# Approximate Group Actions and Ulam Stability

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## Contents



# 1 Introduction to stability

**Definition 1.1** (Hamming metric). *For $n \in \mathbb{N}$, the* normalised Hamming metric *is the metric $d_n$ on $\mathfrak{S}_n$ defined by*

$$d_n(\sigma,\tau) = \frac{1}{n}\left|\left\{i \in \{1,\dots,n\}\,,\ \sigma(i) \neq \tau(i)\right\}\right|.$$

**Theorem 1.2** (Arzhantseva-Paunescu, 2015)**.** *Nearly commuting permutations are near commuting permutations.*

*Or more precisely: for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $n \in \mathbb{N}$ *and for all* $\sigma, \tau \in \mathfrak{S}_n$*, if* $d_n(\sigma\tau, \tau\sigma) < \delta$*, then there exist* $\sigma', \tau' \in \mathfrak{S}_n$ *such that* $\sigma'\tau' = \tau'\sigma'$ *and* $d_n(\sigma, \sigma') + d_n(\tau, \tau') < \varepsilon$*.*

## 1.1 Basic definitions

**Notation 1.3.** *If* $S$ *is a finite (or infinite) set, we shall denote by* $\mathbb{F} = \mathbb{F}_S$ *the free group on* $S$*.*

**Definition 1.4** (Local and global defect)**.** *Let* $E \subseteq \mathbb{F}$ *be a set of reduced words in* $\mathbb{F}$*. Let* $n \in \mathbb{N}$ *and* $f : S \to \mathfrak{S}_n$*.*

(i) *We say that* $f$ *is a* solution *for* $E$ *if* $\widetilde{f}(\omega) = \mathrm{id}_{\mathfrak{S}_n}$ *for all* $\omega \in E$*, where* $\widetilde{f} : \mathbb{F} \to \mathfrak{S}_n$ *is the unique extension of* $f$ *to* $\mathbb{F}$*.*

(ii) *The* local defect *of* $f$ *with respect to* $E$ *is*

$$L_E(f) = \sum_{\omega \in E} d_n\left(\widetilde{f}(\omega), \mathrm{id}_{\mathfrak{S}_n}\right).$$

(iii) *The* global defect *of* $f$ *with respect to* $E$ *is*

$$G_E(f) = \inf_{\substack{h:S\to\mathfrak{S}_n \\ \text{solution for } E}} \sum_{s \in S} d_n\left(f(s), h(s)\right).$$

**Definition 1.5** (Stability)**.** *Let* $E \subseteq \mathbb{F}$ *be a finite set. We say that the family of equations* $(\omega = 1)_{\omega \in E}$ *is* stable (in permutations) *(or that* $E$ *is stable) if there exists* $F : [0, \infty) \to [0, \infty)$ *with* $\lim_0 F = 0$ *such that for all* $n \in \mathbb{N}$ *and for all* $f : S \to \mathfrak{S}_n$*, we have*

$$G_E(f) \leqslant F\left(L_E(f)\right).$$

*In other words, if the local defect is small, then so is the global defect.*

**Remark 1.6.** *The Arzhantseva-Paunescu Theorem says that* $\left\{s_1 s_2 s_1^{-1} s_2^{-1}\right\}$ *is stable (in permutations).*

## 1.2 Connection with group theory

**Definition 1.7** (Stability of groups)**.** *Let* $\Gamma$ *be a group.*

(i) *A sequence of functions* $(f_n : \Gamma \to \mathfrak{S}_n)_{n \in \mathbb{N}}$ *is an* asymptotic homomorphism *is for all* $\gamma_1, \gamma_2 \in \Gamma$*,*

$$d_n\left(f_n(\gamma_1\gamma_2), f_n(\gamma_1) f_n(\gamma_2)\right) \xrightarrow[n\to\infty]{} 0.$$

(ii) *The group* $\Gamma$ *is* stable (in permutations) *if for any asymptotic homomorphism* $(f_n : \Gamma \to \mathfrak{S}_n)_{n \in \mathbb{N}}$*, there is a sequence of homomorphisms* $(h_n : \Gamma \to \mathfrak{S}_n)_{n \in \mathbb{N}}$ *such that, for all* $\gamma \in \Gamma$*,*

$$d_n\left(f_n(\gamma), h_n(\gamma)\right) \xrightarrow[n\to\infty]{} 0.$$

**Proposition 1.8.** *Let* $S$ *be a finite set and let* $E$ *be a finite subset of* $\mathbb{F}_S$*. Then the following assertions are equivalent:*

(i) *The family of equations* $(\omega = 1)_{\omega \in E}$ *is stable.*

(ii) *The group* $\langle S \mid E \rangle$ *is stable.*

*Proof.* Use the fact that $d_n(cad, cbd) = d_n(a, b)$, and fix for each $\gamma \in \Gamma = \langle S \mid E \rangle$ a word over $S^{\pm 1}$ in the class of $\gamma$. □

**Remark 1.9.** *The Arzhantseva-Paunescu Theorem says that* $\mathbb{Z}^2$ *is stable.*

**Theorem 1.10.** *Let* $\Gamma$ *be a finitely generated amenable group. Then* $\Gamma$ *is stable if and only if* $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$*.*

## 1.3 Property testing

**Remark 1.11.** *Suppose given two permutations $a, b \in \mathfrak{S}_n$ with $n$ very large, such that one of the following holds:*

(i) $ab = ba$*,*

(ii) *The pair $(a, b)$ is at a distance at least $\varepsilon$ from the closest commuting pair $(a', b') \in \mathfrak{S}_n \times \mathfrak{S}_n$.*

*We wish to know whether we are in Case* (i) *or* (ii)*.*

*We may use a* "sample and substitute" *algorithm: sample $x_1, \ldots, x_k \in \{1, \ldots, n\}$ uniformly and independently. Then report "Case* (i)*" if $ab\,(x_i) = ba\,(x_i)$ for all $1 \leqslant i \leqslant k$, or "Case* (ii)*" otherwise.*

*Then*

$$\mathbb{P}\left(\textit{Reporting "Case (ii)"} \mid \text{(i)}\right) = 0.$$

*Moreover, the fact that $\left\{s_1 s_2 s_1^{-1} s_2^{-1}\right\}$ is stable implies that we can choose $k$ independently of $n$ such that*

$$\mathbb{P}\left(\textit{Reporting "Case (i)"} \mid \text{(ii)}\right) < \delta.$$

## 1.4 Stability in terms of group actions

**Remark 1.12.** *Functions $f : S \to \mathfrak{S}_n$ correspond bijectively to actions $\mathbb{F}_S \curvearrowright \{1, \ldots, n\}$.*

*We would like to define the local and global defect of an action of $\mathbb{F}_S$ on a finite set $X$.*

**Definition 1.13** (Action graph)**.** *Given an action $\mathbb{F} \curvearrowright X$, the* action graph *is the graph with vertex set $X$, and with an edge labelled by $s$ from $x$ to $s \cdot x$ for all $s \in S$ and $x \in X$.*

**Definition 1.14** (Local and global defect of an action)**.** *Let $E \subseteq \mathbb{F}$. Suppose given an action $\mathbb{F} \curvearrowright X$, where $X$ is a finite set.*

(i) *The* local defect *of $X$ with respect to $E$ is*

$$L_E(X) = \frac{1}{|X|} \left|\{(\omega, x) \in E \times X,\ \omega \cdot x \neq x\}\right|.$$

*This measures how many words in $E$ differ from loops in the action graph.*

(ii) *Consider another action $\mathbb{F} \curvearrowright Y$ with $|Y| = |X|$. If $h : X \to Y$ is a bijection, we set*

$$\|h\|_S = \frac{1}{|X|} \left|\{(s, x) \in S \times X,\ h(s \cdot x) \neq s \cdot h(x)\}\right|.$$

*This measures how far $h$ is from inducing a graph homomorphism on actions graphs. We now define*

$$d_S(X, Y) = \inf_{\substack{h : X \to Y \\ \textit{bijection}}} \|h\|_S\,.$$

*The* global defect *of $X$ with respect to $E$ is*

$$G_E(X) = \inf_{\substack{\mathbb{F} \curvearrowright Y \\ |Y| = |X| \\ L_E(Y) = 0}} d_S(X, Y).$$

**Proposition 1.15.** *Let $f : S \to \mathfrak{S}_n$ and $X = \{1, \ldots, n\}$, equipped with the action induced by $f$. Then*

$$L_E(f) = L_E(X) \qquad \textit{and} \qquad G_E(f) = G_E(X).$$

# 2 Invariant random subgroups and stability

## 2.1 Invariant random subgroups and random stabilisers

**Definition 2.1** (Invariant random subgroups)**.** *Consider a discrete countable group* $\Gamma$*. Denote by* $\mathrm{Sub}(\Gamma)$ *the set of subgroups of* $\Gamma$*. Note that* $\mathrm{Sub}(\Gamma)$ *is a closed subset of the space* $\{0,1\}^\Gamma$ *of all subsets of* $\Gamma$*. The latter, when equipped with the product topology, is a compact metrisable space. Therefore,* $\mathrm{Sub}(\Gamma)$ *is also a compact metrisable space when equipped with the induced topology.*

*Now consider the space* $\mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$ *of Borel probability measures on* $\mathrm{Sub}(\Gamma)$*. We define an action* $\Gamma \curvearrowright \mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$ *by*

$$(\gamma \cdot \mu)(H) = \mu\left(\gamma^{-1} H \gamma\right).$$

*The space of* invariant random subgroups *of* $\Gamma$ *is*

$$\mathrm{IRS}(\Gamma) = \left\{\mu \in \mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right),\ \forall \gamma \in \Gamma,\ \gamma \cdot \mu = \mu\right\}.$$

**Example 2.2.** (i) *If* $N \trianglelefteq \Gamma$ *is a normal subgroup and* $\delta_N$ *is the Dirac measure at* $N$*, then* $\delta_N \in \mathrm{IRS}(\Gamma)$*.*

(ii) *If* $H \leqslant_{fi} \Gamma$ *is a subgroup of finite index, then* $H$ *has finitely many conjugates; write* $\mathcal{H} = \{\gamma H \gamma^{-1},\ \gamma \in \Gamma\}$*. Then*

$$\mu = \frac{1}{k} \sum_{H' \in \mathcal{H}} \delta_{H'} \in \mathrm{IRS}(\Gamma).$$

(iii) *Let* $(X, \nu)$ *be a Borel probability space. Consider an action* $\Gamma \curvearrowright X$ *that is* probability measure preserving*, i.e. such that for all Borel sets* $A \subseteq X$*,* $\nu(\gamma A) = \nu(A)$*. Define a map*

$$\mathrm{st} : x \in X \longmapsto \mathrm{Stab}_\Gamma(x) \in \mathrm{Sub}(\Gamma).$$

*The* random stabiliser *is the probability measure* $\mu$ *on* $\mathrm{Sub}(\Gamma)$ *defined by* $\mu = \mathrm{st}_* \nu$*, i.e.* $\mu(A) = \nu\left(\mathrm{st}^{-1}(A)\right)$*. Then* $\mu \in \mathrm{IRS}(\Gamma)$*.*

*It turns out that every invariant random subgroup arises from a probability measure preserving action in this way (c.f. Proposition 2.4).*

(iv) *Let* $H \leqslant_{fi} \Gamma$*. Consider the action* $\Gamma \curvearrowright \Gamma/H$*. If* $\Gamma/H$ *is equipped with the uniform probability, then it is probability measure preserving. The random stabiliser is the invariant random subgroup of* (ii)*.*

**Remark 2.3.** *The space* $\mathrm{Sub}(\Gamma)$ *is compact and metrisable; it follows that every Borel probability measure* $\mu$ *on* $\mathrm{Sub}(\Gamma)$ *is* outer regular*: for every Borel set* $A \subseteq \mathrm{Sub}(\Gamma)$*,*

$$\mu(A) = \inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U).$$

**Proposition 2.4.** *Let* $\Gamma$ *be a countable group and* $\mu \in \mathrm{IRS}(\Gamma)$*. Then there is a probability space* $(A, \nu)$ *and an action* $\Gamma \curvearrowright (A, \nu)$ *that is probability measure preserving such that the random stabiliser of* $\Gamma \curvearrowright (A, \nu)$ *is* $\mu$*.*

*Proof. First attempt.* Let $A$ be the space of pointed transitive $\Gamma$-spaces up to isomorphism; in other words,

$$A = \left\{[(X, x)],\ \Gamma \curvearrowright X \text{ transitively},\ x \in X\right\},$$

where a bijection $f : (X, x) \to (Y, y)$ is said to be an isomorphism if it is $\gamma$-invariant and satisfies $f(x) = y$. Define an action $\Gamma \curvearrowright A$ by $\gamma \cdot [(X, x)] = [(X, \gamma x)]$, and define a probability measure $\nu$ on

$A$ by the following law: "choose a random subgroup $H \leqslant \Gamma$ according to $\mu$ and output $[(\Gamma/H, H)]$". Let us determine the random stabiliser of $\Gamma \curvearrowright (A, \nu)$. Given $H \subseteq \Gamma$,

$$\begin{aligned}\gamma \in \mathrm{Stab}_\Gamma\left([(\Gamma/H, H)]\right) &\iff (\Gamma/H, \gamma H) \cong (\Gamma/H, H) \\ &\iff \exists f : \Gamma/H \xrightarrow{\cong} \Gamma/H,\ f(\gamma H) = H \\ &\implies \mathrm{Stab}_\Gamma\left(\gamma H\right) = \mathrm{Stab}_\Gamma(H) \\ &\iff \gamma H \gamma^{-1} = H \iff \gamma \in N_\Gamma(H),\end{aligned}$$

where $N_\Gamma(H)$ is the normaliser of $H$ in $\Gamma$. The converse implication is aso true, so the random stabiliser can be described as $N_*\mu$, where $N : H \in \mathrm{Sub}(\Gamma) \mapsto N_\Gamma(H) \in \mathrm{Sub}(\Gamma)$. This does not work; we wanted to restore $\mu$.

The reason why the above attemps fails is that there is too much symmetry: for instance, if $\Gamma = \mathbb{Z}$ and $\mu = \delta_{100\mathbb{Z}}$, then the measure $\nu$ on $A$ will almost surely pick the action $\mathbb{Z} \curvearrowright (\mathbb{Z}/100\mathbb{Z}, 0)$, and the stabiliser of a point is always $\mathbb{Z}$ because

$$x \cdot (\mathbb{Z}/100\mathbb{Z}, 0) = (\mathbb{Z}/100\mathbb{Z}, [x]) \cong (\mathbb{Z}/100\mathbb{Z}, 0)\,.$$

In order to reduce the symmetry, we shall introduce a colouring on the $\Gamma$-spaces $(X, x)$.

*Second attempt.* We define

$$A = \left\{[(X, x, \sigma)]\,,\ \Gamma \curvearrowright X \text{ transitively},\ x \in X,\ \sigma : X \to \{0,1\}^{\mathbb{N}}\right\},$$

where a bijection $f : (X, x, \sigma) \to (Y, y, \tau)$ is said to be an isomorphism if it is $\gamma$-invariant, satisfies $f(x) = y$ and $\sigma = \tau \circ f$. Define an action $\Gamma \curvearrowright A$ by $\gamma \cdot [(X, x, \sigma)] = [(X, \gamma x, \sigma)]$, and define a probability measure $\nu$ on $A$ by the following law: "choose a random subgroup $H \leqslant \Gamma$ according to $\mu$, choose a random function $\sigma : \Gamma/H \to \{0,1\}^{\mathbb{N}}$ uniformly at random and output $[(\Gamma/H, H, \sigma)]$". Now the random stabiliser of $\Gamma \curvearrowright A$ is indeed $\mu$. ☐

## 2.2 Weak-$*$ convergence in $\mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$

**Definition 2.5** (Convergence in $\mathrm{Prob}\left(\mathrm{Sub}\left(\Gamma\right)\right)$)**.** *We equip* $\mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$ *with the weak-$*$ topology.*

*Explicitly, given $(\mu_n)_{n\geqslant 1}$ and $\mu$ in* $\mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$*, we say that $\mu_n \xrightarrow[n\to\infty]{w*} \mu$ (we shall later omit the mention "$w*$" from the notation) if for every $A \subseteq B \subseteq \Gamma$ with $B$ finite,*

$$\mu_n\left(U_{A,B}\right) \xrightarrow[n\to\infty]{} \mu\left(U_{A,B}\right),$$

*where*

$$U_{A,B} = \{H \leqslant \Gamma,\ H \cap B = A\}\,.$$

*Note that $(U_{A,B})_{\substack{A\subseteq B\subseteq\Gamma \\ |B|<\infty}}$ is a basis of open subsets of* $\mathrm{Sub}(\Gamma) \subseteq \{0,1\}^\Gamma$ *that are also closed.*

**Remark 2.6.** *If $\mu$ is a probability measure on a set $X$, then for $A \subseteq B \subseteq C \subseteq X$ with $C$ finite,*

$$\mu\left(U_{B,A}\right) = \sum_{D\subseteq C\setminus B} \mu\left(U_{C,A\cup D}\right).$$

*This equality is called the* consistency relation *for $A, B, C$.*

*Proof.* Note that $U_{B,A} = \coprod_{D\subseteq C\setminus B} U_{C,A\cup D}$. ☐

**Lemma 2.7.** *Let $X$ be a set. Suppose given, for each $A \subseteq B \subseteq X$ with $B$ finite, a real number $\mu\left(U_{A,B}\right) \leqslant 1$ and assume that all the consistency relations are satisfied (c.f. Remark 2.6). Then we can extend $\mu$ to a probability measure on $X$.*

*Proof.* Use the Kolmogorov Extension Theorem. ☐

**Proposition 2.8.** (i) $\mathrm{IRS}(\Gamma)$ *is closed in* $\mathrm{Prob}\,(\mathrm{Sub}(\Gamma))$.

(ii) *Both* $\mathrm{IRS}(\Gamma)$ *and* $\mathrm{Prob}\,(\mathrm{Sub}(\Gamma))$ *are sequentially compact.*

*Proof.* (ii) If $(\mu_n)_{n\geqslant 1}$ is a sequence of probability measures on $\mathrm{Sub}(\Gamma)$, then we can extract a subsequence (using a diagonal argument) such that $(\mu_n(U_{A,B}))_{n\geqslant 1}$ converges to some real number $\mu(U_{A,B})$ for all $A \subseteq B \subseteq \Gamma$ with $B$ finite. It then suffices to apply Lemma 2.7. □

**Example 2.9.** *If* $\Gamma = \mathbb{Z}$, *then* $\delta_{n\mathbb{Z}} \xrightarrow[n\to\infty]{w*} \delta_0$.

**Remark 2.10.** *We now assume that the group* $\Gamma$ *is finitely generated, so there exists a surjective map* $\pi : \mathbb{F} \twoheadrightarrow \Gamma$, *where* $\mathbb{F} = \mathbb{F}_S$ *for some finite set* $S$. *We define a map*

$$\theta : H \in \mathrm{Sub}(\Gamma) \longmapsto \pi^{-1}(H) \in \mathrm{Sub}(\mathbb{F}).$$

*Note that* $\theta$ *is injective and that* $\mathrm{Im}\,\theta$ *is closed in* $\mathrm{Sub}(\mathbb{F})$, *because*

$$\mathrm{Im}\,\theta = \{K \leqslant \mathbb{F},\ \mathrm{Ker}\,\pi \leqslant K\} = \bigcap_{\omega\in\mathrm{Ker}\,\pi} \{K \leqslant F,\ \omega \in K\} = \bigcap_{\omega\in\mathrm{Ker}\,\pi} U^{\mathbb{F}}_{\{\omega\},\{\omega\}}.$$

*We will therefore consider* $\mathrm{Sub}(\Gamma)$ *as a closed subset of* $\mathrm{Sub}(\mathbb{F})$ *(identifying it with* $\bigcap_{\omega\in\mathrm{Ker}\,\pi} U^{\mathbb{F}}_{\{\omega\},\{\omega\}}$*).*

## 2.3 Invariant random subgroups of finite index

**Lemma 2.11.** *If* $\Gamma$ *is a finitely generated group, then for all* $n \in \mathbb{N}$, *the set* $\{H \leqslant \Gamma,\ [\Gamma : H] = n\}$ *is finite.*

*Proof.* Note that the set $\{H \leqslant \Gamma,\ [\Gamma : H] = n\}$ injects into the set of morphisms $\Gamma \to \mathfrak{S}_n$ (which is finite) *via* the action $\Gamma \curvearrowright \Gamma/H$. □

**Definition 2.12** (Invariant random subgroups of finite index)**.** *Let* $\Gamma$ *be a finitely generated group. By Lemma 2.11, we may enumerate the finite index subgroups of* $\Gamma$ *as* $(K_i)_{i\geqslant 1}$. *We then define:*

$$\mathrm{IRS}_{fi}(\Gamma) = \left\{ \mu \in \mathrm{IRS}(\Gamma),\ \exists\, (\alpha_i)_{i\geqslant 1} \in (\mathbb{R}_+)^{\mathbb{N}},\ \sum_{i=1}^{\infty} \alpha_i \delta_{K_i} = \mu_i \text{ and } \sum_{i=1}^{\infty} \alpha_i = 1 \right\}.$$

**Proposition 2.13.** *Let* $\Gamma$ *be a finitely generated group and* $\mu \in \mathrm{IRS}(\Gamma)$. *Then* $\mu \in \overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*}$ *if and only if there is a sequence* $(X_n)_{n\geqslant 1}$ *of finite sets with* $\Gamma$*-action such that*

$$\mu_{X_n} \xrightarrow[n\to\infty]{w*} \mu,$$

*where* $\mu_{X_n}$ *is the random stabiliser of* $\Gamma \curvearrowright X_n$ *(the set* $X_n$ *being equipped with the uniform distribution).*

*Proof.* ($\Leftarrow$) It is clear that if $X_n$ is finite, then the random stabiliser $\mu_{X_n}$ is in $\mathrm{IRS}_{fi}(\Gamma)$.

($\Rightarrow$) If $\mu \in \overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*}$, then there is a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathrm{IRS}_{fi}(\Gamma)$ such that $\nu_n \xrightarrow[n\to\infty]{w*} \mu$. For $n \in \mathbb{N}$, write

$$\nu_n = \sum_{i=1}^{\infty} \alpha_i \frac{1}{|C_i|} \sum_{H\in C_i} \delta_H,$$

where $(C_i)_{i\geqslant 1}$ is the set of conjugacy classes of finite index subgroups of $\Gamma$. Then set

$$\lambda_m = \left(\sum_{i=1}^{m} \alpha_i\right)^{-1} \sum_{i=1}^{m} \alpha_i \frac{1}{|C_i|} \sum_{H\in C_i} \delta_H.$$

Hence $\lambda_m \xrightarrow[m\to\infty]{w*} \nu_n$. Now fix $m$ and choose integers $r_1, \dots, r_m \geqslant 0$ such that $\sum_{i=1}^m r_i = k$ and minimising $\sum_{i=1}^m \left|\alpha_i - \frac{r_i}{k}\right|$; define

$$\kappa_k = \sum_{i=1}^m \frac{r_i}{k} \cdot \frac{1}{|C_i|} \sum_{H \in C_i} \delta_H .$$

Thus $\kappa_k \xrightarrow[k\to\infty]{} \lambda_m$. After having fixed a representative $H_i$ of $C_i$ for all $i$, set

$$X = \coprod_{i=1}^m \left(\Gamma/H_i\right)^{\amalg r_i} .$$

Then $X$ is a finite probability space (equipped with the uniform measure) and its random stabiliser is $\kappa_k$. □

## 2.4 Stability and sequences of actions

**Remark 2.14.** *Let $E \subseteq \mathbb{F}$ and consider an action $\mathbb{F} \curvearrowright X$. Note that the local defect of $X$ with respect to $E$ can be written as*

$$L_E(X) = \sum_{\omega \in E} \mathbb{P}\left(\omega x \neq x\right),$$

*where $x$ follows the uniform distribution on $X$.*

**Proposition 2.15.** *Assume that the group $\Gamma$ has a finite presentation $\langle S \mid E\rangle$ and let $\pi : \mathbb{F} \twoheadrightarrow \Gamma$ be the corresponding homomorphism. Let $(X_n)_{n\geqslant 1}$ be a sequence of finite sets equipped with an $\mathbb{F}$-action. Assume that the sequence $(\mu_{X_n})_{n\geqslant 1}$ of random stabilisers converges in the weak-$*$ topology to $\mu \in \mathrm{IRS}(\mathbb{F})$.*

*Then $L_E\left(X_n\right) \xrightarrow[n\to\infty]{} 0$ if and only if $\mu \in \mathrm{IRS}(\Gamma)$.*

*Proof.* ($\Rightarrow$) Let $\omega \in \operatorname{Ker}\pi = \langle\langle E\rangle\rangle$. Write $\omega = \prod_{i=1}^r g_i \omega_i^{\varepsilon_i} g_i^{-1}$, with $g_i \in \mathbb{F}$, $\omega_i \in E$ and $\varepsilon_i \in \{\pm 1\}$. Thus

$$\begin{aligned}
\mathbb{P}_{X_n}\left(\omega x \neq x\right) &\leqslant \mathbb{P}_{X_n}\left(\bigcup_{i=1}^r \left(g_i \omega_i^{\varepsilon_i} g_i^{-1}\right) x \neq x\right) \leqslant \sum_{i=1}^r \mathbb{P}_{X_n}\left(\omega_i^{\varepsilon_i}\left(g_i^{-1} x\right) \neq g_i^{-1} x\right) \\
&= \sum_{i=1}^r \mathbb{P}_{X_n}\left(\omega_i^{\varepsilon_i} x \neq x\right) = \sum_{i=1}^r \mathbb{P}_{X_n}\left(\omega_i x \neq x\right) \\
&\leqslant r \sum_{w \in E} \mathbb{P}_{X_n}\left(wx \neq x\right) = r L_E\left(X_n\right) \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

It follows that, for $\omega \in \operatorname{Ker}\pi$,

$$\mu_{X_n}\left(U_{\{\omega\},\{\omega\}}\right) = \mathbb{P}_{X_n}\left(\omega \in \operatorname{Stab}_{\mathbb{F}}(x)\right) \xrightarrow[n\to\infty]{} 1.$$

But this also converges to $\mu\left(U_{\{\omega\},\{\omega\}}\right)$, so the latter is equal to 1. Hence,

$$\mu\left(\mathrm{IRS}(\Gamma)\right) = \mu\left(\bigcap_{\omega \in \operatorname{Ker}\pi} \mu\left(U_{\{\omega\},\{\omega\}}\right)\right) = 1,$$

or in other words $\mu \in \mathrm{IRS}(\Gamma)$. □

**Lemma 2.16.** *Let $(Z_n)_{n\geqslant 1}$ be a sequence of finite sets equipped with a $\Gamma$-action (where $\Gamma$ is a discrete countable group) such that $\mu_{Z_n} \xrightarrow[n\to\infty]{w*} \mu \in \mathrm{IRS}(\Gamma)$. Given a sequence of integers $(m_k)_{k\geqslant 1}$ such that $m_k \xrightarrow[k\to\infty]{} \infty$, there is a sequence $(Y_k)_{k\geqslant 1}$ of finite sets equipped with a $\Gamma$-action such that $|Y_k| = m_k$ and $\mu_{Y_k} \xrightarrow[k\to\infty]{w*} \mu$.*

*Proof.* We use the following two ideas:

(i) If $\Gamma \curvearrowright Z$, with $Z$ finite, then $\mu_{Z^{\amalg n}} = \mu_Z$.

(ii) If $\Gamma \curvearrowright T$, with $|T| \ll |Z|$, then $\mu_{Z \amalg T} \approx \mu_Z$.

More precisely, define indices $(i_n)_{n \geqslant 1}$ such that

$$\forall k \in \{i_n, \dots, i_{n+1} - 1\}, \ \frac{|Z_n|}{m_k} < \frac{1}{n}.$$

For $k \geqslant i_1$, write the Euclidean division of $m_k$ by $|Z_{n_k}|$: $m_k = q_k |Z_{n_k}| + r_k$, with $0 \leqslant r_k < |Z_{n_k}|$. Now set

$$Y_k = Z_{n_k}^{\amalg q_k} \amalg T_{r_k},$$

where $T_{r_k}$ is the trivial $\Gamma$-action on $r_k$ points. Hence $|Y_k| = m_k$ and, for all $A \subseteq B \subseteq \Gamma$ with $B$ finite,

$$\mu_{Y_k}(U_{B,A}) = \frac{r_k}{m_k} \mu_{T_{r_k}}(U_{B,A}) + \left(1 - \frac{r_k}{m_k}\right) \mu_{Z_{n_k}}(U_{B,A}) \xrightarrow[k \to \infty]{} \mu(U_{B,A}). \qquad \square$$

**Proposition 2.17.** *Assume that the group $\Gamma$ has a finite presentation $\langle S \mid E \rangle$. Let $(X_n)_{n \geqslant 1}$ be a sequence of finite sets equipped with an $\mathbb{F}$-action such that the sequence $(\mu_{X_n})_{n \geqslant 1}$ of random stabilisers converges in the weak-$*$ topology to $\mu \in \mathrm{IRS}(\mathbb{F})$ with $L_E(X_n) \xrightarrow[n \to \infty]{} 0$. If we assume in addition that*

$$\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma),$$

*then there are finite sets $(Y_n)_{n \in \mathbb{N}}$ equipped with a $\Gamma$-action, such that $|Y_n| = |X_n|$, and*

$$\mu_{Y_n} \xrightarrow[n \to \infty]{w*} \mu.$$

*Proof.* By Proposition 2.15, $\mu \in \mathrm{IRS}(\Gamma) = \overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*}$. Therefore, by Proposition 2.13, there is a sequence $(Z_n)_{n \geqslant 1}$ of finite sets with a $\Gamma$-action such that $\mu_{Z_n} \xrightarrow[n \to \infty]{w*} \mu$. The result now follows from Lemma 2.16. $\square$

**Notation 2.18.** *Given a group $\Gamma$ with a (finite) generating set $S$, we let*

$$B_\Gamma(r) = \left\{ s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}, \ k \leqslant r, \ (s_1, \dots, s_k) \in S^k, \ (\varepsilon_1, \dots, \varepsilon_k) \in \{\pm 1\}^k \right\}.$$

*Moreover, if $A \subseteq B_\Gamma(r)$, we write*

$$U_{r,A} = U_{B_\Gamma(r),A} = \{H \leqslant \Gamma, \ H \cap B_\Gamma(r) = A\}.$$

**Proposition 2.19.** *Let $\Gamma$ be a group with a finite generating set $S$. Given $(\mu_n)_{n \geqslant 1}$ and $\mu$ in $\mathrm{IRS}(\Gamma)$, we have*

$$\mu_n \xrightarrow[n \to \infty]{w*} \mu \iff \forall r \geqslant 1, \ \forall A \subseteq B_\Gamma(r), \ \mu_n(U_{r,A}) \xrightarrow[n \to \infty]{} \mu(U_{r,A}).$$

## 2.5 Expander graphs

**Remark 2.20.** *We wish to show that, in the context of Proposition 2.17, we have $d_S(X_n, Y_n) \xrightarrow[n \to \infty]{} 0$ (c.f. Definition 1.14). This will prove the reverse implication of Theorem 1.10: if $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$, then $\Gamma$ is stable.*

*We first give an example showing that this does not hold without the assumption that $\Gamma$ is amenable.*

**Proposition 2.21.** (i) *If $p$ is a prime number, then the homomorphism $\pi_p : SL_2\mathbb{Z} \to SL_2(\mathbb{Z}/p)$ of reduction modulo $p$ is surjective, and its kernel is*

$$\Gamma(p) = \{I + pA, \ A \in M_2(\mathbb{Z})\} \cap SL_2\mathbb{Z}.$$

(ii) *There are sequences* $(\ell_n)_{n\geqslant 1}$ *and* $(q_n)_{n\geqslant 1}$ *of prime numbers such that the sequence* $\left(\frac{\ell_n}{q_n}\right)_{n\geqslant 1}$ *is decreasing and converges to* 2.

(iii) $|SL_2(\mathbb{Z}/p)| = (p+1)p(p-1)$.

*Proof.* (ii) Use the Prime Number Theorem: if $p_n$ is the numbers of prime numbers at most $n$, then $p_n \sim \frac{n}{\log n}$. □

**Definition 2.22** (Expander graphs)**.** *Let* $X = (V, E)$ *be a finite graph. The* Cheeger constant *of* $X$ *is*

$$h(X) = \min_{\substack{U\subseteq V\\ |U|\leqslant\frac{1}{2}|V|}} \frac{|E(U, V\backslash U)|}{|U|},$$

*where* $E(U_1, U_2)$ *is the set of edges with one endpoint in* $U_1$ *and the other in* $U_2$.

*We say that* $X$ *is an* $\varepsilon$-expander *if* $h(X) \geqslant \varepsilon$.

*A sequence* $(X_n)_{n\geqslant 1}$ *is said to be an* expanding family *(or a sequence of* expander graphs*) if* $|X_n| \xrightarrow[n\to\infty]{} \infty$ *and there exists* $\varepsilon_0 > 0$ *such that* $X_n$ *is an* $\varepsilon_0$*-expander for all* $n$.

**Example 2.23.** *Let* $C_n$ *denote the cycle graph on* $n$ *vertices. Then* $(C_n)_{n\in\mathbb{N}}$ *is not an expanding family because* $h(C_n) \sim \frac{2}{n} \to 0$.

*Margulis used Kazhdan's Property* $(T)$ *to show that, if* $(X_n)_{n\geqslant 1}$ *is a sequence of finite sets equipped with transitive* $SL_3\mathbb{Z}$*-actions, then the graphs of the actions form an expanding family (where* $SL_3\mathbb{Z}$ *is equipped with a finite generating set).*

**Proposition 2.24.** *Equip* $SL_2(\mathbb{Z}/p)$ *with the action of* $SL_2\mathbb{Z}$ *by left multiplication for all primes* $p$*. Then the sequence* $(SL_2(\mathbb{Z}/p))_{p \text{ prime}}$ *is an expanding family (where* $SL_2\mathbb{Z}$ *is equipped with a finite generating set).*

**Example 2.25.** *Let* $(\ell_n)_{n\geqslant 1}$ *and* $(q_n)_{n\geqslant 1}$ *be sequences of prime numbers such that the sequence* $\left(\frac{\ell_n}{q_n}\right)_{n\geqslant 1}$ *is decreasing and converges to* 2*. Define*

$$X_n = SL_2(\mathbb{Z}/\ell_n) \qquad \text{and} \qquad Y_n = SL_2(\mathbb{Z}/q_n)^{\amalg 8} \amalg T_{r_n},$$

*where* $T_{r_n}$ *is the trivial action on* $r_n$ *points, with* $r_n$ *chosen such that* $|X_n| = |Y_n|$.

*Make* $SL_2\mathbb{Z}$ *act on* $(X_n)_{n\geqslant 1}$ *and* $(Y_n)_{n\geqslant 1}$ *in the natural way. Then*

$$\lim_{n\to\infty} \mu_{X_n} = \lim_{n\to\infty} \mu_{Y_n} = \delta_{\{I\}} \in \mathrm{IRS}(SL_2\mathbb{Z}).$$

*However, there is a constant* $\eta_0 > 0$ *such that* $d_S(X_n, Y_n) \geqslant \eta_0$ *for all* $n \in \mathbb{N}$.

*Proof.* Note that $\mu_{X_n} = \delta_{\Gamma(\ell_n)}$. Since $\ell_n \xrightarrow[n\to\infty]{} \infty$, it follows that

$$\mu_{X_n} \xrightarrow[n\to\infty]{w*} \delta_{\{I\}},$$

and similarly for $\mu_{Y_n}$ because $\frac{r_n}{|Y_n|} \xrightarrow[n\to\infty]{} 0$. To find a lower bound for $d_S(X_n, Y_n)$, let $f_n : Y_n \to X_n$ be a bijection. Consider a copy $Z_n$ of $SL_2(\mathbb{Z}/q_n)$ inside $Y_n$. Note that, in the action graph, $E(Z_n, Y_n\backslash Z_n) = \varnothing$. However, since $(X_n)_{n\geqslant 1}$ is an expanding family by Proposition 2.24,

$$|E(f_n(Z_n), X_n\backslash f_n(Z_n))| \geqslant \varepsilon_0 |f_n(Z_n)| \geqslant \frac{1}{16}\varepsilon_0 |X_n|.$$

It follows that $d_S(X_n, Y_n) \geqslant \frac{\varepsilon_0}{16}$. □

## 2.6 Convergence of $\mathbb{F}$-spaces

**Definition 2.26** (Pointed $S$-graphs)**.** *A* pointed $S$-graph of radius at most $r$ *is an oriented graph $Y$ with a distinguished vertex $y \in Y$, where the edges are labelled by $S$, such that every vertex has at most one incoming $s$-edge and one outgoing $s$-edge for all $s \in S$, and every vertex of $Y$ is at a distance at most $r$ from $y$.*

*We denote by $\mathcal{X}_{\bullet,\leqslant r}$ the set of isomorphism classes of pointed $S$-graphs of radius at most $r$. This is a finite space.*

*Given a set $C$, we denote by $\mathcal{X}^C_{\bullet,\leqslant r}$ the set of isomorphism classes of pointed $S$-graphs of radius at most $r$ with a colouring $\sigma : Y \to C$.*

**Definition 2.27** (Convergence of $\mathbb{F}$-spaces)**.** *Given a $\mathbb{F}$-space $(X,\nu)$, there is a map $f_{(X,\nu)} : \mathcal{X}_{\bullet,\leqslant r} \to \mathbb{R}$ defined by*

$$f_{(X,\nu)}\left([(Y,y)]\right) = \mathbb{P}\left(B_X(x,r) \cong (Y,y)\right),$$

*where $x$ is chosen randomly according to $\nu$ and $B_X(x,r)$ is the ball of centre $x$ and radius $r$ in the action graph of $\mathbb{F} \curvearrowright X$ (c.f. Definition 1.13).*

*Now given $\mathbb{F}$-spaces $(X_n)_{n\geqslant 1}$ and $(X,\nu)$, we say that $X_n \xrightarrow[n\to\infty]{} X$ if*

$$\forall r \geqslant 1,\ \forall\, [(Y,y)] \in \mathcal{X}_{\bullet,\leqslant r},\ f_{X_n}\left([(Y,y)]\right) \xrightarrow[n\to\infty]{} f_{(X,\nu)}\left([(Y,y)]\right).$$

*This notion of convergence is equivalent to the convergence of random stabilisers by Proposition 2.19.*

*Similarly, if there are colourings $\sigma_n : X_n \to C$ and $\sigma : X \to C$, then we can define $X_n \xrightarrow[n\to\infty]{} X$ as before, by replacing $\mathcal{X}_{\bullet,\leqslant r}$ by $\mathcal{X}^C_{\bullet,\leqslant r}$.*

**Remark 2.28.** *Recall from Proposition 2.4 that, given an invariant random subgroup $\mu \in \mathrm{IRS}(\Gamma)$, there is a canonical space $(A,\nu)$ associated to $\mu$ whose random stabiliser is $\mu$: $A = \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ is the set of isomorphism classes of spaces $(Y,y,\sigma)$, where $Y$ is equipped with a transitive action of $\mathbb{F}$, $y \in Y$ is a distinguished point, and $\sigma : Y \to \{0,1\}^{\mathbb{N}}$ is a colouring. This set $\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ is equipped with an action of $\mathbb{F}$ given by $\omega \cdot [(Y,y,\sigma)] = [(Y,\omega y,\sigma)]$ and the proof of Proposition 2.4 described the construction of a probability measure $\mu_{\{0,1\}^{\mathbb{N}}}$ on $\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ without mentioning the $\sigma$-algebra.*

*To make the $\sigma$-algebra explicit, we shall equip $\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ with a metric and use the Borel $\sigma$-algebra. We say that two spaces $[(Y_1,y_1,\sigma_1)], [(Y_2,y_2,\sigma_2)] \in \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ are* $r$-locally isomorphic*, and we write $(Y_1,y_1,\sigma_1) \simeq_r (Y_2,y_2,\sigma_2)$ (for $r \geqslant 0$) if*

$$\left(B_1, y_1, (\pi\circ\sigma_1)_{|B_1}\right) \cong \left(B_2, y_2, (\pi\circ\sigma_2)_{|B_2}\right),$$

*where $B_i$ is the ball of centre $y_i$ and radius $r$ in $Y_i$, and $\pi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^r$ is the projection map. Then we set*

$$d\left([(Y_1,y_1,\sigma_1)], [(Y_2,y_2,\sigma_2)]\right) = \exp\left(-r_0\right),$$

*with $r_0 = \sup\{r \geqslant 0,\ (Y_1,y_1,\sigma_1) \simeq_r (Y_2,y_2,\sigma_2)\}$.*

**Remark 2.29.** *The space $\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ is compact and metrisable; it follows that $\mu_{\{0,1\}^{\mathbb{N}}}$ is* outer regular*: for every Borel set $A \subseteq \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$,*

$$\mu(A) = \inf_{\substack{U \supseteq A \\ U \text{ open}}} \mu(U).$$

**Definition 2.30** ($r$-local map)**.** *Let $r \geqslant 0$. A Borel map $f : \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}} \to C$ (with $C$ finite) is said to be* $r$-local *if*

$$f\left([(X,x,\sigma)]\right) = f\left([(Y,y,\tau)]\right)$$

*whenever $(X,x,\sigma) \simeq_r (Y,y,\tau)$.*

**Remark 2.31.** *If $U \subseteq \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}$ is open, then we can write $U = \bigcup_{r\geqslant 0} U_r$, with $U_r \subseteq U_{r+1}$ open, and where $\mathbb{1}_{U_r} : \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}} \to \{0,1\}$ is $r$-local.*

*Proof.* Given $[(X,x,\sigma)] \in \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$ and $r \geqslant 0$, consider the ball of radius $e^{-r}$ centred at $(X,x,\sigma)$:

$$B_{(X,x,\sigma),r} = \left\{(Y,y,\tau) \in \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}},\ (X,x,\sigma) \text{ and } (Y,y,\tau) \text{ are } r\text{-locally isomorphic}\right\}.$$

Given an open set $U \subseteq \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$, define

$$U_r = \bigcup_{B_{(X,x,\sigma),r} \subseteq U} B_{(X,x,\sigma),r}.$$

Then $U_r$ is open, $U_r \subseteq U_{r+1}$, $U = \bigcup_{r\geqslant 0} U_r$ and $\mathbb{1}_{U_r}$ is $r$-local. □

**Remark 2.32.** *By Remarks 2.29 and 2.31, we have the following: if $B \subseteq \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$ is Borel and $\varepsilon > 0$, then:*

(i) *There exists an open set $U \supseteq B$ such that $\mu_{\{0,1\}^\mathbb{N}}(U \backslash B) < \frac{\varepsilon}{2}$.*

(ii) *There exists $r \geqslant 0$ and an open set $U_r \subseteq U$ with $\mathbb{1}_{U_r}$ $r$-local such that $\mu_{\{0,1\}^\mathbb{N}}(U \backslash U_r) < \frac{\varepsilon}{2}$.*

*Hence $\mu_{\{0,1\}^\mathbb{N}}(B \triangle U_r) < \varepsilon$.*

**Lemma 2.33.** *Let $\sigma : \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}} \to C$ be a Borel map with $C$ finite. Let $\varepsilon > 0$. Then there exists $r \geqslant 0$ and an $r$-local map $\ell : \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}} \to C$ such that*

$$\forall c \in C,\ \mu_{\{0,1\}^\mathbb{N}}\left(\sigma^{-1}(c) \triangle \ell^{-1}(c)\right) < \varepsilon.$$

**Definition 2.34** (Typical points of $\mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$)**.** *The subset of* typical points *of $\mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$ is defined by*

$$\mathcal{T}_\bullet^{\{0,1\}^\mathbb{N}} = \left\{[(X,x,\sigma)] \in \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}},\ \sigma \textit{ is injective}\right\}.$$

*We have $\mu_{\{0,1\}^\mathbb{N}}\left(\mathcal{T}_\bullet^{\{0,1\}^\mathbb{N}}\right) = 1$.*

**Remark 2.35.** *Let $[(X,x,\sigma)] \in \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$. Consider the action graph of $\mathbb{F} \curvearrowright \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$, and equip it with the root-colouring: the colour of $[(Y,y,\tau)]$ is $\tau(y)$. Now take the connected component of $[(X,x,\sigma)]$. If $(X,x,\sigma)$ is a typical point, then this component is a coloured pointed $S$-graph isomorphic to $(X,x,\sigma)$.*

## 2.7 The Elek Transfer Theorem

**Proposition 2.36.** *Let $(X_n)_{n\geqslant 1}$ be a sequence of finite $\mathbb{F}$-spaces. For every $n \geqslant 1$ and $y \in X_n$, choose an element $\alpha_n(y) \in \{0,1\}^\mathbb{N}$ uniformly independently at random. This yields a random sequence $\left(\alpha_n : X_n \to \{0,1\}^\mathbb{N}\right)_{n\geqslant 1}$ of colourings. Assume that*

$$\mu_{X_n} \xrightarrow[n\to\infty]{w*} \mu \in \mathrm{IRS}(\mathbb{F}).$$

*Let $\mu_{(X_n,\alpha_n)}$ be the probability measure on $\mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$ defined as follows: choose $x \in X_n$ uniformly at random, and output $(\mathbb{F}\cdot x, x, \alpha_n) \in \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$; let $\mu_{\{0,1\}^\mathbb{N}}$ be the probability measure defined as follows: choose $H \sim \mu \in \mathrm{IRS}(\mathbb{F})$, choose $\alpha : \mathbb{F}/H \to \{0,1\}^\mathbb{N}$ uniformly at random, and output $(\mathbb{F}/H, H, \alpha)$.*

*Then, with probability* 1,

$$\mu_{(X_n,\alpha_n)} \xrightarrow[n\to\infty]{w*} \mu_{\{0,1\}^\mathbb{N}}.$$

*Proof.* Fix $r_0 \geqslant 0$ and $(B,b,\beta) \in \mathcal{X}_{\bullet,\leqslant r_0}^{\{0,1\}^{r_0}}$. Define

$$p_n = \mathbb{P}_{x\in X_n}\left((X_n,x) \simeq_r (B,b)\right) \quad \text{and} \quad p = \mathbb{P}_{(Y,y)\sim\mu}\left((Y,y) \simeq_r (B,b)\right).$$

Hence, $\mu_{X_n} \xrightarrow[n\to\infty]{} \mu$ (without colourings) means that $p_n \xrightarrow[n\to\infty]{} p$. Similarly, let

$$p'_n = \mathbb{P}_{x\in X_n}\left((X_n,x,\alpha_n) \simeq_r (B,b,\beta)\right) \quad \text{and} \quad p' = \mathbb{P}_{(Y,y)\sim\mu}\left((Y,y,\sigma) \simeq_r (B,b,\beta)\right) = 2^{-|B|r_0} p.$$

We want to show that $p'_n \xrightarrow[n\to\infty]{} p'$ with probability 1. Note that

$$p'_n = p_n \cdot \frac{1}{|A_n|} \sum_{a\in A_n} I_a,$$

where $A_n = \{a \in X_n,\ (X_n, a) \simeq_r (B, b)\}$ and, for $a \in A_n$, $I_a = \mathbb{1}\left((X_n, a, \alpha_n) \simeq_r (B, b, \beta)\right)$. Therefore it suffices to show that

$$\frac{1}{|A_n|} \sum_{a\in A_n} I_a \xrightarrow[n\to\infty]{} 2^{-|B|r_0},$$

with probability 1 under the random choice of $(\alpha_n)_{n\geqslant 1}$. Note that $\mathbb{E}\left(I_a\right) = 2^{-|B|r_0}$. However, for $X_n$ fixed, the $(I_a)_{a\in A}$ are not independent, but there are only few dependencies, allowing us to use a modified version of the Law of Large Numbers (due to Elek). □

**Theorem 2.37** (Elek Transfer Theorem)**.** *Let $(X_n)_{n\geqslant 1}$ be a sequence of finite $\mathbb{F}$-spaces. Assume that*

$$\mu_{X_n} \xrightarrow[n\to\infty]{w*} \mu \in \mathrm{IRS}(\mathbb{F}),$$

*or equivalently, $X_n \xrightarrow[n\to\infty]{} \left(\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}, \mu_{\{0,1\}^{\mathbb{N}}}\right)$. Let $\sigma : \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}} \to C$ be a colouring (with $C$ finite). Then there exists a sequence $(\sigma_n : X_n \to C)_{n\geqslant 1}$ of colourings such that*

$$(X_n, \sigma_n) \xrightarrow[n\to\infty]{} \left(\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}, \mu_{\{0,1\}^{\mathbb{N}}}, \sigma\right).$$

*Proof.* By Lemma 2.33, for every $\varepsilon > 0$, there exists $r \geqslant 0$ and an $r$-local map $\ell : \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}} \to C$ such that

$$\mu_{\{0,1\}^{\mathbb{N}}}\left(\ell^{-1}(c) \triangle \sigma^{-1}(c)\right) < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it suffices to show that there is a sequence $(\tau_n : X_n \to C)_{n\geqslant 1}$ of colourings such that

$$(X_n, \tau_n) \xrightarrow[n\to\infty]{} \left(\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}, \mu_{\{0,1\}^{\mathbb{N}}}, \ell\right).$$

*Step 1.* For $n \geqslant 1$, choose $\alpha_n : X_n \to \{0,1\}^{\mathbb{N}}$ uniformly and independently at random. By Proposition 2.36, $\mu_{(X_n,\alpha_n)} \xrightarrow[n\to\infty]{} \mu_{\{0,1\}^{\mathbb{N}}}$ with probability 1. In particular, there exists a sequence $\left(\alpha_n : X_n \to \{0,1\}^{\mathbb{N}}\right)_{n\geqslant 1}$ for which the above holds.

*Step 2.* We have a sequence $(\alpha_n)_{n\geqslant 1}$ of $\{0,1\}^{\mathbb{N}}$-colourings from which we want to deduce $C$-colourings. Define $\tau_n : X_n \to C$ by

$$\tau_n(x) = \ell\left(\underbrace{(X_n, x, \alpha_n)}_{\in \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}}\right) \in C.$$

We want $\mu_{(X_n,\tau_n)} \xrightarrow[n\to\infty]{} \mu_{\left(\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}, \mu_{\{0,1\}^{\mathbb{N}}}, \ell\right)}$. Define

$$L : (X, x, \alpha) \in \mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}} \longmapsto (X, x, y \mapsto \ell\,(X, y, \alpha)) \in \mathcal{X}_\bullet^{C}.$$

Then $\mu_{(X_n,\tau_n)} = L_*\mu_{(X_n,\alpha_n)}$. The fact that $\ell$ is $r$-local implies the continuity of $L_*$, i.e.

$$\lim_{n\to\infty} \mu_{(X_n,\tau_n)} = \lim_{n\to\infty} L_*\mu_{(X_n,\alpha_n)} = L_*\mu_{\{0,1\}^{\mathbb{N}}}.$$

Now, since $\mu_{\{0,1\}^{\mathbb{N}}}\left(\mathcal{T}^{\{0,1\}^{\mathbb{N}}}\right) = 1$, we obtain $L_*\mu_{\{0,1\}^{\mathbb{N}}} = \mu_{\left(\mathcal{X}_\bullet^{\{0,1\}^{\mathbb{N}}}, \mu_{\{0,1\}^{\mathbb{N}}}, \ell\right)}$. □

## 2.8 The Ornstein-Weiss Theorem for amenable groups

**Example 2.38.** *Consider the group $\mathbb{Z}^2$ with its generating set $S = \{(1,0),(0,1)\}$. If $B_n$ is the ball centred at* 0 *and with radius $n$ in $\mathbb{Z}^2$, then $|B_n| = (2n+1)^2$ and $|\partial B_n| = 4(2n+1)$. Therefore*

$$\frac{|\partial B_n|}{|B_n|} \sim \frac{2}{n} \xrightarrow[n\to\infty]{} 0.$$

*Hence, it is "relatively cheap" to disconnect many points in* $\mathrm{Cay}(\mathbb{Z}^2, S)$.

**Definition 2.39** (Amenable group)**.** *Let $\Gamma$ be a group with a finite generating set $S$.*

(i) *A* Følner sequence *for $\Gamma$ is a sequence $(F_n)_{n\geqslant 1}$ of finite subsets of $\Gamma$ such that $\Gamma = \bigcup_{n\geqslant 1} F_n$ and, for all $s \in S^{\pm 1}$,*

$$\frac{|F_n \triangle s F_n|}{|F_n|} \xrightarrow[n\to\infty]{} 0.$$

(ii) *$\Gamma$ is* amenable *if it has a* Følner sequence.

**Example 2.40.** *The free group $\mathbb{F}_2$ is not amenable.*

**Definition 2.41** (Borel equivalence relation)**.** *Let $X$ be a compact metric space. A* Borel equivalence relation *is an equivalence relation $E \subseteq X \times X$ that is Borel as a subset of $X \times X$ (note that it does not matter whether $X \times X$ is equipped with the product $\sigma$-algebra or with the Borel $\sigma$-algebra).*

**Notation 2.42.** *Let $\Gamma \curvearrowright (X,\nu)$ be a probability measure preserving action. Denote*

$$E_X = \{(x,\gamma x),\ x \in X,\ \gamma \in \Gamma\} \subseteq X \times X.$$

*Then $E_X$ is a Borel equivalence relation on $X$. If $\Gamma$ is countable, then every equivalence class of $E_X$ is countable.*

**Definition 2.43** (Hyperfinite equivalence relations)**.** *Let $X$ be a compact metric space.*

(i) *An equivalence relation $E \subseteq X \times X$ is* finite *if every equivalence class is finite.*

(ii) *An equivalence relation $E \subseteq X \times X$ is* hyperfinite *if there is a sequence $(E_n)_{n\geqslant 1} \subseteq X \times X$ such that $E_n$ is a finite Borel equivalence relation on $X$, $E_n \subseteq E_{n+1}$ for all $n$, and $E = \bigcup_{n\geqslant 1} E_n$.*

**Example 2.44.** *Let $X = \{0,1\}^{\mathbb{N}}$. Define $E_0 \subseteq X \times X$ by $(x,y) \in E_0$ if and only if there exists $n \in \mathbb{N}$ such that $x_m = y_m$ for all $m \geqslant n$. Then $E_0$ is hyperfinite, because $E_0 = \bigcup_{n\geqslant 1} E_n$, where $E_n = \{(x,y) \in X \times X,\ \forall m \geqslant n,\ x_m = y_m\}$.*

**Definition 2.45** (Hyperfinite action)**.** *A probability measure preserving action $\Gamma \curvearrowright (X,\nu)$ is* hyperfinite *if there exists $X_0 \subseteq X$ Borel such that*

(i) *$\nu(X_0) = 1$,*

(ii) *$X_0$ is a union of orbits of $\Gamma \curvearrowright (X,\nu)$,*

(iii) *The equivalence relation $E_{X_0}$ is hyperfinite.*

**Theorem 2.46** (Ornstein-Weiss)**.** *Let $\Gamma$ be a countable amenable group. Then any probability measure preserving action $\Gamma \curvearrowright (X,\nu)$ is hyperfinite.*

**Remark 2.47.** (i) *The original motivation (due to Ornstein-Weiss and Dye) for the Ornstein-Weiss Theorem was to show that all monoatomic ergodic probability measure preserving actions of amenable groups are orbit equivalent.*

(ii) *The Ornstein-Weiss Theorem is usually proved as a special case of the Connes-Feldmann-Weiss Theorem.*

**Proposition 2.48.** *Let $\mathbb{F} = \mathbb{F}_S$ be the free group on a finite set $S$. Let $\varepsilon > 0$. Given an action $\mathbb{F} \curvearrowright (X, \nu)$ that is probability measure preserving and hyperfinite, there is $\sigma : X \to \mathcal{P}(S)$ and an integer $k \geqslant 1$ such that*

(i) *Every connected component has at most $k$ vertices in the graph with vertex set $X$, and with edge set $\{(x, sx), \ x \in X, \ s \in \sigma(x)\}$.*

(ii) $\nu(\sigma^{-1}(\{S\})) > 1 - \varepsilon$.

*Proof.* Let $X_0 \subseteq X$ such that $\nu(X_0) = 1$, $\Gamma \curvearrowright X_0$ (i.e. $X_0$ is a union of $\Gamma$-orbits) and $E_{X_0}$ is hyperfinite. There is a sequence $(E_n)_{n\geqslant 1}$ of finite Borel equivalence relations on $X_0$, $E_n \subseteq E_{n+1}$ for all $n$, and $E_{X_0} = \bigcup_{n\geqslant 1} E_n$. Define $\tau_n : X_0 \to \mathcal{P}(S)$ by

$$\tau_n(x) = \{s \in S, \ (x, sx) \in E_n\}.$$

Note that $X_0 = \bigcup_{n\geqslant 1} \tau_n^{-1}(\{S\})$, and the sequence $(\tau_n^{-1}(\{S\}))_{n\geqslant 1}$ is increasing; it follows that

$$\nu\left(\tau_n^{-1}(\{S\})\right) \xrightarrow[n\to\infty]{} \nu(X_0) = 1.$$

In particular, there exists $N \geqslant 1$ such that $\nu\left(\tau_N^{-1}(\{S\})\right) > 1 - \frac{\varepsilon}{2}$. Consider the graph with vertex set $X$ and with edge set $\{(x, sx), \ x \in X, \ s \in \tau_N(x)\}$. All its components are finite, but there may not be a uniform bound on the size of components. For $k \geqslant 1$, let

$$L_k = \left\{(x_1, x_2) \in E_N, \ [x_1]_{E_N} \leqslant k\right\}.$$

Again, $E_N = \bigcup_{k\geqslant 1} L_k$. Consider

$$\sigma_k(x) = \{s \in S, \ (x, sx) \in L_k\}.$$

Then $\tau_N^{-1}(\{S\}) = \bigcup_{k\geqslant 1} \sigma_k^{-1}(\{S\})$, so

$$\nu\left(\sigma_k^{-1}(\{S\})\right) \xrightarrow[k\to\infty]{} \nu\left(\tau_N^{-1}(\{S\})\right) > 1 - \frac{\varepsilon}{2},$$

so there exists $K \geqslant 1$ such that $\nu\left(\sigma_K^{-1}(\{S\})\right) > 1 - \varepsilon$. □

## 2.9 The Newman-Sohler-Elek Theorem

**Theorem 2.49** (Newman-Sohler-Elek)**.** *Let $\Gamma = \langle S \mid E \rangle$ be a finitely generated amenable group, $(X_n)_{n\geqslant 1}$ and $(Y_n)_{n\geqslant 1}$ be sequences of finite $\Gamma$ spaces with $|X_n| = |Y_n|$. Assume that*

$$\lim_{n\to\infty} \mu_{X_n} = \lim_{n\to\infty} \mu_{Y_n} = \mu \in \mathrm{IRS}(\Gamma).$$

*Then $d_S(X_n, Y_n) \xrightarrow[n\to\infty]{} 0$, where $d_S$ is the distance introduced in Definition 1.14.*

*Proof.* We have an action $\mathbb{F} \curvearrowright \left(\mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}, \mu_{\{0,1\}^\mathbb{N}}\right)$, with $\mu_{\left(\mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}, \mu_{\{0,1\}^\mathbb{N}}\right)} = \mu$. Since $\mu \in \mathrm{IRS}(\Gamma)$, there exists $A \subseteq \mathcal{X}_\bullet^{\{0,1\}^\mathbb{N}}$ with $\mu_{\{0,1\}^\mathbb{N}}(A) = 1$ and $\mu_{\left(A, \mu_{\{0,1\}^\mathbb{N}}\right)} = \mu$, such that $\Gamma \curvearrowright \left(A, \mu_{\{0,1\}^\mathbb{N}}\right)$. Now,

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = \left(A, \mu_{\{0,1\}^\mathbb{N}}\right).$$

By the Ornstein-Weiss Theorem (Theorem 2.46), the action $\Gamma \curvearrowright \left(A, \mu_{\{0,1\}^\mathbb{N}}\right)$ is hyperfinite. Therefore, given $\varepsilon > 0$, there is $\sigma : A \to \mathcal{P}(S)$ and $k \geqslant 1$ as in Proposition 2.48. We now think of $\mathcal{P}(S)$

as a set of colours and use the Elek Transfer Theorem (Theorem 2.37) twice: we obtain sequences $(\sigma_n : X_n \to \mathcal{P}(S))_{n\geqslant 1}$ and $(\tau_n : Y_n \to \mathcal{P}(S))_{n\geqslant 1}$ such that

$$\lim_{n\to\infty} (X_n, \sigma_n) = \lim_{n\to\infty} (Y_n, \tau_n) = \left(A, \mu_{\{0,1\}^{\mathbb{N}}}, \sigma\right).$$

By $(*)$ with radius 0, we obtain

$$\lim_{n\to\infty} \frac{|\sigma_n^{-1}(S)|}{|X_n|} = \lim_{n\to\infty} \frac{|\tau_n^{-1}(S)|}{|Y_n|} = \mu_{\{0,1\}^{\mathbb{N}}}\left(\sigma^{-1}(S)\right) \geqslant 1-\varepsilon.$$

By $(*)$ with radius $k$, almost all connected components are of size at most $k$ in the graph with vertex set $X_n$ and edge set $\{(x, sx)\,,\ x \in X_n,\ s \in \sigma_n(x)\}$, and similarly for $Y_n$. This yields

$$d_S(X_n, Y_n) \leqslant 2\,|S|\,\varepsilon$$

for $n$ large enough. □

**Remark 2.50.** *Work of Elek and Szalo shows that Theorem 2.49 is actually a characterisation of amenability.*

**Corollary 2.51.** *Let $\Gamma = \langle S \mid E\rangle$ be a finitely generated amenable group. If $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$, then $\Gamma$ is stable.*

*Proof.* If $\Gamma$ is not stable, then there exists an $\varepsilon_0 > 0$ and a sequence $(X_n)_{n\geqslant 1}$ of finite $\mathbb{F}$-spaces such that $L_E(X_n) < \frac{1}{n}$ and $G_E(X_n) \geqslant \varepsilon_0$. Since $\mathrm{IRS}(\mathbb{F})$ is compact (by Proposition 2.8), we can replace $(X_n)_{n\geqslant 1}$ by a subsequence such that $(\mu_{X_n})_{n\geqslant 1}$ converges to some $\mu \in \mathrm{IRS}(\mathbb{F})$. By Proposition 2.15, $\mu \in \mathrm{IRS}(\Gamma)$ because $L_E(X_n) \xrightarrow[n\to\infty]{} 0$. By Proposition 2.17, since $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$, there is a sequence $(Y_n)_{n\geqslant 1}$ of $\Gamma$-spaces such that $|Y_n| = |X_n|$ and $\mu_{Y_n} \xrightarrow[n\to\infty]{} \mu$. Now, by Theorem 2.49, $d_S(X_n, Y_n) \xrightarrow[n\to\infty]{} 0$, so $\varepsilon_0 \leqslant G_E(X_n) \xrightarrow[n\to\infty]{} 0$, a contradiction. □

# 3 Examples of stable groups

## 3.1 Stability of $\mathbb{Z}^d$

**Definition 3.1** (Almost normal subgroup, profinitely closed subgroup)**.** *Let $\Gamma$ be a group and let $H \leqslant \Gamma$ be a subgroup.*

(i) *$H$ is* almost normal *if $[\Gamma : N_\Gamma(H)] < \infty$, where $N_\Gamma(H) = \{\gamma \in \Gamma,\ \gamma H \gamma^{-1} = H\}$.*

(ii) *$H$ is* profinitely closed *if $H = \bigcap_{H\leqslant K \leqslant_{fi} \Gamma} K$.*

**Proposition 3.2.** *Let $\Gamma$ be an amenable group. Assume that $\mathrm{Sub}(\Gamma)$ is countable and every almost normal subgroup of $\Gamma$ is profinitely closed. Then $\Gamma$ is stable.*

*Proof.* By Corollary 2.51, it suffices to show that, given $\mu \in \mathrm{IRS}(\Gamma)$, we have $\mu \in \overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*}$. Since $\mathrm{Sub}(\Gamma)$ is countable, we can write

$$\mu = \sum_{C\in\mathcal{C}} \alpha_C \frac{1}{|C|} \sum_{H\in C} \delta_H$$

for some $(\alpha_C)_{C\in\mathcal{C}}$ nonnegative such that $\sum_{C\in\mathcal{C}} \alpha_C = 1$, where $\mathcal{C}$ is the set of all conjugacy classes of subgroups of $\Gamma$ which are almost normal.

It is therefore enough to show that $\frac{1}{|C|}\sum_{H\in C} \delta_H \in \overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*}$ for all $C \in \mathcal{C}$. But $H$ is profinitely closed by assumption, so there are subgroups $K_i \trianglelefteq_{fi} N_\Gamma(H) \leqslant_{fi} \Gamma$ for $i \in I$, with $H = \bigcap_{i\in I} K_i$. Hence, if

$$\nu_i = \frac{1}{[\Gamma : N_\Gamma(H)]} \sum_{g\in\Gamma/N_\Gamma(H)} \delta_{gK_ig^{-1}},$$

then $\nu_i \xrightarrow[i\to\infty]{} \frac{1}{|C|}\sum_{g\in\Gamma/N_\Gamma(H)} \delta_{gHg^{-1}}$. □

**Corollary 3.3.** *$\mathbb{Z}^d$ is stable.*

*Proof.* We know that $\mathbb{Z}^d$ is amenable, $\mathrm{Sub}\left(\mathbb{Z}^d\right)$ is countable. Given $H \leqslant \mathbb{Z}^d$, we can write $H = m_1\mathbb{Z} \oplus \cdots \oplus m_\ell\mathbb{Z}$ with $\mathbb{Z}^d = H \oplus \mathbb{Z}^r$ after a change of basis in $\mathbb{Z}^d$. If $H_i = m_1\mathbb{Z} \oplus \cdots \oplus m_\ell\mathbb{Z} \oplus (i\mathbb{Z})^r$ for $i \geqslant 1$, then $H \leqslant H_i \leqslant_{fi} \mathbb{Z}^d$ and $H = \bigcap_{i\geqslant 1} H_i$, so we can apply Proposition 3.2. ☐

## 3.2 Stability of virtually polycyclic groups

**Definition 3.4** (Polycyclic group)**.** *A group $\Gamma$ is* polycyclic *if there is a sequence*

$$1 = \Gamma_0 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma_2 \trianglelefteq \cdots \trianglelefteq \Gamma_n = \Gamma,$$

*such that $\Gamma_{i+1}/\Gamma_i$ is cyclic for all $0 \leqslant i < n$.*

*The* Hirsch length *of $\Gamma$, denoted by $h(\Gamma)$, is defined to be the number of infinite cyclic factors in the above sequence.*

**Example 3.5.** (i) *Finitely generated nilpotent groups are polycyclic.*

(ii) *$D_\infty$ is polycyclic.*

**Proposition 3.6.** *Let $\Gamma$ be a polycyclic group.*

(i) *Every subgroup $H \leqslant \Gamma$ is finitely generated. In particular,* $\mathrm{Sub}(\Gamma)$ *is countable.*

(ii) *If $\Gamma$ is infinite, then there is a subgroup $A \leqslant \Gamma$ such that $A \cong \mathbb{Z}^r$ for some $r \geqslant 1$.*

**Proposition 3.7.** *If $1 \to N \to \Gamma \to Q \to 1$ is an exact sequence of polycyclic groups, then*

$$h(\Gamma) = h(N) + h(Q).$$

**Definition 3.8** (LERF group)**.** *A group is said to be* LERF *if every finitely generated subgroup is profinitely closed.*

**Example 3.9.** *Free groups are LERF.*

**Theorem 3.10** (Maltsev)**.** *Polycyclic groups are LERF.*

*Proof.* Let $\Gamma$ be a polycyclic group. Note that all subgroups of $\Gamma$ are finitely generated, so we want to show that, if $H \leqslant \Gamma$, then $H = \bigcap_{H\leqslant K\leqslant_{fi}\Gamma} K$.

Let $g \in \Gamma\backslash H$. We want to construct $H \leqslant K \leqslant_{fi} \Gamma$ such that $g \notin K$. If $\Gamma$ is abelian, this is the proof of Corollary 3.3. If $h(\Gamma) = 0$, the result is clear. We proceed by induction on $h(\Gamma)$, assuming that $h(\Gamma) \geqslant 1$. By Proposition 3.6, there exists $A \trianglelefteq \Gamma$ such that $A \cong \mathbb{Z}^r$ for some $r \geqslant 1$.

We claim that there exists $m \geqslant 1$ such that

$$g \notin HA^m.$$

If this were false, then in particular $g \in HA$, so we could write $g = ha$ with $h \in H$ and $a \in A$. Since $g \notin H$, $a \notin H \cap A$. Hence, we have

$$a \notin H \cap A \leqslant A.$$

By the abelian case, there exists $H \cap A \leqslant B \leqslant_{fi} A$ such that $a \notin B$. Since $[A : B] < \infty$, there exists $m \geqslant 1$ such that $A^m \leqslant B$. By assumption, $g \in HA^m$, so we can write $g = h_1a_1$ with $h_1 \in H$ and $a_1 \in A^m \leqslant B$. It follows that $ha = g = h_1a_1$, so

$$a_1a^{-1} = h_1^{-1}h \in A \cap B \leqslant B,$$

so $a \in B$, a contradiction.

Therefore, there exists $m \geqslant 1$ such that $g \notin HA^m$. Now we have

$$gA^m \notin HA^m/A^m \leqslant \Gamma/A^m.$$

Since $h\left(\Gamma/A^m\right) = h(\Gamma) - h\left(A^m\right) < h(\Gamma)$, the induction hypothesis implies that there is $HA^m/A^m \leqslant K/A^m \leqslant_{fi} \Gamma/A^m$ such that $gA^m \notin K/A^m$. In particular, $g \notin K$ and $H \leqslant K \leqslant_{fi} \Gamma$. ☐

**Corollary 3.11.** *Polycyclic groups are stable.*

*In fact, virtually polycyclic groups are stable.*

**Example 3.12.** *For all $n \in \mathbb{Z}$, the group* $\mathrm{BS}(1,n) = \langle x, y \mid yxy^{-1} = x^n \rangle$ *is stable.*

## 3.3 Sufficient condition for instability

**Definition 3.13** (Sofic group). *Let $\Gamma$ be a finitely generated group equipped with a surjective homomorphism $\pi : \mathbb{F} \to \Gamma$, where $\mathbb{F} = \mathbb{F}_S$ for some finite set $S$. We say that $\Gamma$ is* sofic *if*

$$\delta_{\mathrm{Ker}\,\pi} = \delta_{1_\Gamma} \in \overline{\mathrm{IRS}_{fi}(\mathbb{F})}^{w*}.$$

*Equivalently, $\Gamma$ is sofic if there is a sequence $(X_n)_{n\geqslant 1}$ of finite $\mathbb{F}$-sets such that*

$$\mu_{X_n} \xrightarrow[n\to\infty]{} \delta_{1_\Gamma}.$$

*It is an open problem to know whether or not there exist non-sofic groups.*

**Definition 3.14** (Residually finite group). *A group $\Gamma$ is said to be* residually finite *if*

$$\bigcap_{K \trianglelefteq_{fi} \Gamma} = 1_\Gamma.$$

**Proposition 3.15.** *Residually finite groups are sofic.*

**Proposition 3.16.** *If a group $\Gamma$ is sofic but not residually finite, then $\Gamma$ is not stable.*

*Proof.* Since $\Gamma$ is not residually finite, there exists $\gamma_0 \in \left(\bigcap_{K \leqslant_{fi} \Gamma} K\right) \backslash \{1\}$. Pick $w_0 \in \mathbb{F}$ such that $\pi(w_0) = \gamma_0$, and denote by $\ell_0$ the length (relative to $S$) of $w_0$. Since $\Gamma$ is sofic, there is a sequence $(X_n)_{n\geqslant 1}$ of finite $\mathbb{F}$-spaces such that

$$\mu_{X_n} \xrightarrow[n\to\infty]{} \delta_{\mathrm{Ker}\,\pi}.$$

We consider balls of radius $\ell_0$, and we define

$$A_n = \{x \in X_n,\ (X_n, x) \simeq_{\ell_0} (\Gamma, 1)\}.$$

Therefore $\frac{|A_n|}{|X_n|} \xrightarrow[n\to\infty]{} 1$. Let $B_n$ be a maximal subset of $A_n$ such that the balls $(B(x, \ell_0))_{x\in B_n}$ are disjoint. By maximality, $|B_n| \geqslant (2|S|)^{-2\ell_0} |A_n|$; it follows that for $n$ large enough,

$$|B_n| \geqslant \underbrace{\frac{1}{2}(2|S|)^{-2\ell_0}}_{C} |X_n|.$$

But for every $x \in B_n$, we have $w_0 x \neq x$ because $\pi(w_0) \neq 1$. Hence, if $Y$ is a finite $\Gamma$-space with $|Y| = |X_n|$, and $y \in Y$, then $[\Gamma : \mathrm{Stab}_\Gamma(y)] < \infty$, so $\gamma_0 \in \mathrm{Stab}_\Gamma(y)$, i.e. $\gamma_0 y = y$, so $w_0 y = y$. It follows that

$$G_E(X_n) \geqslant d_S(X_n, Y) \geqslant C > 0,$$

but $L_E(X_n) \xrightarrow[n\to\infty]{} 0$. Hence, $\Gamma$ is not stable. □

## 3.4 Instability of BS(2, 3)

**Definition 3.17** (Baumslag–Solitar groups). *We define* $\mathrm{BS}(m,n) = \langle x, y \mid yx^m y^{-1} = x^n \rangle$.

**Definition 3.18** (Metabelian group). *A group $\Gamma$ is said to be* metabelian *if one of the following two equivalent assertions is satisfied:*

(i) *$\Gamma$ is nilpotent of class at most 2.*

(ii) *There is an exact sequence* $1 \to A_1 \to \Gamma \to A_2 \to 1$ *with* $A_1, A_2$ *abelian.*

**Proposition 3.19.** $\mathrm{BS}(2,3)$ *is free by metabelian, i.e. there is an exact sequence*

$$1 \to \mathbb{F} \to \mathrm{BS}(2,3) \to Q \to 1,$$

*with* $\mathbb{F}$ *free and* $Q$ *metabelian.*

*Proof.* Write $\Gamma = \mathrm{BS}(2,3) = \langle x, y \mid yx^2y^{-1} = x^3 \rangle$. Consider $\Gamma'' \trianglelefteq \Gamma$. It is clear that $Q = \Gamma/\Gamma''$ is metabelian, so it suffices to prove that $\Gamma''$ is free. Define $\varphi : \Gamma \to GL_2\mathbb{Q}$ by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad y \mapsto \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives a well-defined group homomorphism because $\varphi(y)\varphi(x)^2\varphi(y)^{-1} = \varphi(x)^3$.

Moreover, $\operatorname{Im}\varphi$ is included in the subgroup $T \subseteq GL_2\mathbb{Q}$ of upper triangular matrices. But

$$T' \subseteq \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \, c \in \mathbb{Q} \right\} \cong \mathbb{Q},$$

so $T'$ is abelian and $T$ is metabelian. It follows that $\operatorname{Im}\varphi = \Gamma/\operatorname{Ker}\varphi$ is metabelian, so

$$\Gamma'' \subseteq \operatorname{Ker}\varphi.$$

Now consider the action $\Gamma'' \curvearrowright \Gamma/\langle x \rangle$. Note that this action is free: if $\gamma \in \Gamma''$, then

$$\gamma\gamma_0 \langle x \rangle = \gamma_0 \langle x \rangle \implies \gamma_0^{-1}\gamma\gamma_0 \langle x \rangle = \langle x \rangle \implies \gamma_0^{-1}\gamma\gamma_0 \in \langle x \rangle \cap \Gamma'' \subseteq \langle x \rangle \cap \operatorname{Ker}\varphi = \{1\}.$$

Note that $\Gamma$ can be written as the HNN extension $\Gamma = \mathbb{Z}*_\phi$, where $\phi : 3\mathbb{Z} \xrightarrow{\cong} 2\mathbb{Z}$, and the action $\Gamma \curvearrowright \Gamma/\langle x \rangle$ corresponds to $\mathbb{Z}*_\phi \curvearrowright \mathbb{Z}*_\phi/\mathbb{Z}$. Bass-Serre theory tells us that $\Gamma'' \curvearrowright \Gamma/\langle x \rangle$ is a free action on a tree by graph automorphisms without edge inversions, so $\Gamma''$ is a free group; in fact, $\Gamma'' \cong \mathbb{F}_{\aleph_0}$. □

**Definition 3.20** (Residually amenable group)**.** *A group* $\Gamma$ *is said to be* residually amenable *if there is a sequence* $(H_n)_{n\geqslant 1}$ *of normal subgroups of* $\Gamma$ *such that* $\bigcap_{n\geqslant 1} H_n = \{1\}$, $H_{n+1} \leqslant H_n$ *and* $\Gamma/H_n$ *is amenable for all* $n \geqslant 1$.

**Proposition 3.21.** *Free by metabelian groups are residually solvable hence residually amenable.*

*Sketch of proof.* Consider an exact sequence $1 \to \mathbb{F} \to \Gamma \to Q \to 1$.

First note that $\mathbb{F}$ is residually finite: take a freely generating set $S = \{s_1, s_2, \dots\}$ for $\mathbb{F}$ and let $w = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}$, $\varepsilon_j \in \{\pm 1\}$. Consider the line graph $X$ with $(k+1)$ vertices $v_0, \dots, v_k$, where the $v_{j-1}$ and $v_j$ are linked by an edge labelled by $s_{i_j}$, going towards $v_j$ if $\varepsilon_j = 1$, or towards $v_{j-1}$ otherwise. This graph can be completed to an action of $\mathbb{F}$ on $X$. This gives a group homomorphism $\rho_w : \mathbb{F} \to \mathfrak{S}_X$ with $\rho_w(w) \neq 1$. Hence, $\operatorname{Ker}\rho_w \trianglelefteq_{fi} \mathbb{F}$ and $w \notin \operatorname{Ker}\rho_w$, so $\mathbb{F}$ is residually finite.

Iwasawa used this idea to prove a stronger result: if $w \in \mathbb{F}\backslash\{1\}$, then there exists $r \geqslant 1$, $n \geqslant 1$ and

$$\rho_w : \mathbb{F} \to UT_r\left(\mathbb{Z}/p^n\right)$$

such that $\rho_w(w) \neq 1$, where $UT_r\left(\mathbb{Z}/p^n\right)$ is the subgroup of $GL_r\left(\mathbb{Z}/p^n\right)$ of upper triangular matrices with ones on the diagonal (c.f. Robinson for more details).

Now $UT_r\left(\mathbb{Z}/p^n\right)$ is a finite $p$-group, so it is nilpotent and therefore solvable. Now take $\rho_w : \mathbb{F} \to UT_r\left(\mathbb{Z}/p^n\right)$ as above. Since $UT_r\left(\mathbb{Z}/p^n\right)$ is step-$\ell$ solvable for some $\ell$, we have $\mathbb{F}^{(\ell)} \leqslant \operatorname{Ker}\rho_w$. But $w \notin \operatorname{Ker}\rho_w$, so $w \notin \mathbb{F}^{(\ell)}$. Hence,

$$\bigcap_{\ell=1}^{\infty} \mathbb{F}^{(\ell)} = \{1\},$$

and $\mathbb{F}/\mathbb{F}^{(\ell)}$ is solvable. □

**Proposition 3.22.** *Residually amenable groups are sofic.*

*Sketch of proof.* Let $\Gamma$ be residually amenable. Take a radius $r \geqslant 1$. Then there exists $H \trianglelefteq \Gamma$ such that $\Gamma/H$ is amenable and $\mathrm{Cay}(\Gamma) \simeq_r \mathrm{Cay}(\Gamma/H)$. Since $\Gamma/H$ is amenable, it has a Følner sequence $(F_\ell)_{\ell\geqslant 1}$. Take $\ell$ very large and consider the action $\mathbb{F}_S \curvearrowright F_\ell$. Then almost all $r$-balls in $F_\ell$ look like the $r$-ball in $\Gamma/H$, which is the $r$-ball in $\Gamma$. □

**Corollary 3.23.** $\mathrm{BS}(2,3)$ *is sofic.*

**Lemma 3.24.** *Let $\Delta$ be a finitely generated group and let $\alpha : \Delta \to \Delta$ be a surjective homomorphism. Then for every finite set $X$ and $\rho : \Delta \to \mathfrak{S}_X$, we have* $\mathrm{Ker}\,\alpha \leqslant \mathrm{Ker}\,\rho$.

*Proof.* Set $\mathcal{A}_X = \mathrm{Hom}(\Delta, \mathfrak{S}_X)$, and define $\alpha^* : \mathcal{A}_X \to \mathcal{A}_X$ by $\alpha^*(\varphi) = \varphi \circ \alpha$. Since $\alpha$ is surjective, $\alpha^*$ is injective. But $|\mathcal{A}_X| < \infty$ because $\Delta$ is finitely generated, so $\alpha^*$ is also surjective. Therefore, $\rho = \varphi \circ \alpha$ for some $\varphi \in \mathcal{A}_X$, so $\mathrm{Ker}\,\alpha \leqslant \mathrm{Ker}\,\rho$. □

**Proposition 3.25.** $\mathrm{BS}(2,3)$ *is not residually finite.*

*Proof.* Write $\Gamma = \mathrm{BS}(2,3) = \langle x, y \mid yx^2y^{-1} = x^3\rangle$. Define $\alpha : \Gamma \to \Gamma$ by $x \mapsto x^2$ and $y \mapsto y$. This is a surjective homomorphism (because $y = \alpha(y) \in \mathrm{Im}\,\alpha$ and $x = \alpha(yxy^{-1}x^{-1}) \in \mathrm{Im}\,\alpha$). However, we have

$$\alpha\left(\left(yxy^{-1}x^{-1}\right)^2 x^{-1}\right) = 1,$$

and $(yxy^{-1}x^{-1})^2\, x^{-1}$ by Britton's Lemma, so $\alpha$ is not injective. Therefore, $\Gamma$ is not Hopf, and hence not residually finite (by Lemma 3.24). □

**Corollary 3.26.** $\mathrm{BS}(2,3)$ *is not stable.*

*Proof.* This follows from Proposition 3.16. □

## 3.5 The lamplighter group

**Proposition 3.27.** *The lamplighter group $\mathbb{Z}/2 \wr \mathbb{Z}$ is metabelian hence amenable.*

**Theorem 3.28** (Levit-Lubotzky)**.** *Let $\Gamma = \mathbb{Z}/2 \wr \mathbb{Z}$. Then* $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$.

*Sketch of proof.* The main ingredients of the proof are:

(i) Weiss' Monotilability Theorem: if $\Gamma$ is amenable and residually finite (or solvable), then there exists a Følner sequence $(F_n)_{n\geqslant 1}$ with $|F_n| < \infty$ and a sequence $(H_n)_{n\geqslant 1}$ of finite-index subgroups such that each $F_n$ is a transversal for the left cosets of $H_n$ in $\Gamma$.

(ii) Lindenstrauss' Pointwise Ergodic Theorem.

□

**Conjecture 3.29.** *If $\Gamma$ is a metabelian group, then* $\overline{\mathrm{IRS}_{fi}(\Gamma)}^{w*} = \mathrm{IRS}(\Gamma)$.

**Remark 3.30.** *There are step-3 solvable groups that are not stable, for instance*

$$\Gamma = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & p^n & * & * \\ 0 & 0 & p^m & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL_4\left(\mathbb{Z}\left[\frac{1}{p}\right]\right),\ m, n \in \mathbb{Z} \right\}.$$

**Remark 3.31.** *Let $\Gamma = \mathbb{Z}/2 \wr \mathbb{Z}$. Consider the map*

$$F : x \in \{0,1\}^{\mathbb{Z}} \longmapsto \bigoplus_{\substack{n\in\mathbb{Z} \\ x_n = 1}} \mathbb{Z}/2 \in \mathrm{Sub}(\Gamma).$$

*Take the product measure $\nu$ on $\{0,1\}^{\mathbb{Z}}$. Then the pushforward $F_*\nu \in \mathrm{Prob}\left(\mathrm{Sub}(\Gamma)\right)$ is an invariant random subgroup and has no atoms.*

**Remark 3.32.** *The lamplighter group $\mathbb{Z}/2 \wr \mathbb{Z}$ is not finitely presented. However, there are two (equivalent) ways in which we can give a meaning to stability:*

- *Definition 1.7 does not rely on finite presentability.*
- *We may modify Definition 1.14 and say that $\Gamma$ is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ and a finite subset $E_0 \subseteq E$ such that for all finite $\mathbb{F}$-space $X$ with $L_{E_0}(X) < \delta$, there exists a finite $\Gamma$-space $Y$ with $|Y| = |X|$ and $d(X, Y) < \varepsilon$.*

# 4 Open questions

**Question 4.1.** *Are metabelian groups stable?*

**Question 4.2.** *Are amenable LERF groups stable?*

**Definition 4.3** (Flexible stability)**.** *Given $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_N$, $n \leqslant N$, we define*

$$d(\sigma, \tau) = \frac{1}{N} \left( |\{x \in \{1, \ldots, n\}, \sigma(x) \neq \tau(x)\}| + (N - n) \right).$$

*We say that $\Gamma = \langle S \mid E \rangle$ is* flexibly stable *if given $f : \mathbb{F} \to \mathfrak{S}_n$ with $L_E(f)$ small, there is $n \leqslant N \leqslant (1 + \varepsilon)n$ and $h : \Gamma \to \mathfrak{S}_N$ such that $d(h, f)$ is small.*

**Theorem 4.4** (Becker-Lubotzky)**.** *$SL_n(\mathbb{Z})$ is not stable for $n \geqslant 3$.*

**Question 4.5.** *Is $SL_n(\mathbb{Z})$ flexibly stable?*

**Theorem 4.6** (Bowen-Burton)**.** *If $SL_5(\mathbb{Z})$ is flexibly stable, then there exists a non-sofic group.*

**Question 4.7.** *Study stability when $\mathfrak{S}_n$ is replaced by $\left(U(n), \frac{1}{\sqrt{n}} \|\cdot\|_2\right)$.*

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