$. Hence

$$\prod_{B_i \in \pi_0 B} R \simeq \prod_{B_i \in \pi_0 B} H^0\left(B_i\right) \simeq H^0(B) \simeq H^n\left(E, E^\sharp\right) \simeq \prod_{B_i \in \pi_0 B} H^n\left(E_{|B_i}, E^\sharp_{|B_i}\right),$$

and the map $\prod_{B_i \in \pi_0 B} R \overset{\simeq}{\to} \prod_{B_i \in \pi_0 B} H^n\left(E_{|B_i}, E^\sharp_{|B_i}\right)$ is given by $\vec{r} \mapsto (r_i c)_{B_i \in \pi_0 B}$. It follows that $\vec{r} \times c \in H^n\left(E, E^\sharp\right)$ is a Thom class iff *r_i generates $R \simeq H^0\left(B_i\right)$ for all i. Therefore:*

- *If $R = \mathbb{Z}/2$, there is a unique Thom class.*

- *If $R = \mathbb{Z}$, there are $2^{|\pi_0 B|}$ Thom classes.*

Lemma 4.30. *If $f : B' \to B$, then there is a morphism*

$$\begin{array}{ccc} f^*(E) & \xrightarrow{f_E} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

given by $f_E : (b, \vec{v}) \in f^*(E) \mapsto \vec{v} \in E$.

If $u \in H^n\left(E, E^\sharp\right)$ is a Thom class for E, then $f_E^*(u) \in H^n\left(f^*(E), f^*(E)^\sharp\right)$ is a Thom class for $f^*(E)$.

Proof. The diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{f_E} & E \\ {\scriptstyle \iota_{b'}}\uparrow & & \uparrow{\scriptstyle \iota_{f(b')}} \\ f^*(E)_{|b'} & \xrightarrow{\simeq} & E_{|f(b')} \end{array}$$

commutes, so the fact that $\iota_{f(b')}^*(u)$ generates $H^n\left(E_{f(b')}, E_{f(b')}^\sharp\right)$ implies that $\iota_{b'}^*\left(f_E^*(u)\right)$ generates $H^n\left(f^*(E)_{b'}, f^*(E)_{b'}^\sharp\right)$. □

Lemma 4.31. *Suppose that $B = B_1 \cup B_2$ and $u \in H^n\left(E, E^\sharp\right)$. If $u_{|B_i} = \iota_{B_i}^*(u)$ is a Thom class for $E_{|B_i}$ for $i = 1, 2$, then u is a Thom class for E.*

Proof. If $b \in B$, there exists $i \in \{1, 2\}$ such that $b \in B_i$, and if we write $u_{|b} = \iota_b^*(u)$, then $u_{|b} = \left(u_{|B_i}\right)_{|b}$ generates $H^n\left(E_b, E_b^\sharp\right)$ since $u_{|B_i}$ is a Thom class. □

Theorem 4.32 (Thom isomorphism)**.** *If $E \xrightarrow{\pi} B$ is an n-dimensional vector bundle, then:*

(i) *E has a unique $\mathbb{Z}/2$-Thom class.*

(ii) *If E has an R-Thom class u (i.e. E is R-*oriented*), then the map*

$$\psi : a \in H^*(B; R) \longmapsto \pi^*(a) \cup u \in H^{*+n}\left(E, E^\sharp; R\right)$$

is an isomorphism.

Proof. We prove the result when B is compact.

Step 1: The theorem holds if E is trivial. This is Example 4.29.

Step 2: Suppose $B_1, B_2 \subseteq B$ and let $B_\cap = B_1 \cap B_2$ and $B_\cup = B_1 \cup B_2$. If the theorem holds for $E_1 = E_{|B_1}$, $E_2 = E_{|B_2}$ and $E_\cap = E_{|B_\cap}$, then it holds for $E_\cup = E_{|B_\cup}$.

(i) Consider the Mayer-Vietoris Sequence over $R = \mathbb{Z}/2$:

$$\cdots \to H^{n-1}\left(E_\cap, E_\cap^\sharp\right) \to H^n\left(E_\cup, E_\cup^\sharp\right) \xrightarrow{\alpha} H^n\left(E_1, E_1^\sharp\right) \oplus H^n\left(E_2, E_2^\sharp\right) \xrightarrow{\beta} H^n\left(E_\cap, E_\cap^\sharp\right) \to \cdots.$$

Note that $H^{n-1}\left(E_\cap, E_\cap^\sharp\right) = 0$ since (ii) holds for $E_\cap$, so α is injective. Since (i) holds for E_1 and E_2, they have Thom classes $u_i \in H^n\left(E_i, E_i^\sharp\right)$. By Lemma 4.30, $(u_i)_{|E_\cap}$ is a Thom class for $E_\cap$. By (i), $(u_i)_{|E_\cap} = u_\cap$ is the unique Thom class for $E_\cap$, so $\beta\left(u_1 \oplus u_2\right) = u_\cap - u_\cap = 0$. By exactness, $u_1 \oplus u_2 \in \operatorname{Im}\alpha$, i.e. there exists $u_\cup \in H^n\left(E_\cup, E_\cup^\sharp\right)$ with $(u_\cup)_{|E_i} = u_i$. By Lemma 4.31, $u_\cup$ is a Thom class for $E_\cup$, which proves the existence. For uniqueness, note that if $u'_\cup$ is a Thom class for $E_\cup$, then $(u'_\cup)_{|E_i}$ is a Thom class for E_i, so $(u'_\cup)_{|E_i} = u_i$, i.e. $\alpha\left(u'_\cup\right) = \alpha\left(u_\cup\right)$, so $u'_\cup = u_\cup$ since α is injective.

(ii) Use the Mayer-Vietoris Sequence:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^*(B_\cup) & \longrightarrow & H^*(B_1)\oplus H^*(B_2) & \longrightarrow & H^*(B_\cap) & \longrightarrow \cdots \\
& \psi_\cup \downarrow & & \psi_1\oplus\psi_2 \downarrow & & \psi_\cap \downarrow & \\
\cdots \to & H^{*+n}\left(E_\cup, E_\cup^\sharp\right) & \longrightarrow & H^{*+n}\left(E_1, E_1^\sharp\right)\oplus H^{*+n}\left(E_2, E_2^\sharp\right) & \longrightarrow & H^{*+n}\left(E_n, E_n^\sharp\right) & \to \cdots
\end{array}$$

This diagram commutes, and $\psi_1 \oplus \psi_2$ and $\psi_\cap$ are isomorphisms, so $\psi_\cup$ is an isomorphism by the Five Lemma (Lemma 2.59).

Step 3: The theorem holds for all compact spaces B. Consider an open cover $\{V_1, \dots, V_k\}$ of B such that $E_{|V_i}$ is trivial for all i. Let $W_j = \bigcup_{i=1}^j V_i$. We prove by induction on j that the theorem holds for $E_{|W_j}$: for $j = 1$, $W_1 = V_1$ so the theorem holds by Step 1. If the theorem holds for $E_{|W_{j-1}}$, then it also holds for $E_{|V_j}$, and $E_{|V_j \cap W_{j-1}}$ since $E_{|V_j}$ is trivial, so it holds for $E_{|W_j} = E_{|W_{j-1} \cup V_j}$ by Step 2. $\square$

4.5 Sphere bundles

Definition 4.33 (Riemannian metric)**.** *A* Riemannian metric g *on a vector bundle* $E \xrightarrow{\pi} B$ *is a map* $g : E \oplus E \to \mathbb{R}$ *such that the map* $g_{|(E\oplus E)_b} : E_b \times E_b \to \mathbb{R}$ *is an inner product on* E_b *for all* $b \in B$.

Lemma 4.34. *If B admits partitions of unity, then B also admits (lots of) Riemannian metrics.*

Definition 4.35 (Sphere and disc bundles)**.** *If g is a Riemannian metric on the vector bundle* $E \xrightarrow{\pi} B$, *we define*

- *The* unit sphere bundle *of E by* $\mathbb{S}(E, g) = \{\vec{v} \in E,\ g(\vec{v}, \vec{v}) = 1\}$,
- *The* unit disc bundle *of E by* $\mathbb{D}(E, g) = \{\vec{v} \in E,\ g(\vec{v}, \vec{v}) \leqslant 1\}$.

Hence $\mathbb{S}(E, g) \cap E_b \simeq \mathbb{S}^{n-1}$ *and* $\mathbb{D}(E, g) \cap E_b \simeq \mathbb{D}^n$.

Remark 4.36. *If g, g' are two Riemannian metrics on E, then* $\mathbb{S}(E, g) \simeq \mathbb{S}(E, g')$ *and* $\mathbb{D}(E, g) \simeq \mathbb{D}(E, g')$. *We may therefore write* $\mathbb{S}(E)$ *and* $\mathbb{D}(E)$ *instead of* $\mathbb{S}(E, g)$ *and* $\mathbb{D}(E, g)$.

Remark 4.37. *There is a homotopy equivalence*

$$\mathbb{S}(E) \sim E^\sharp,$$

given by the inclusion $i : \mathbb{S}(E) \hookrightarrow E^\sharp$ *and by the map* $E^\sharp \to \mathbb{S}(E)$ *defined by* $\vec{v} \mapsto \frac{\vec{v}}{\sqrt{g(\vec{v},\vec{v})}}$.

Likewise, there is a homotopy equivalence

$$\mathbb{D}(E) \sim B,$$

given by the projection $\pi : \mathbb{D}(E) \to B$ *and by the zero section* $s_0 : B \to \mathbb{D}(E)$.

Example 4.38. (i) *If* $E = B \times \mathbb{R}^n$ *is the trivial bundle, then* $\mathbb{S}(E) = B \times \mathbb{S}^{n-1}$ *and* $\mathbb{D}(E) = B \times \mathbb{D}^n$.

(ii) *If* $M \xrightarrow{\pi} \mathbb{S}^1$ *is the Möbius bundle, then* $\mathbb{D}(M)$ *is the Möbius band and* $\mathbb{S}(M) = \partial\mathbb{D}(M) \simeq \mathbb{S}^1$. *Note that* $\mathbb{S}(M) \not\simeq B \times \mathbb{S}^0$, *which gives another proof of the fact that M is a nontrivial vector bundle.*

Moreover, use the homotopy equivalences $\mathbb{S}^1 \simeq \mathbb{S}(M) \sim M^\sharp$ *and* $\mathbb{S}^1 \simeq B \sim M$ *to define a map* $M^\sharp \to M$ *induced by* $z \mapsto z^2$ *on* $\mathbb{S}^1 \to \mathbb{S}^1$. *Since this map has degree 2, the long exact sequence of* $\left(M, M^\sharp\right)$ *gives*

$$H^*\left(M, M^\sharp; \mathbb{Z}\right) = \begin{cases} \mathbb{Z}/2 & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Since this is not isomorphic to $H^{*-1}(B)$, *it follows by Theorem 4.32 that M is not $\mathbb{Z}$-orientable.*

4.6 Gysin sequence

Remark 4.39. *Assume $E \xrightarrow{\pi} B$ is an R-oriented vector bundle with Thom class u. By Theorem 4.32, the long exact sequence of $\left(E, E^{\sharp}\right)$ with coefficients in R can be written as:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^*\left(E, E^{\sharp}\right) \arrow[r, "j^*"] & H^*(E) \arrow[r] & H^*\left(E^{\sharp}\right) \arrow[r] & H^{*+1}\left(E, E^{\sharp}\right) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{*-n}(B) \arrow[u, "\psi", "\simeq"'] \arrow[r, "\alpha"] & H^*(B) \arrow[u, "\pi^* = (s_0^*)^{-1}", "\simeq"'] \arrow[r] & H^*(\mathbb{S}(E)) \arrow[u, "\simeq"'] \arrow[r] & H^{*+1-n}(B) \arrow[u, "\simeq"'] \arrow[r] & \cdots
\end{tikzcd}$$

where $j : (E, \varnothing) \to \left(E, E^{\sharp}\right)$ is the inclusion. Given $a \in H^*(B)$, we have

$$\begin{aligned}
\alpha(a) &= s_0^* j^* \psi(a) = s_0^* j^* \left(\pi^*(a) \cup u\right) \\
&= s_0^* \left(\pi^*(a) \cup j^*(u)\right) = s_0^* \pi^*(a) \cup s_0^* j^*(u) = a \cup s_0^* j^*(u).
\end{aligned}$$

Definition 4.40 (Euler class)**.** *If $E \xrightarrow{\pi} B$ is an R-oriented n-dimensional vector bundle with Thom class $u \in H^n\left(E, E^{\sharp}; R\right)$, its* Euler class *is*

$$e(E) = s_0^* j^*(u) \in H^n(B).$$

Theorem 4.41 (Gysin sequence)**.** *If $E \xrightarrow{\pi} B$ is an R-oriented n-dimensional vector bundle, then there is a long exact sequence with coefficients in R:*

$$\cdots \to H^{*-n}(B) \xrightarrow{\beta} H^*(B) \xrightarrow{\pi^*} H^*(\mathbb{S}(E)) \to H^{*+1-n}(B) \to \cdots,$$

where $\beta : a \mapsto a \cup e(E)$.

Proposition 4.42. *Assume $E \xrightarrow{\pi} B$ is R-oriented.*

(i) *If $f : B' \to B$, then $f^*(E)$ is R-oriented and*

$$e\left(f^*(E)\right) = f^*\left(e(E)\right).$$

(ii) *If E is trivial and $n > 0$, then $e(E) = 0$.*

(iii) *If $E_i \xrightarrow{\pi_i} B$ are R-oriented for $i = 1, 2$, so is $E_1 \oplus E_2$, and*

$$e\left(E_1 \oplus E_2\right) = e\left(E_1\right) \cup e\left(E_2\right).$$

(iv) *If $B \xrightarrow{s} E$ is a nonvanishing section and $n > 0$, then*

$$e(E) = 0.$$

Proof. (i) There is a commuting diagram

$$\begin{tikzcd}
(B, \varnothing) \arrow[r, "s_0"] & (E, \varnothing) \arrow[r, "j"] & \left(E, E^{\sharp}\right) \\
(B', \varnothing) \arrow[u, "f"] \arrow[r, "s_0'"] & (f^*E, \varnothing) \arrow[u, "f_E"] \arrow[r, "j'"] & \left(f^*E, f^*E^{\sharp}\right) \arrow[u, "f_E"]
\end{tikzcd}$$

By Lemma 4.30, $f_E^*(u)$ is a Thom class for f^*E, so f^*E is oriented and

$$e\left(f^*E\right) = {s'_0}^* {j'}^* f_E^*(u) = f^* s_0^* j^*(u) = f^*\left(e(E)\right).$$

(ii) Let $E_0 = \mathbb{R}^n$ and consider the trivial bundle $E_0 \xrightarrow{\pi} \{p\}$. We have $e\left(E_0\right) \in H^n\left(\{p\}\right) = 0$. Now if $E \xrightarrow{\pi} B$ is trivial, then we can write $E = f^*E_0$ where $f : B \to \{p\}$, so that $e(E) = f^*\left(e\left(E_0\right)\right) = 0$ by (i).

(iv) If s is a nonvanishing section, then $\langle s\rangle = \{\vec{v} \in E,\ \vec{v} \in \langle s\left(\pi\left(\vec{v}\right)\right)\rangle\}$ is a 1-dimensional subbundle of E, so $E \simeq \langle s\rangle \oplus \langle s\rangle^\perp$. By (iii), we have $e(E) = e\left(\langle s\rangle\right) \cup e\left(\langle s\rangle^\perp\right) = 0$ since $\langle s\rangle$ is trivial. □

Example 4.43. $H^*\left(\mathbb{RP}^n; \mathbb{Z}/2\right) \simeq \left(\mathbb{Z}/2\right)[a]/\left(a^{n+1}\right)$ *as a ring, with* $|a| = 1$.

Proof. Using Example 3.9 and Proposition 3.35, we have, as groups,

$$H^*\left(\mathbb{RP}^n; \mathbb{Z}/2\right) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leqslant * \leqslant n \\ 0 & \text{otherwise} \end{cases}.$$

We equip the tautological bundle $\mathcal{T}_{\mathbb{RP}^n} = \{([x], \vec{v}) \in \mathbb{RP}^n \times \mathbb{R}^{n+1},\ \vec{v} \in \mathbb{R}x\}$ with a Riemannian metric g defined by:

$$g\left(\left([x], \vec{v}_1\right), \left([x], \vec{v}_2\right)\right) = \langle \vec{v}_1, \vec{v}_2\rangle \in \mathbb{R}.$$

Hence

$$\mathbb{S}\left(\mathcal{T}_{\mathbb{RP}^n}\right) = \{\left([x], \vec{v}\right),\ \vec{v} \in \mathbb{R}x,\ \|\vec{v}\| = 1\} \simeq \mathbb{S}^n,$$

and the map $\mathbb{S}\left(\mathcal{T}_{\mathbb{RP}^n}\right) \xrightarrow{\pi} \mathbb{RP}^n$ corresponds under this isomorphism to the projection $\mathbb{S}^n \to \mathbb{RP}^n$. We write the Gysin sequence for $\mathcal{T}_{\mathbb{RP}^n}$ with $\mathbb{Z}/2$-coefficients:

$$\cdots \to H^{*-1}\left(\mathbb{RP}^n\right) \xrightarrow{\beta} H^*\left(\mathbb{RP}^n\right) \to H^*\left(\mathbb{S}^n\right) \to H^*\left(\mathbb{RP}^n\right) \to \cdots.$$

We claim that β is an isomorphism for $1 \leqslant * \leqslant n$. We may assume that $n > 2$. For $* = 1$, we have

$$0 \to H^0\left(\mathbb{RP}^n\right) \xrightarrow{\simeq} H^0\left(\mathbb{S}^n\right) \xrightarrow{0} H^0\left(\mathbb{RP}^n\right) \xrightarrow{\simeq} H^1\left(\mathbb{RP}^n\right) \to H^1\left(\mathbb{S}^n\right) = 0;$$

for $1 < * < n$,

$$0 = H^{*-1}\left(\mathbb{S}^n\right) \to H^{*-1}\left(\mathbb{RP}^n\right) \xrightarrow{\simeq} H^*\left(\mathbb{RP}^n\right) \to H^*\left(\mathbb{S}^n\right) = 0;$$

for $* = n$,

$$0 = H^{n-1}\left(\mathbb{S}^n\right) \to H^{n-1}\left(\mathbb{RP}^n\right) \xrightarrow{\simeq} H^n\left(\mathbb{RP}^n\right) \xrightarrow{0} H^n\left(\mathbb{S}^n\right) \xrightarrow{\simeq} H^n\left(\mathbb{RP}^n\right) \to H^{n+1}\left(\mathbb{RP}^n\right) = 0.$$

That proves the claim.

Now let $a = e\left(\mathcal{T}_{\mathbb{RP}^n}\right) \in H^1\left(\mathbb{RP}^n; \mathbb{Z}/2\right)$. We prove by induction on k that $\left\langle a^k\right\rangle = H^k\left(\mathbb{RP}^n; \mathbb{Z}/2\right)$. For $k = 0$, this is obvious; if it holds for $k-1$, then the isomorphism $\beta : H^{k-1}\left(\mathbb{RP}^n\right) \xrightarrow{\simeq} H^k\left(\mathbb{RP}^n\right)$ sends $\left\langle a^{k-1}\right\rangle$ to $\left\langle a^k\right\rangle$, hence the result. Since $H^{n+1}\left(\mathbb{RP}^n\right) = 0$, it follows that $a^{n+1} = 0$ and therefore $H^*\left(\mathbb{RP}^n; \mathbb{Z}/2\right) \simeq \left(\mathbb{Z}/2\right)[a]/\left(a^{n+1}\right)$ as claimed. □

4.7 Orientations and orientability

Definition 4.44 (Orientability)**.** *A vector bundle* $E \xrightarrow{\pi} B$ *is said to be* orientable *if it is* $\mathbb{Z}$*-orientable (i.e. it admits a* $\mathbb{Z}$*-Thom class).*

Remark 4.45. *Every vector bundle over* $\mathbb{S}^1$ *is isomorphic to* $[0,1] \times \mathbb{R}^n/\sim$*, where* $\sim$ *is given by* $(0, \vec{v}) \sim (1, A\vec{v})$ *for some* $A \in GL_n\mathbb{R}$*. This gives two isomorphism classes:*

(i) *Either* $\det A > 0$ *and the vector bundle is trivial,*

(ii) *Or* $\det A < 0$ *and the vector bundle is not trivial and not orientable.*

Now given any vector bundle $E \xrightarrow{\pi} B$, define for $\gamma : \mathbb{S}^1 \to B$,

$$\varphi_E(\gamma) = \begin{cases} 0 & \text{if } \gamma^* E \text{ is trivial} \\ 1 & \text{otherwise} \end{cases} \in \mathbb{Z}/2.$$

If $\gamma_0 \sim \gamma_1$, then $\gamma_0^* E \simeq \gamma_1^* E$ by Theorem 4.23, so φ_E descends to a map

$$\varphi_E : \pi_1 B \to \mathbb{Z}/2,$$

which is a homomorphism. Since $\mathbb{Z}/2$ is abelian, φ_E induces a map $\overline{\varphi}_E$ on the abelianisation of $\pi_1 B$, which is isomorphic to $H_1(B)$ by the Hurewicz Theorem (Theorem 2.83). Therefore, we have a homomorphism

$$\overline{\varphi}_E \in \mathrm{Hom}\left(H_1(B), \mathbb{Z}/2\right) \simeq H^1\left(B; \mathbb{Z}/2\right).$$

It turns out that E is orientable iff $\overline{\varphi}_E = 0$.

Corollary 4.46. *If $E \xrightarrow{\pi} B$ is a vector bundle such that $H^1\left(B; \mathbb{Z}/2\right) = 0$, then E is orientable.*

Example 4.47. *$\mathcal{T}_{\mathbb{CP}^n}$ is orientable, and the same argument as for $\mathbb{RP}^n$ (c.f. Example 4.43) yields*

$$H^*\left(\mathbb{CP}^n; \mathbb{Z}\right) \simeq \mathbb{Z}[a] / \left(a^{n+1}\right),$$

with $a = e\left(\mathcal{T}_{\mathbb{CP}^n}\right)$ and $|a| = 2$.

4.8 Manifolds

Definition 4.48 (Topological manifold)**.** *An n-dimensional* topological manifold *M is a second-countable Hausdorff space which admits an open cover $\{U_\alpha,\ \alpha \in A\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n$, called* charts. *The maps $\psi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta\left(U_\alpha \cap U_\beta\right) \xrightarrow{\simeq} \varphi_\alpha\left(U_\alpha \cap U_\beta\right)$ are called* transition functions. *They satisfy*

(i) $\psi_{\alpha\alpha} = \mathrm{id}$,

(ii) $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$,

(iii) $\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$.

Definition 4.49 (Smooth manifold)**.** *A* smooth manifold *is a topological manifold M together with an open cover $\{U_\alpha,\ \alpha \in A\}$ and charts $\varphi_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n$ such that all the transition functions $\psi_{\alpha\beta}$ are smooth maps. Note that the open cover and the charts are part of the data, as opposed to the definition of topological manifolds.*

If M, M' are smooth manifolds, a map $f : M \to M'$ is said to be smooth *if $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ is smooth where defined for all charts φ_α of M and φ'_β of M'. We say that f is a* diffeomorphism *if it is a homeomorphism and f, f^{-1} are smooth.*

Example 4.50. $\mathbb{S}^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{T}^n, \Sigma_g$ *are all smooth manifolds.*

Remark 4.51. *If M is an n-manifold, the set of smooth manifolds homeomorphic to M quotiented by the relation of diffeomorphism has only 1 element for $n \leqslant 3$, but may have many for $n > 3$.*

Definition 4.52 (Tangent bundle)**.** *If M is a smooth manifold with charts $\varphi_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n$, define*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R}),$$

by $g_{\alpha\beta}(x) = \left(\mathrm{d}\psi_{\alpha\beta}\right)_{|\varphi_\beta(x)}$. The chain rule implies that

(i) $g_{\alpha\alpha}(x) = I_n$,

(ii) $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$,

(iii) $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$.

The tangent bundle *TM of M is the n-dimensional vector bundle over M with transition functions $g_{\alpha\beta}$.*

4.9 Fundamental class

Notation 4.53. *Suppose M is an n-manifold and $A \subseteq M$ is compact. We write*

$$(M \mid A) = (M, M \backslash A).$$

If $B \subseteq A$, we have an inclusion map $\iota : (M \mid A) \to (M \mid B)$; if $w \in H_*(M \mid A)$, we write $w_{|B} = \iota_*(w)$.

Remark 4.54. *If M is an n-manifold and $x \in M$, choose a chart $U_\alpha \ni x$. Then, by excision,*

$$H_*(M \mid x; R) \simeq H_*(U_\alpha \mid x; R) \simeq H_*(\mathbb{R}^n \mid \varphi_\alpha(x); R) = \begin{cases} R & \text{if } * = n \\ 0 & \text{otherwise} \end{cases}.$$

Definition 4.55 (Fundamental class)**.** *An R-*fundamental class *(or R-*orientation*) for an n-manifold M is a class $[M] \in H_n(M; R) = H_n(M \mid M; R)$ such that $[M]_{|x}$ generates $H_n(M \mid x; R) \simeq R$ for all $x \in M$.*

Definition 4.56 (Closed manifold)**.** *A manifold is said to be* closed *if it is compact.*

Theorem 4.57. *Any closed manifold M has a unique $\mathbb{Z}/2$-fundamental class.*

Theorem 4.58. *If M is a closed and connected n-dimensional manifold, then*

(i) $H_n(M; \mathbb{Z}/2) \simeq \mathbb{Z}/2 = \langle [M] \rangle$.

(ii) $H_n(M; \mathbb{Z}) \simeq \mathbb{Z}$ *or* 0. *If M is $\mathbb{Z}$-oriented, then* $H_n(M; \mathbb{Z}) \simeq \mathbb{Z} = \langle [M] \rangle$.

(iii) $H_i(M) = 0$ *for all* $i > n$.

Notation 4.59. *If M is closed, connected and R-oriented, then $H^n(M; R) \simeq R$ by the Universal Coefficient Theorem. We define $[M]^*$ to be the generator of $H^n(M; R)$ such that*

$$\langle [M]^*, [M] \rangle = 1.$$

4.10 Submanifolds

Definition 4.60 (Submanifold)**.** *Suppose M is a smooth n-manifold. A subset $N \subseteq M$ is a k-dimensional* submanifold *of M if for every $x \in N$, there is a chart $\varphi_x : U_x \to \mathbb{R}^n$ such that $x \in U_x$ and $\varphi_x(U_x \cap N) = \mathbb{R}^k \times 0 \subseteq \mathbb{R}^n$. If so, N is a smooth k-manifold.*

If $N \subseteq M$ is a submanifold, then $TN \subseteq TM_{|N}$ is a subbundle.

Example 4.61. $\mathbb{S}^{n-1} \subseteq \mathbb{S}^n$, $\mathbb{RP}^{n-1} \subseteq \mathbb{RP}^n$ *and* $\mathbb{S}^n \times \{p\} \subseteq \mathbb{S}^n \times \mathbb{S}^m$ *are all submanifolds.*

Definition 4.62 (Normal bundle)**.** *Let M be a smooth n-manifold and $N \subseteq M$ be a submanifold. The* normal bundle *is defined by*

$$\nu_{M|N} = TN^\perp \subseteq TM_{|N}.$$

Hence $TM_{|N} = \nu_{M|N} \oplus TN$.

Note that, to define $TN^\perp$, we need to choose a Riemannian metric g on TM; the isomorphism type of $\nu_{M|N}$ does not depend on the choice of g since $\nu_{M|N} \simeq TM_{|N}/TN$.

Example 4.63. (i) *Let $M = \mathbb{R}^{n+1}$, $N = \mathbb{S}^n$. Then $\nu_{\mathbb{R}^{n+1}|\mathbb{S}^n}$ is trivial since it has a nonvanishing section $\mathbb{S}^n \xrightarrow{s} \nu_{\mathbb{R}^{n+1}|\mathbb{S}^n}$ given by $\vec{x} \mapsto \vec{x}$.*

Note that $T\mathbb{R}^{n+1}_{|\mathbb{S}^n} \simeq \nu_{\mathbb{R}^{n+1}|\mathbb{S}^n} \oplus T\mathbb{S}^n$, and $T\mathbb{R}^{n+1}_{|\mathbb{S}^n}$ and $\nu_{\mathbb{R}^{n+1}|\mathbb{S}^n}$ are both trivial, but $T\mathbb{S}^n$ need not be trivial.

(ii) *Let M be the Möbius band, $N = \mathbb{S}^1$. Then $\nu_{M|\mathbb{S}^1}$ is the Möbius bundle.*

(iii) *Let* $M = \mathbb{RP}^{n+1}$, $N = \mathbb{RP}^n$. *Then* $\nu_{\mathbb{RP}^{n+1}|\mathbb{RP}^n} = \mathcal{T}_{\mathbb{RP}^n}$.

(iv) *Let* $M = \mathbb{CP}^{n+1}$, $N = \mathbb{CP}^n$. *Then* $\nu_{\mathbb{CP}^{n+1}|\mathbb{CP}^n} = \mathcal{T}_{\mathbb{CP}^n}$.

Theorem 4.64 (Tubular neighbourhood)**.** *If $N \subseteq M$ is a submanifold of a smooth manifold, then there is an open set $N \subseteq V \subseteq M$ such that $(V, N) \simeq \left(\nu_{M|N}, s_0(N)\right)$.*

Sketch of proof. Use the exponential map $\nu_{M|N} \to M$. $\square$

Proposition 4.65. *A smooth manifold M is $\mathbb{Z}$-orientable (in the sense of manifolds)* iff *TM is $\mathbb{Z}$-orientable (in the sense of vector bundles).*

Sketch of proof. If $\mathbb{S}^1 \simeq \gamma \subseteq M$ is a submanifold, we have a tubular neighbourhood $V(\gamma) \subseteq M$. Now M is orientable iff $V(\gamma)$ is orientable for all γ, iff $TM_{|V(\gamma)}$ is orientable for all γ, iff $TM_{|\gamma}$ is orientable for all γ, iff TM is orientable. $\square$

4.11 Poincaré duality

Notation 4.66. *From now on, R is either $\mathbb{Z}$ or a field (mainly $\mathbb{Q}$ or $\mathbb{Z}/p$). We shall work with R-coefficients throughout.*

We shall consider a closed, connected smooth manifold M, with an R-fundamental class $[M] \in H_n(M; R)$.

Proposition 4.67. *The facts that M is connected and R-oriented imply that $H_n(M) \simeq R$.*

Corollary 4.68. $H^n(M) \simeq R$.

Proof. If R is a field, then $H^n(M) \simeq \operatorname{Hom}(H_n(M), R)$ by the Universal Coefficient Theorem. If $R = \mathbb{Z}$, then M is $\mathbb{Z}/p$-oriented for every prime p since the image of $[M]$ under $H_n(M; \mathbb{Z}) \to H_n(M; \mathbb{Z}/p)$ is a $\mathbb{Z}/p$-fundamental class, so $H_n(M; \mathbb{Z}/p) \simeq \mathbb{Z}/p$. This implies by the Universal Coefficient Theorem (Proposition 3.28) that $H_{n-1}(M; \mathbb{Z})$ has no p-torsion. It follows that $H_{n-1}(M; \mathbb{Z})$ is free, so $H^n(M; \mathbb{Z}) \simeq \mathbb{Z}$ by the Universal Coefficient Theorem. $\square$

Notation 4.69. *From now on, we consider $N \subseteq M$ a k-dimensional closed submanifold, we write $\nu = \nu_{M|N}$ for its normal bundle, and we choose a tubular neighbourhood V for N. Hence $(V \mid N) \simeq \left(\nu, \nu^\sharp\right)$.*

Lemma 4.70. *The submanifold N is orientable* iff *its normal bundle ν is orientable.*

Sketch of proof. Since M is orientable, TM is orientable and so is $TM_{|N}$. Therefore, $\overline{\varphi}_{TM_{|N}} = 0$ (c.f. Remark 4.45). But $TM_{|N} = TN \oplus \nu$, so

$$0 = \overline{\varphi}_{TM_{|N}} = \overline{\varphi}_{TN} + \overline{\varphi}_{\nu},$$

which implies that $\overline{\varphi}_{TN} = 0$ iff $\overline{\varphi}_{\nu} = 0$. $\square$

Remark 4.71. *We have the following commutative diagram:*

$$\begin{tikzcd}
(M, \varnothing) \arrow[r, "j"] \arrow[dr, "\beta"'] & (M \mid N) \arrow[d, "\alpha"] & (V \mid N) \arrow[l] & \left(\nu, \nu^\sharp\right) \arrow[l, "\simeq"'] \arrow[ll, bend left, "i"] \\
& (M \mid x) & &
\end{tikzcd}$$

We know that the maps i_* and i^* are isomorphisms by excision.

Notation 4.72. *We now assume that N is oriented and we define $[N]^* \in H^n(N)$ by*

$$\langle [N]^*, [N] \rangle = 1 \in R.$$

Lemma 4.73. *$j_* [M]$ generates $H_n (M \mid N) \simeq R$.*

Proof. By excision and the Thom isomorphism (Theorem 4.32),

$$H^* (M \mid N) \simeq H^* (V \mid N) \simeq H^* \left(\nu, \nu^\sharp\right) \simeq H^{*-n+k}(N) = \begin{cases} R & \text{if } * = n \\ 0 & \text{if } * > n \end{cases}.$$

It follows by the Universal Coefficient Theorem (Proposition 3.28) that $H_n (M \mid N) \simeq R$. But $[M]$ is a fundamental class, so $\beta_*[M] = \alpha_* j_*[M]$ generates $H_n (M \mid x) \simeq R$, hence $j_*[M]$ generates $H_n(M \mid N)$. □

Corollary 4.74. *There is a unique R-orientation $u_{M|N} \in H^{n-k} \left(\nu, \nu^\sharp\right)$ on ν s.t.*

$$\left\langle \pi^*[N]^* \cup u_{M|N}, i_*^{-1} j_*[M] \right\rangle = 1 \in R.$$

Proof. We know that $i_*^{-1} j_*[M]$ generates $H_n \left(\nu, \nu^\sharp\right) \simeq R$ by Lemma 4.73. Let $u \in H^{n-k} \left(\nu, \nu^\sharp\right)$ be some Thom class for ν (which is orientable by Lemma 4.70 because N is). Then $[N]^*$ generates $H^k(N)$, so $\pi^*[N]^* \cup u$ generates $H^n \left(\nu, \nu^\sharp\right)$ (by the Thom isomorphism, Theorem 4.32), so $r = \langle \pi^*[N]^* \cup u, i_*^{-1} j_*[M] \rangle$ generates R. It suffices to take $u_{M|N} = r^{-1} u$. □

Definition 4.75 (Poincaré dual)**.** *If $[M]$ and $[N]$ are R-orientations on M and N, the* Poincaré dual *of $[N]$ is*

$$\mathrm{PD}_{[M]} [N] = j^* (i^*)^{-1} \left(u_{M|N}\right) \in H^{n-k}(M).$$

Proposition 4.76. *If $a \in H^k(M)$, then*

$$\langle a, i_{0*}[N] \rangle = \left\langle a \cup \mathrm{PD}_{[M]} [N], [M] \right\rangle,$$

where $i_0 : N \hookrightarrow M$.

Proof. $[N]^*$ generates $H^k(N) \simeq R$, so if $c = \langle a, i_{0*}[N] \rangle = \langle i_0^* a, [N] \rangle$, then $i_0^* a = c[N]^*$. Moreover, we have maps

$$\begin{array}{ccc} \nu \simeq V & \xrightarrow{i} & M \\ \pi \big\downarrow & \nearrow_{i_0} & \\ N & & \end{array}$$

with $i \sim i_0 \circ \pi$, so $i^* a = \pi^* i_0^* a = c\pi^*[N]^*$. Finally,

$$\begin{aligned} \left\langle a \cup \mathrm{PD}_{[M]} [N], [M] \right\rangle &= \left\langle a \cup j^* (i^*)^{-1} u_{M|N}, [M] \right\rangle \\ &= \left\langle a \cup (i^*)^{-1} u_{M|N}, j_*[M] \right\rangle \\ &= \left\langle i^* a \cup u_{M|N}, i_*^{-1} j_*[M] \right\rangle \\ &= \left\langle c\pi^*[N]^* \cup u_{M|N}, i_*^{-1} j_*[M] \right\rangle \\ &= c = \langle a, i_{0*}[N] \rangle, \end{aligned}$$

using Corollary 4.74. □

Definition 4.77 (Cup product pairing)**.** *The* cup product pairing *on $H^*(M)$ is the bilinear map*

$$(\cdot, \cdot) : H^*(M) \times H^*(M) \to R,$$

given by $(a, b) = \langle a \cup b, [M] \rangle$.

Hence $\langle a, i_{0*}[N] \rangle = \left(a, \mathrm{PD}_{[M]}[N]\right)$.

4.12 Intermission – Nonsingular bilinear pairings

Definition 4.78 (Nonsingular bilinear pairing)**.** *Let V, W be vector spaces over a field $\mathbb{F}$. A bilinear pairing $(\cdot,\cdot) : V \times W \to \mathbb{F}$ is* nonsingular *if*

(i) $(\forall \vec{v} \in V,\ (\vec{v}, \vec{w}) = 0) \implies \vec{w} = 0$.

(ii) $(\forall \vec{w} \in W,\ (\vec{v}, \vec{w}) = 0) \implies \vec{v} = 0$.

Lemma 4.79. *Assume that V and W are finite-dimensional. If the bilinear pairing $(\cdot,\cdot) : V \times W \to \mathbb{F}$ is nonsingular, then the induced linear maps $\varphi : V \to W^*$ and $\psi : W \to V^*$ are isomorphisms.*

Proof. Note that φ, ψ are both injective, so $\dim V \leqslant \dim W^* = \dim W$, and likewise $\dim W \leqslant \dim V$. It follows that φ, ψ are isomorphisms by injectivity. □

4.13 Poincaré duality (continued)

Remark 4.80. *The cup product pairing splits as a sum of pairings*

$$(\cdot,\cdot) : H^k(M) \times H^{n-k}(M) \to R.$$

Example 4.81. *If $N = \{p\} \subseteq M$, Proposition 4.76 implies that $\left\langle 1 \cup \mathrm{PD}_{[M]}\left[\{p\}\right], [M] \right\rangle = \langle 1, [p] \rangle = 1$, from which it follows that*

$$\mathrm{PD}_{[M]}\left[\{p\}\right] = [M]^*.$$

Definition 4.82 (Transverse submanifolds)**.** *Submanifolds $N_1, N_2 \subseteq M$ are said to be* transverse *(and we write $N_1 \pitchfork N_2$) if for every $x \in N_1 \cap N_2$, there is a chart $\varphi_x : U_x \to \mathbb{R}^n$ with $\varphi_x(x) = 0$ such that*

$$\begin{aligned} \varphi_x\left(N_1 \cap U_x\right) &= \mathbb{R}^k \times \mathbb{R}^{n_1-k} \times 0 \subseteq \mathbb{R}^{n-n_1-n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \\ \varphi_x\left(N_2 \cap U_x\right) &= \mathbb{R}^k \times 0 \times \mathbb{R}^{n_2-k} \subseteq \mathbb{R}^{n-n_1-n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned}$$

If so $N' = N_1 \cap N_2$ is a k-dimensional submanifold of N_1, N_2 and M.

In fact, $N_1 \pitchfork N_2$ if $TN_{1|x} + TN_{2|x} = TM_{|x}$ for all $x \in N'$.

Proposition 4.83. *If $N_1 \pitchfork N_2$ and $i_2 : N_2 \hookrightarrow M$ is the inclusion, then*

$$i_2^*\left(\mathrm{PD}_{[M]}\left[N_1\right]\right) = \mathrm{PD}_{[N_2]}\left[N_1 \cap N_2\right].$$

Proof. Let V be a tubular neighbourhood of N_1. If V is small enough, then $V' = N_2 \cap V$ is a tubular neighbourhood of $N' = N_2 \cap N_1$ in N_2. Now consider the diagram:

$$\begin{tikzcd}
 & & & (\mathbb{R}^{n-n_1} \mid 0) \arrow[dl, "\iota_x"'] \arrow[d] \\
(M, \varnothing) \arrow[r, "j"] & (M \mid N_1) & (V \mid N_1) \arrow[l, "i"'] & \left(\nu, \nu^\sharp\right) \arrow[l, "\simeq"'] \\
(N_2, \varnothing) \arrow[u, "i_2"] \arrow[r, "j'"] & (N_2 \mid N') \arrow[u] & (V' \mid N') \arrow[l, "i'"'] \arrow[u, "i_2"] & \left(\nu, \nu^\sharp\right) \arrow[l, "\simeq"'] \\
 & & & (\mathbb{R}^{n-n_1} \mid 0) \arrow[ul, "\iota'_x"] \arrow[u]
\end{tikzcd}$$

We have $i_2 \circ \iota'_x \sim \iota_x$. If u is a Thom class for $(V \mid N_1)$, then ${\iota'_x}^* i_2^*(u) = \iota_x^*(u)$ generates $H^*\left(\mathbb{R}^{n-n_1} \mid 0\right)$, so $i_2^*(u)$ is a Thom class for $(V' \mid N')$. Therefore

$$i_2^*\left(\mathrm{PD}_{[M]}\left[N_1\right]\right) = i_2^* j^* \left(i^*\right)^{-1} u = j^* \left({i'}^*\right)^{-1} i_2^* u = \mathrm{PD}_{[N_2]}\left[N'\right].$$

□

Notation 4.84. *We now assume that $R = \mathbb{F}$ is a field.*

If M is orientable with dual fundamental class $[M]^*$, then $M \times M$ is orientable with dual fundamental class $[M]^* \times [M]^*$.

We shall write $\Delta = \{(x, x),\ x \in M\} \subseteq M \times M$ and $D = \mathrm{PD}_{[M \times M]}[\Delta]$.

Lemma 4.85. *If $a \in H^*(M)$, then*

$$(1 \times a) \cup D = (a \times 1) \cup D.$$

Proof. Consider

$$\begin{tikzcd} V \arrow[r, "i"] & M \times M \\ M \arrow[u, "s_0"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

where V is a tubular neighbourhood of Δ and $\Delta(x) = (x, x)$. Then $s_0 : M \to V$ is a homotopy equivalence, so s_0^* is an isomorphism; therefore

$$s_0^* i^* (a \times 1) = \Delta^* (a \times 1) = a \cup 1 = 1 \cup a = \Delta^* (1 \times a) = s_0^* i^* (1 \times a),$$

hence $i^* (a \times 1) = i^* (1 \times a)$. Therefore $i^* (a \times 1) \cup u = i^* (1 \times a) \cup u$, so $(a \times 1) \cup (i^*)^{-1} u = (1 \times a) \cup (i^*)^{-1} u$, so $j^* \left((a \times 1) \cup (i^*)^{-1} u\right) = j^* \left((1 \times a) \cup (i^*)^{-1} u\right)$, or in other words,

$$(a \times 1) \cup D = (a \times 1) \cup j^* (i^*)^{-1} u = (1 \times a) \cup j^* (i^*)^{-1} u = (1 \times a) \cup D,$$

where $j : (M \times M, \varnothing) \to (M \times M \mid \Delta)$. ☐

Remark 4.86. *Since $R = \mathbb{F}$ is a field, we have*

$$H^*(M \times M) \simeq H^*(M) \otimes H^*(M).$$

Choose a basis $(a_i)_{i \in I}$ of $H^*(M)$; thus $(a_i \times a_j)_{i,j \in I}$ is a basis of $H^*(M \times M)$, so we can write

$$D = \mathrm{PD}_{[M \times M]}[\Delta] = \sum_{i,j \in I} c_{ij} a_i \times a_j = \sum_{i \in I} a_i \times b_i,$$

where $b_i = \sum_{j \in I} c_{ij} a_j \in H^{n - |a_i|}(M)$.

Lemma 4.87. *We have the identity*

$$D = [M]^* \times 1 + \sum_{|a_i| < n} a_i \times b_i.$$

Proof. Consider $i_y : x \in M \mapsto (x, y) \in M \times M$. We have $M \times y \pitchfork \Delta$, so Proposition 4.83 implies that

$$i_y^*(D) = i_y^* \left(\mathrm{PD}_{[M \times M]}[\Delta]\right) = \mathrm{PD}_{[M \times y]}[\Delta \cap M \times y] = \mathrm{PD}_{[M]}[\{y\}] = [M]^*,$$

using Example 4.81. Now

$$i_y^* (a_i \times b_i) = i_y^* (\pi_1^* (a_i) \cup \pi_2^* (b_i)) = (\pi_1 \circ i_y)^* (a_i) \cup (\pi_2 \circ i_y)^* (b_i) = \begin{cases} a_i b_i & \text{if } b_i \in H^0(M) \simeq \mathbb{F} \\ 0 & \text{otherwise} \end{cases}.$$

Write $D = [M]^* \times b_0 + \sum_{|a_i| < n} a_i \times b_i$. Then

$$[M]^* = i_y^*(D) = [M]^* b_0 + 0,$$

so $b_0 = 1$. ☐

Lemma 4.88. *If $a \in H^*(M)$, then*

$$a = (-1)^{n|a|} \sum_{i\in I} (a, a_i)\, b_i.$$

Proof. By Lemma 4.85, we have $(1 \times a) \cup D = (a \times 1) \cup D$ for all a. Therefore

$$\sum_{i\in I} (1 \times a) \cup (a_i \times b_i) = \sum_{i\in I} (a \times 1) \cup (a_i \times b_i)\,,$$

or in other words

$$\sum_{i\in I} (-1)^{|a_i||a|} a_i \times (a \cup b_i) = \sum_{i\in I} (a \cup a_i) \times b_i.$$

Looking at terms of the form $[M]^* \times c$ for $c \in H^0(M) = \mathbb{F}$ and using Lemma 4.87, we have

$$(-1)^{n|a|}\, [M]^* \times a = \sum_{i\in I} \langle a \cup a_i, [M]\rangle\, [M]^* \times b_i.$$

The result follows from the definition of the cup pairing. □

Theorem 4.89 (Poincaré duality)**.** *Suppose $\mathbb{F}$ is a field and M is $\mathbb{F}$-oriented. Then:*

(i) *The cup product pairing $(\cdot,\cdot) : H^k(M;\mathbb{F}) \times H^{n-k}(M;\mathbb{F}) \to \mathbb{F}$ is nonsingular.*

(ii) *There is an isomorphism*

$$\mathrm{PD} : H_k(M;\mathbb{F}) \xrightarrow{\simeq} H^{n-k}(M;\mathbb{F})\,,$$

satisfying $\langle a, x\rangle = (a, \mathrm{PD}(x))$.

Proof. (i) If $(a, b) = 0$ for all b, then $a = 0$ by Lemma 4.88. Moreover,

$$(a, b) = (-1)^{|a||b|}\,(b, a)\,,$$

so $(\cdot,\cdot)$ is nonsingular.

(ii) We have isomorphisms

$$\alpha : H^{n-k}(M) \to \mathrm{Hom}_{\mathbb{F}}\left(H^k(M), \mathbb{F}\right) \qquad \text{and} \qquad \beta : H_k(M) \to \mathrm{Hom}_{\mathbb{F}}\left(H^k(M), \mathbb{F}\right)$$

defined by $\alpha(b)(a) = (a, b)$ and $\beta(x)(a) = \langle a, x\rangle$. It suffices to take $\mathrm{PD} = \alpha^{-1} \circ \beta$. □

4.14 Three more facts

Proposition 4.90. *We have the identity $(a_i, b_j) = (-1)^{|b_j|}\delta_{ij}$.*

Proof. Apply Lemma 4.88 with $a = b_j$. □

Proposition 4.91. *If $E \xrightarrow{\pi} M$ is a vector bundle and $s, s_0 : M \to E$ are sections, then*

$$e(E) = s_0^*\left(\mathrm{PD}_{[E]}[s]\right) = \mathrm{PD}_{[M]}\left[s^{-1}(0)\right]$$

if $s \pitchfork s_0$.

Proof. Use Proposition 4.83. □

Proposition 4.92. *We have*

$$e\,(TM) = \chi(M)\,[M]^*\,.$$

Proof. Note that

$$\langle e(TM), [M]\rangle = (D, D) = \left(\sum_{i\in I} a_i \times b_i, \sum_{i\in I} (-1)^{|a_i||b_i|} b_i \times a_i\right) = \sum_{i\in I} (-1)^{|b_i|} = \chi(M).$$ □

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