Mapping Class Groups

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1 Introduction

1.1 Surfaces

Definition 1.1 (Manifold of finite type)**.** *A manifold M will be called* of finite type *if it is a compact manifold punctured at a finite number of points.*

Notation 1.2. *We shall consider connected, smooth, oriented surfaces (i.e. 2-manifolds) of finite type.*

Theorem 1.3 (Classification of surfaces of finite type)**.** *Every connected, oriented surface of finite type is diffeomorphic to some $S_{g,n,b}$ for some $g, n, b \geqslant 0$, where $S_{g,n,b}$ is a surface with g holes, n punctures and b boundary components.*

Proposition 1.4. *Let $g, n, b \geqslant 0$. The* Euler characteristic *of $S_{g,n,b}$ is given by*

$$\chi\left(S_{g,n,b}\right) = 2 - 2g - (n + b).$$

Example 1.5.

(i) *There are three surfaces S with $\chi(S) > 0$: the sphere $\mathbb{S}^2$, the plane $\mathbb{C}$ and the (closed) disc $\mathbb{D}^2$.*

(ii) *There are four surfaces S with $\chi(S) = 0$: the torus $\mathbb{T}^2$, the punctured plane $\mathbb{C}^*$, the annulus $\mathbb{S}^1 \times I$ and the punctured (closed) disc $\mathbb{D}^2_*$.*

1.2 Mapping class groups

Definition 1.6 (Group of homeomorhisms)**.** *Let S be a surface. Consider the group* $\mathrm{Homeo}^+(S)$ *of orientation-preserving homeomorphisms of S. We equip this group with the* compact-open topology*, i.e. the topology of uniform convergence on all compact subsets. Moreover, if $A \subseteq S$ is a subset, we define* $\mathrm{Homeo}^+(S, A) = \left\{ f \in \mathrm{Homeo}^+(S),\ f_{|A} = \mathrm{id}_A \right\}$.

Remark 1.7. *A path $\gamma : [0,1] \to \mathrm{Homeo}^+(S)$ is equivalent to an* isotopy *$\varphi : [0,1] \times S \to S$, i.e. a homotopy s.t. $\varphi(t, \cdot)$ is a homeomorphism for all $t \in [0,1]$.*

Definition 1.8 (Mapping class group)**.** *If S is a surface, we denote by* $\mathrm{Homeo}_0(S, \partial S)$ *the path-connected component of* id_S *in* $\mathrm{Homeo}^+(S, \partial S)$. *Then* $\mathrm{Homeo}_0(S, \partial S)$ *is a normal subgroup of* $\mathrm{Homeo}^+(S, \partial S)$ *and we define the* mapping class group *of S by*

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S) / \mathrm{Homeo}_0(S, \partial S).$$

Theorem 1.9 (Baer, Munkres)**.** *Let S be a surface of finite type. Then* $\mathrm{Mod}(S)$ *can be defined using diffeomorphisms instead of homeomorphisms:*

$$\mathrm{Mod}(S) \cong \mathrm{Diffeo}^+(S, \partial S) / \mathrm{Diffeo}_0(S, \partial S).$$

Moreover, $\mathrm{Mod}(S)$ *can also be defined as the quotient of* $\mathrm{Homeo}^+(S, \partial S)$ *by the relation of homotopy (instead of isotopy) relative to ∂S.*

Note that this result is only true for surfaces, and not for manifolds of higher dimensions.

1.3 Context and motivation

Example 1.10. *Let S be a surface and $\phi \in \mathrm{Diffeo}(S)$. Consider $M_\phi = S \times [0,1]/\sim$ where $\sim$ is defined by $(x,1) \sim (\phi(x),0)$. The manifold M_ϕ is called a* surface bundle *over $\mathbb{S}^1$, and it only depends on the class of ϕ in the quotient group* $\mathrm{Mod}(S)$*.*

Remark 1.11. *There is an analogy between surfaces and n-dimensional tori. Both are generalisations of the 2-dimensional torus, and the fundamental group $\pi_1 S$ of a surface S corresponds to $\pi_1 \mathbb{T}^n = \mathbb{Z}^n$. Likewise, the mapping class group $\mathrm{Mod}(S)$ corresponds to $SL_n\mathbb{Z}$, and the closed curves on S (up to isotopy) correspond to vectors in $\mathbb{R}^n$.*

2 Curves, surfaces and hyperbolic geometry

2.1 The hyperbolic plane

Definition 2.1 (Hyperbolic plane)**.** *We consider two (equivalent) models for the* hyperbolic plane*:*

(i) *The* upper-half-plane model*: we equip $\mathbb{H}^2 = \{z \in \mathbb{C},\, \Im(z) > 0\}$ with the Riemannian metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. In this model, geodesics of $\mathbb{H}^2$ are vertical lines and semi-circles orthogonal to the x-axis. The isometries of $\mathbb{H}^2$ are the* Möbius transformations *$z \mapsto \frac{az+b}{cz+d}$ with real coefficients. In other words,* $\mathrm{Isom}^+(\mathbb{H}^2) = PSL_2\mathbb{R}$*.*

(ii) *The* Poincaré disc model *(which can be obtained from the upper-half-plane model* via *the map $z \mapsto \frac{z-i}{z+i}$): we equip $\mathbb{H}^2 = \{z \in \mathbb{C},\, |z| < 1\}$ with the Riemannian metric $\mathrm{d}s^2 = 4\frac{\mathrm{d}x^2+\mathrm{d}y^2}{(1-r^2)^2}$. We define the* Gromov boundary *(at infinity) by $\partial\mathbb{H}^2 = \mathbb{S}^1 \subseteq \mathbb{C}$ and we set $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2$. We note that isometries of $\mathbb{H}^2$ extend uniquely to Möbius transformations on $\overline{\mathbb{H}}^2$.*

Proposition 2.2. *Let $f \in \mathrm{Isom}^+\,\mathbb{H}^2 \setminus \{\mathrm{id}\}$. Then f is of one of the three following types:*

(i) *f is a* hyperbolic *(or* loxodromic*) isometry: f preserves a geodesic line $\mathcal{A}$, called its* axis*, on which it acts by translation of parameter τ, called the* translation length *of f. Moreover, one can check that for every $z \in \mathbb{H}^2 \setminus \mathcal{A}$, $d(x, f(x)) > \tau$.*

(ii) *f is an* elliptic *isometry: f has a unique fixed point in $\mathbb{H}^2$ and acts by rotation around that point in the Poincaré disc model.*

(iii) *f is a* parabolic *isometry: up to conjugacy, $f(z) = z \pm 1$ in the upper-half-plane model.*

Moreover, the above classification is invariant under conjugacy.

Proof. By Brouwer's Fixed Point Theorem, $\overline{f} : \overline{\mathbb{H}}^2 \to \overline{\mathbb{H}}^2$ has at least one fixed point. But since $\overline{f}$ is a nontrivial Möbius transformation, it has at most two fixed points. If it has two fixed points, show that both these fixed points lie on $\partial\mathbb{H}^2$ (for otherwise f would fix a geodesic line and have infinitely many fixed points). In that case, f is hyperbolic. Otherwise, $\overline{f}$ has exactly one fixed point. If it lies in $\mathbb{H}^2$, then f is elliptic, otherwise it is parabolic. □

2.2 Hyperbolic structures

Definition 2.3 (Geometric structure)**.** *A* geometric structure *on a surface S is a complete, finite-area Riemannian metric of constant curvature $\kappa \in \{-1, 0, +1\}$ in which every boundary component is a geodesic.*

Theorem 2.4 (Gauß-Bonnet)**.** *Let S be a surface with a geometric structure. Then:*

$$\int_S \kappa \,\mathrm{d}\mathcal{A} = 2\pi\chi(s).$$

Corollary 2.5. *If the surface S has a geometric structure, then it must satisfy* $\operatorname{sign}(\kappa) = \operatorname{sign}(\chi(S))$.

Example 2.6. *Using Example 1.5, we see that:*

(i) *There are three surfaces S with $\chi(S) > 0$: the sphere $\mathbb{S}^2$ with its usual geometric structure, the disc $\mathbb{D}^2$ with the geometric structure of a hemisphere, and the plane $\mathbb{C}$, which has no complete finite-area metric.*

(ii) *There are three surfaces S with $\chi(S) = 0$: the torus $\mathbb{T}^2$ with the Euclidean geometric structure induced by the quotient $\mathbb{R}^2/\mathbb{Z}^2$, the annulus $\mathbb{S}^1 \times I$ with the geometric structure of a cylinder and the punctured plane and disc, which have no comple finite-area metric.*

Most surfaces of interest will have a negative Euler characteristic and therefore a hyperbolic geometric structure.

Theorem 2.7. *Assume that the surface S is connected, oriented, of finite type, with $\chi(S) < 0$. Then there is a convex subspace $\widetilde{S} \subseteq \mathbb{H}^2$ with geodesic boundary, and an action $\pi_1(S) \curvearrowright \widetilde{S}$ by isometries s.t. $S \cong \pi_1 S \backslash \widetilde{S}$ has finite area. In particular, S has curvature -1 everywhere. The space $\widetilde{S}$ is the universal covering of S.*

Such a surface S is said to be hyperbolic.

Moreover, if S is closed or indeed has no boundary component, $\widetilde{S} = \mathbb{H}^2$.

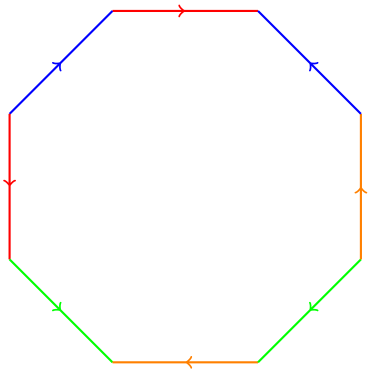

Figure 1: A two-holed torus obtained as a quotient of an octogon

Proof. We shall assume that $S = S_{g,0,0}$.

Note that the theorem is a generalisation of the fact that the torus $\mathbb{T}^2$ can be obtained as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

For $\mathbb{T}^2$, viewing it as the quotient of a square with opposite edges identified allows one to equip it with a Euclidean metric. A g-holed torus can be viewed a quotient of a $4g$-gon, as in Figure 1. This $4g$-gon cannot be equipped with a Euclidean metric because the total interior angle (i.e. the sum of the angles of four corners) is greater than 2π. It can however be equipped with a hyperbolic metric: indeed, note that for the ideal $4g$-gon with vertices on $\partial\mathbb{H}^2$, the total interior angle is 0, while this angle converges to $(4g-2)\pi$ for small regular $4g$-gons. By the Intermediate Value Theorem, there exists a regular hyperbolic $4g$-gon with total interior angle 2π (because $g > 1$). We use this $4g$-gon to equip S with a hyperbolic metric. With this metric, the universal covering $\widetilde{S}$ of S will be the hyperbolic plane tesselated by regular $4g$-gons with total interior angle 2π (as in Figure 2). □

2.3 Curves on hyperbolic surfaces

Definition 2.8 (Closed curve)**.** *A* closed curve *on a surface S is a continuous (or smooth) map $\alpha : \mathbb{S}^1 \to S$.*

To each closed curve is associated a conjugacy class $[\alpha]$ in $\pi_1 S$, and therefore an isometry (up to conjugacy) of $\mathbb{H}^2$ if S is hyperbolic.

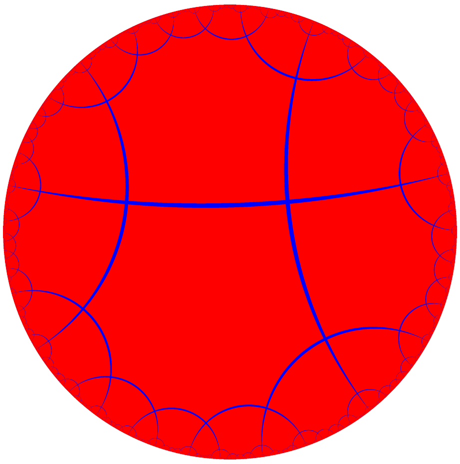

Figure 2: Tesselation of the hyperbolic plane by regular octogons

Definition 2.9 (Essential and inessential curves)**.** *Let S be a hyperbolic surface and consider a closed curve α on S.*

- *The curve α is said to be* inessential *if it is homotopic to a point or a puncture.*
- *Otherwise, α is said to be* essential.

Lemma 2.10. *Let S be a hyperbolic surface and let α be a closed curve on S. We identify α with the induced isometry of $\mathbb{H}^2$.*

(i) *If α is elliptic, then it is homotopic to a point.*

(ii) *If α is parabolic, then it is homotopic to a puncture.*

(iii) *If α is hyperbolic, then it is essential.*

Proof. (i) If α is elliptic, then it fixes a point of $\mathbb{H}^2$. But since $\pi_1 S$ acts freely on $\widetilde{S}$, it follows that α acts as the identity, so α is homotopic to a point.

(ii) If α is parabolic, then we may assume that it is given by $z \mapsto z+1$ in the upper-half-plane-model. Choose $x_0 = \alpha(0)$ as a basepoint and let $\widetilde{x}_0 \in \mathbb{H}^2$ be a lift of x_0. If $\widetilde{\alpha}$ is a lift of α at $\widetilde{x}_0$, we know that $\widetilde{\alpha}(1) = \widetilde{x}_0 + 1$. For $s \in [0, +\infty)$, set $\widetilde{\alpha}_s(t) = \widetilde{\alpha}(t) + is$. We have $\widetilde{\alpha}_s(1) = \widetilde{\alpha}_s(0) + 1$ for all s, so $\widetilde{\alpha}_s$ descends to a loop α_s in S. By compactness of $\overline{\mathbb{H}}^2$, α_s must converge to a puncture of S as $s \to \infty$.

(iii) Knowing (i) and (ii), it suffices to prove that if α is homotopic to a puncture, then it is parabolic. Assume that α is homotopic to a puncture. Homotopies from α to the puncture allows one to construct annuli around that puncture with outer boundary α. Since S is complete by assumption, every Cauchy sequence converges so the heights of the annuli must diverge to ∞. Because S has finite area, the girths of the annuli must converge to 0. In other words, there exist paths $(\alpha_n)_{n\in\mathbb{N}}$ homotopic to α s.t. $\ell(\alpha_n) \xrightarrow[n\to+\infty]{} 0$. Lift α to a path $\widetilde{\alpha} : [0,1] \to \mathbb{H}^2$, each α_n to a path $\widetilde{\alpha}_n$. Set $\widetilde{x}_n = \widetilde{\alpha}_n(0)$ and note that $\widetilde{\alpha}_n(1) = \alpha \cdot \widetilde{x}_n$. If α were hyperbolic, its translation length would satisfy

$$\tau(\alpha) \leqslant d(\widetilde{x}_n, \alpha\widetilde{x}_n) = d(\widetilde{\alpha}_n(0), \widetilde{\alpha}_n(1)) \leqslant \ell(\widetilde{\alpha}_n) = \ell(\alpha_n) \xrightarrow[n\to+\infty]{} 0.$$

This is a contradiction, therefore α must be parabolic. □

Lemma 2.11. *Let S be a hyperbolic surface and let α be an essential closed curve on S. Then there exists a unique geodesic representative of the homotopy class of α.*

Proof. Existence. Lift α to a map $\widetilde{\alpha}$ between universal covers as in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R} & \overset{\widetilde{\alpha}}{\dashrightarrow} & \widetilde{S} \subseteq \mathbb{H}^2 \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & \xrightarrow{\alpha} & S \end{array}$$

Note that the action of $\mathbb{Z} = \pi_1 \mathbb{S}^1$ on $\mathbb{R}$ induces an action on $\widetilde{S}$, namely the action by $\langle \alpha \rangle \subseteq \pi_1 S$; moreover the map $\widetilde{\alpha} : \mathbb{R} \to \widetilde{S}$ is $\mathbb{Z}$-equivariant.

By Lemma 2.10, we know that α is hyperbolic, so it has an axis $\mathcal{A} \subseteq \mathbb{H}^2$. Consider the orthogonal projection $\pi : \mathbb{H}^2 \to \mathcal{A}$. For $t \in \mathbb{R}$, let $\widetilde{\gamma}_t : [0,1] \to \mathbb{H}^2$ be the unique constant-speed geodesic from $\widetilde{\alpha}(t)$ to $\pi \circ \widetilde{\alpha}(t)$. Since $\langle \alpha \rangle$ acts on both $\widetilde{\alpha}$ and $\mathcal{A}$, and the paths $\widetilde{\gamma}_t$ are defined canonically, taking the quotient by $\mathbb{Z} \cong \langle \alpha \rangle$ defines a homotopy from α to some closed curve β on S in the image of $\mathcal{A}$. Therefore, up to reparametrisation, β is a constant-speed geodesic that is homotopic to α.

Uniqueness. Suppose α, β are two homotopic geodesics on S and lift them to geodesics $\widetilde{\alpha}, \widetilde{\beta} : \mathbb{R} \to \mathbb{H}^2$. These geodesics $\widetilde{\alpha}, \widetilde{\beta}$ are contained in a bounded distance of each other because they are lifts of homotopic curves. It follows that $\widetilde{\alpha}, \widetilde{\beta}$ have the same endpoints in $\partial \mathbb{H}^2$ and therefore $\widetilde{\alpha} = \widetilde{\beta}$. □

Remark 2.12. *The existence assertion in Lemma 2.11 remains true in the Euclidean case, but not the uniqueness.*

3 Simple closed curves and intersection numbers

3.1 Simple closed curves

Definition 3.1 (Simple closed curve)**.** *A* simple closed curve *is a curve $\alpha : \mathbb{S}^1 \to S$ that is injective.*

Definition 3.2 ((Ambient) isotopy of simple closed curves)**.** *Let α_0, α_1 be simple closed curves on a surface S.*

(i) *An* isotopy *from α_0 to α_1 is a homotopy $\alpha_\bullet$ s.t. each α_t is a simple closed curve.*

(ii) *An* ambient isotopy *from α_0 to α_1 is an isotopy $\phi_\bullet : S \to S$ s.t. $\phi_0 = \mathrm{id}_S$ and $\phi_1 \circ \alpha_0 = \alpha_1$.*

Lemma 3.3. *Two essential simple closed curves on an orientable surface S are homotopic relative to ∂S if and only if they are ambient isotopic.*

Proof. See Lemma 3.15. □

Definition 3.4 (Primitive element)**.** *Let G be a group. An element $h \in G$ is said to be* primitive *if it cannot be written in the form $h = g^n$ with $g \in G$ and $n > 1$.*

Lemma 3.5. *Homotopy classes of essential simple closed curves on the torus $\mathbb{T}^2$ correspond to primitive elements of $\pi_1 \mathbb{T}^2 = \mathbb{Z}^2$.*

Lemma 3.6. *If α is an essential simple closed curve on a hyperbolic surface S, then α defines a primitive element of $\pi_1 S$. In fact, the centraliser of α is $C(\alpha) = \langle \alpha \rangle$.*

Proof. Note that it suffices to prove the second assertion. By Lemma 2.11, we may assume without loss of generality that α is geodesic and we may consider its axis $\mathcal{A} \subseteq \mathbb{H}^2$. Let $g \in C(\alpha)$. For every $x \in \mathcal{A}$, we have

$$d\,(gx, \alpha g x) = d\,(gx, g\alpha x) = d\,(x, \alpha x) = \tau(\alpha),$$

which implies that $gx \in \mathcal{A}$. In other words, $g \cdot \mathcal{A} \subseteq \mathcal{A}$. From this it follows that $C(\alpha)$ acts on $\mathcal{A}$, which enables us to consider the following commutative diagram, since $\langle \alpha \rangle \subseteq C\,(\alpha)$:

$$\begin{array}{ccc}
\mathbb{R} \cong \mathcal{A} & \xrightarrow{\subseteq} & \widetilde{S} \subseteq \mathbb{H}^2 \\
\downarrow & & \downarrow \\
\mathbb{S}^1 \cong \langle \alpha \rangle \backslash \mathcal{A} & \xrightarrow{\alpha} & S \\
\downarrow & \nearrow & \\
C(\alpha) \backslash \mathcal{A} & &
\end{array}$$

Since α is injective (as a simple closed curve), it follows that the covering map $\langle \alpha \rangle \backslash \mathcal{A} \to C(\alpha) \backslash \mathcal{A}$ is injective, and therefore $C(\alpha) = \langle \alpha \rangle$. $\square$

3.2 Intersection numbers

Definition 3.7 (Intersection number)**.** *Let α, β be (simple) closed curves on a surface S. The* (geometric) intersection number *of α and β is defined by*

$$i(\alpha, \beta) = \min_{\substack{\alpha' \sim \alpha \\ \beta' \sim \beta}} |\alpha' \cap \beta'| .$$

We say that α and β are in minimal position *if $i(\alpha, \beta) = |\alpha \cap \beta|$.*

Definition 3.8 (Transverse curves)**.** *We say that two curves α and β are* transverse *if, locally, all their intersection points look like two transverse lines.*

Proposition 3.9. *Any two curves can be made transverse by a small isotopy.*

Definition 3.10 (Bigon)**.** *Let α and β be two transverse simple closed curves on a surface S. A* bigon *for α, β is an embedded (closed) disc $D \hookrightarrow S$ such that $D \cap (\alpha \cup \beta) = \partial D = a \cup b$ where $a \subseteq \alpha$ and $b \subseteq \beta$ are arcs.*

Lemma 3.11. *If α and β are transverse simple closed curves on a surface S without bigons, then any pair $\widetilde{\alpha}, \widetilde{\beta}$ of lifts in $\widetilde{S}$ intersect in at most one point.*

Proof. Suppose $\widetilde{\alpha}$ and $\widetilde{\beta}$ intersect in at least 2 points for some lifts $\widetilde{\alpha}, \widetilde{\beta}$. Then $\widetilde{\alpha}, \widetilde{\beta}$ bound some discs $D_0 \hookrightarrow \widetilde{S}$. Pass to an innermost disc D, bounded without loss of generality by $\widetilde{\alpha}, \widetilde{\beta}$ and not intersecting any other lift. We need to prove that the composite $D \hookrightarrow \widetilde{S} \to S$ is an embedding. This is equivalent to

$$\forall g \in \pi_1 S,\ gD \cap D \neq \varnothing \implies g = 1.$$

But because D is innermost, note that $g(\partial D) \cap \mathring{D} = \varnothing$ for all g. Therefore, $D \subseteq gD$ as soon as $gD \cap D \neq \varnothing$, and g^{-1} induces a map $D \to D$. By the Brouwer Fixed Point Theorem, g has a fixed point, so $g = 1$ because the action of $\pi_1 S$ on $\widetilde{S}$ is free. $\square$

Proposition 3.12 (Bigon Criterion)**.** *Two transverse simple closed curves α, β on a surface S are in minimal position if and only if they have no bigon.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) We will assume that S is hyperbolic and closed and that α, β are essential. Suppose there are no bigons and fix a lift $\widetilde{\alpha}$ of α in $\widetilde{S} = \mathbb{H}^2$. Look at all the lifts $\widetilde{\beta}$ of β: they all intersect $\widetilde{\alpha}$ at most once by Lemma 3.11. Moreover, note that $\mathbb{Z} \cong \langle \alpha \rangle$ acts on $\widetilde{\alpha}$ and

$$\alpha \cap \beta = \mathbb{Z} \backslash \left(\widetilde{\alpha} \cap \bigcup_{\widetilde{\beta} \text{ lift of } \beta} \widetilde{\beta} \right).$$

Therefore, to prove the proposition, it suffices to show that modifying α and β by homotopies doesn't alter whether or not a given pair of lifts $\widetilde{\alpha}, \widetilde{\beta}$ intersect. Denote by $\xi_\pm \in \partial \mathbb{H}^2$ (resp. $\eta_\pm \in \partial \mathbb{H}^2$) the endpoints of $\tilde{\alpha}$ (resp. $\tilde{\beta}$). Note that if $\tilde{\alpha}$ and $\tilde{\beta}$ intersect, then $\{\xi_\pm\} \cap \{\eta_\pm\} = \varnothing$. Indeed: if $\{\xi_\pm\} = \{\eta_\pm\}$, then $\langle\alpha\rangle$ acts on $\tilde{\alpha}\cap\tilde{\beta}$ because $\tilde{\alpha}$ and $\tilde{\beta}$ share a common axis, therefore $1 = \left|\tilde{\alpha} \cap \tilde{\beta}\right| \in \{0, +\infty\}$, a contradiction; if on the other hand $\xi_+ = \eta_+$ and $\xi_- \neq \eta_-$, then we can assume without loss of generality that $\xi_+ = \eta_+ = +\infty$ in the upper-half-plane model; an explicit computation shows that $[\alpha, \beta]$ is parabolic, which contradicts the fact that S is closed.

Let us examine how $\xi_\pm$ and $\eta_\pm$ are arranged on $\partial\mathbb{H}^2 = \mathbb{S}^1$.

- If $\tilde{\alpha} \cap \tilde{\beta} \neq \varnothing$, then we have an alternation of elements from $\{\eta_\pm\}$ and $\{\xi_\pm\}$ when going round the circle: we say that $\xi_\pm$ *cross* $\eta_\pm$.
- If $\tilde{\alpha} \cap \tilde{\beta} = \varnothing$, then we have two elements from $\{\eta_\pm\}$ followed by two elements from $\{\xi_\pm\}$ when going round the circle: we say that $\xi_\pm$ *do not cross* $\eta_\pm$.

Now note that homotopies $\alpha_\bullet$ of α and $\beta_\bullet$ of β only move lifts $\tilde{\alpha}$ and $\tilde{\beta}$ by a bounded distance, so they do not move the endpoints $\xi_\pm, \eta_\pm$. Therefore, homotopies don't change whether or not $\tilde{\alpha}$ and $\tilde{\beta}$ intersect, and they don't change the value of $|\alpha \cap \beta|$. □

Corollary 3.13. *Geodesics are always in minimal position.*

Proof. If two geodesics are not in minimal position, then there is a pair of lifts $\tilde{\alpha}, \tilde{\beta}$ in $\tilde{S}$ with a bigon. The uniqueness of geodesics in $\tilde{S}$ implies that $\tilde{\alpha} = \tilde{\beta}$, so $\alpha = \beta$. □

Proposition 3.14 (Annulus Criterion)**.** *Let α, β be disjoint essential simple closed curves on a surface S. If α and β are homotopic, then they bound an embedded annulus in S.*

Proof. We shall assume that S is hyperbolic. Choose lifts $\tilde{\alpha}, \tilde{\beta}$ of α, β to $\tilde{S} \subseteq \mathbb{H}^2$ with the same endpoints $\{\xi_\pm\}$ on $\partial\mathbb{H}^2$. The union $\tilde{\alpha} \cup \tilde{\beta} \cup \{\xi_\pm\}$ forms an embedded circle in $\overline{\mathbb{H}}^2$, bounding a region $R \subseteq \mathbb{H}^2$. The natural action of $\mathbb{Z} = \langle\alpha\rangle = \langle\beta\rangle \subseteq \pi_1 S$ preserves R. Consider the quotient $A = \mathbb{Z}\backslash R$. Since A is a surface with two boundary components and with $\pi_1 A \cong \mathbb{Z}$, it follows that A is an annulus with boundary components α and β. It remains to prove that the map $A \to S$ is an embedding, or equivalently that $\forall g \in \pi_1 S$, $gR \cap R \neq \varnothing \implies g \in \langle\alpha\rangle$. But note that, by Lemma 3.6, $\langle\alpha\rangle = \mathrm{Stab}_{\pi_1 S}(\{\xi_\pm\})$. This implies that, if $g \notin \langle\alpha\rangle$, then g moves either ξ_+ or ξ_-; therefore $g\left(\tilde{\alpha} \cup \tilde{\beta}\right) \cap \left(\tilde{\alpha} \cup \tilde{\beta}\right) = \varnothing$ which implies that $gR \cap R = \varnothing$. □

Lemma 3.15. *Two essential simple closed curves α, β on an orientable surface S are homotopic relative to ∂S if and only if they are ambient isotopic.*

Proof. Assume that α, β are homotopic. After an ambient isotopy, we may assume that α, β are transverse. Since they are homotopic, their intersection number is 0. We may therefore assume that they are disjoint (otherwise, there is a bigon, and we can reduce $|\alpha \cap \beta|$ strictly by an ambient isotopy). Hence, α and β bound an annulus by the Annulus Criterion, and we may push α and β over the annulus. □

3.3 Change of coordinates

Definition 3.16 (Cut surface of a curve)**.** *Any smooth simple closed curve $\alpha : \mathbb{S}^1 \to S$ has a small open regular neighbourhood $N(\alpha)$ s.t. $N(\alpha) \cong \mathbb{S}^1 \times (-1, +1)$. The* cut surface *$S_\alpha$ of α is defined by*

$$S_\alpha = S \backslash N(\alpha).$$

S_α has two new boundary circles α_- and α_+ determined by the orientation of S and α. We can recover S via

$$S = S_\alpha \cup_{(\alpha_- \sqcup \alpha_+)} A,$$

where A is the annulus.

Definition 3.17 (Topological type)**.** *The* topological type *of an essential simple closed curve α on a surface S is the homeomorphism type of S_α. If S_α is connected, α is said to be* nonseparating.

Example 3.18. *Let $S = S_{g,0,0}$. If α is nonseparating, then $S_\alpha \cong S_{g-1,0,2}$. Thus, there is only one topological type of nonseparating curves.*

Moroever, there are $\left\lceil \frac{g}{2} \right\rceil$ topological types of separating curves.

Proof. Note that S_α has two boundary components, no puncture, and

$$2-2g=\chi(S)=\chi\left(S_\alpha\right)-\chi\left(\mathbb{S}^1\right)=\chi\left(S_\alpha\right)=2-2g\left(S_\alpha\right)-2-0,$$

which implies that $g\left(S_\alpha\right)=g-1$. □

Proposition 3.19 (Change of coordinates)**.** *Two simple closed curves α, β have the same topological type* iff *there exists an orientation-preserving homeomorphism $\phi: S \to S$ fixing ∂S and such that $\phi \circ \alpha = \beta$.*

Proof. ($\Leftarrow$) Clear. ($\Rightarrow$) Suppose $\phi : S_\alpha \to S_\beta$ is a homeomorphism. Composing ϕ with an orientation-reversing homeomorphism of S_β, we may assume that ϕ is orientation-preserving. Since $\mathrm{Homeo}^+\left(S_\beta\right)$ acts transitively on the boundary components of each connected component, we may assume that ∂S is preserved and that ϕ sends α_+ to β_+ and α_- to β_-. The Annulus Criterion (Proposition 3.14) now implies that we can extend ϕ over the glueing annulus to a homeomorphism $S \to S$. Finally, since $\phi \circ \alpha$ is homotopic (hence ambient isotopic by Proposition 3.15) to β, we may modify ϕ so that $\phi \circ \alpha = \beta$ as requested. □

Corollary 3.20. (i) *If α is a nonseparating simple closed curve on S, then there exists a simple closed curve β on S s.t. $i(\alpha, \beta) = 1$.*

(ii) *Suppose $\alpha_1, \beta_1, \alpha_2, \beta_2$ are simple closed curves on S such that $i(\alpha_1, \beta_1) = i(\alpha_2, \beta_2) = 1$. Then there exists a homeomorphism $\phi : S \to S$ s.t. $\alpha_2 = \phi \circ \alpha_1$ and $\beta_2 = \phi \circ \beta_1$.*

4 Basic computations of mapping class groups

4.1 The Alexander Lemma

Lemma 4.1. $\mathrm{Mod}\left(\mathbb{D}^2\right) \cong 1$.

Proof. Suppose $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ is a homeomorphism that fixes $\partial\mathbb{D}^2$. Define

$$\phi_t(x)=\begin{cases}(1-t)\phi\left(\frac{x}{1-t}\right) & \text{if } 0 \leqslant |x| \leqslant 1-t \\ x & \text{if } 1-t < x \leqslant 1\end{cases}.$$

Note that ϕ_t is continuous since ϕ fixes $\partial\mathbb{D}^2$; therefore $\phi_\bullet$ defines an isotopy from ϕ to $\mathrm{id}_{\mathbb{D}^2}$. □

Lemma 4.2. $\mathrm{Mod}\left(\mathbb{D}^2_*\right) \cong 1$.

Proof. In the proof of Lemma 4.1, note that if $\phi(0) = 0$, then $\phi_t(0) = 0$ for all t. □

4.2 Spheres with few punctures

Definition 4.3 (Arc)**.** *A* (proper) arc *is a continuous map $\alpha : [0, 1] \to S$ s.t. $\alpha(0), \alpha(1) \in \partial S \cup \{$punctures of $S\}$ and $(0, 1) \subseteq \alpha^{-1}\left(\mathring{S}\right)$. We say that α is*

- Simple *if $\alpha_{|(0,1)}$ is injective,*
- Essential *if α is not homotopic (with fixed endpoints) to a puncture or a boundary component.*

Lemma 4.4. *Let α, β be simple arcs on $S_{0,3,0}$ with distinct endpoints. If α and β have the same endpoints, then they are isotopic.*

Proof. Without loss of generality, we may assume that $S_{0,3,0} = \mathbb{C}\backslash\{0,1\}$ and α, β go from 0 to 1 and are transverse. By finding innermost discs and pushing over bigons, we may assume that $\alpha \cap \beta = \{0,1\}$. Therefore, $\alpha \cup \beta$ is the boundary of a disc, so α and β are isotopic. □

Remark 4.5. *There is a natural homomorphism* $\mathrm{Mod}\,(S_{g,n,b}) \to \mathfrak{S}_n$ *obtained by acting on the punctures, and this homomorphism is surjective if S is connected.*

Definition 4.6 (Pure mapping class group)**.** *The pure mapping class group of $S_{g,n,b}$ is defined by*

$$\mathrm{PMod}\,(S_{g,n,b}) = \mathrm{Ker}\,(\mathrm{Mod}\,(S_{g,n,b}) \to \mathfrak{S}_n)\,.$$

Proposition 4.7. *The natural homomorphism* $\mathrm{Mod}\,(S_{0,3,0}) \to \mathfrak{S}_3$ *is an isomorphism.*

Proof. It suffices to show that the above homomorphism is injective. Therefore, suppose $\phi : S_{0,3,0} \to S_{0,3,0}$ fixes the punctures. We think of $S_{0,3,0}$ as $\mathbb{C}\backslash\{0,1\}$ and we consider the arc α from 0 to 1 given by $\alpha(t) = t$. Now, $\phi \circ \alpha$ is a proper arc from 0 to 1, so it is (ambient) isotopic to α by Lemma 4.4. We may therefore assume that $\phi \circ \alpha = \alpha$. Now, ϕ descends to a self-homeomorphism $\overline{\phi}$ fixing the boundary of $S_\alpha \cong \mathbb{D}^2_*$. By Lemma 4.2, $\overline{\phi}$ is isotopic to id_{S_α}, so we can reglue to see that ϕ is isotopic to id_S. □

Corollary 4.8. $\mathrm{Mod}\,(\mathbb{S}^2) \cong \mathrm{Mod}\,(\mathbb{C}) \cong 1$ *and* $\mathrm{Mod}\,(\mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The above surfaces S are all 2-spheres with at most three punctures, so we may compose $\phi : S \to S$ with an isotopy in the Möbius group until ϕ fixes three points, and then ϕ is isotopic to id_S by Proposition 4.7. □

4.3 The annulus

Proposition 4.9. $\mathrm{Mod}\,(\mathbb{S}^1 \times I) \cong \mathbb{Z}$.

Proof. Denote $A = \mathbb{S}^1 \times I$. Identifying $\mathbb{S}^1$ with the unit circle in $\mathbb{C}$, the universal cover $\widetilde{A}$ is homeomorphic to the infinite strip $\mathbb{R} \times I$, with covering map $\widetilde{A} \to A$ given by $(x, y) \mapsto (e^{2i\pi x}, y)$. Now let $\phi : A \to A$ be a diffeomorphism with $\phi_{|\partial A} = \mathrm{id}_{\partial A}$. Let $\widetilde{\phi} : \widetilde{A} \to \widetilde{A}$ be the unique lift of ϕ fixing the origin $(0,0)$. Denote $\widetilde{\phi}_1 = \widetilde{\phi}_{|\mathbb{R}\times\{1\}}$. Since $\widetilde{\phi}_1$ is a lift of $\mathrm{id}_{\mathbb{S}^1\times\{1\}}$, it is the translation by some integer n. Note that n does not vary when ϕ is replaced by a homotopic diffeomorphism $A \to A$ (because n varies continuously and $\mathbb{Z}$ is discrete), so we have a well-defined map $\mathrm{Mod}(A) \to \mathbb{Z}$ defined by $[\phi] \mapsto n$. It remains to prove that this map is a group isomorphism.

If $\phi, \psi : A \to A$ are two diffeomorphisms, then $\widetilde{\psi \circ \phi} = \widetilde{\psi} \circ \widetilde{\phi}$ by the uniqueness of lifts, from which it follows that $\mathrm{Mod}(A) \to \mathbb{Z}$ is a group homomorphism.

For each $n \in \mathbb{Z}$, the matrix

$$\widetilde{\phi} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : \mathbb{R} \times I \to \mathbb{R} \times I$$

defines a diffeomorphism $\widetilde{A} \to \widetilde{A}$ that descends to the identity on each boundary component and such that $\widetilde{\phi}_1$ is the translation by n. Therefore, the morphism $\mathrm{Mod}(A) \to \mathbb{Z}$ is surjective.

To prove the injectivity, consider a diffeomorphism $\phi : A \to A$ such that $\widetilde{\phi}$ fixes $(0,1)$ (in addition to $(0,0)$). We need to show that ϕ is isotopic to the identity. Consider the arc δ in A defined by $\delta(t) = (1,t)$ and let $\widetilde{\delta}$ be its lift starting at $(0,0)$. Both $\widetilde{\delta}$ and $\widetilde{\phi} \circ \widetilde{\delta}$ end at $(0,1)$. We may assume after a small isotopy that δ and $\phi \circ \delta$ are transverse; therefore, Lemma 3.11 implies that δ and $\phi \circ \delta$ form a bigon. If the corners of that bigon are not $(1,0)$ and $(1,1)$, then we may apply an isotopy to ϕ and reduce the number of intersection points. Otherwise, δ and $\phi \circ \delta$ bound a bigon, and we may modify ϕ by an isotopy until $\phi \circ \delta = \delta$. We now conclude as before: cutting along δ, ϕ defines a diffeomorphism $\overline{\phi}$ of the cut surface A_δ that fixes the boundary. By Lemma 4.1, $\overline{\phi}$ is isotopic to id_{A_δ}, so ϕ is isotopic to id_A. □

Definition 4.10 (Dehn twist)**.** *The generator of* Mod $(\mathbb{S}^1 \times I) \cong \mathbb{Z}$ *is called a* Dehn twist*.*

Since many surfaces contain essential annuli, we will see that they usually also contain Dehn twists.

4.4 The torus and the punctured torus

Remark 4.11. *Consider the once-punctured torus $\mathbb{T}^2_* = S_{1,1,0}$. A self-diffeomorphism of $\mathbb{T}^2_*$ can be thought of as a diffeomorphism of $\mathbb{T}^2$ fixing a point; it therefore induces an automorphism of $\pi_1\mathbb{T}^2 \cong \mathbb{Z}^2$ by functoriality. Therefore, we have a group homomorphism*

$$\mathrm{Mod}\left(\mathbb{T}^2_*\right) \to GL_2\left(\mathbb{Z}\right).$$

Theorem 4.12. *For the once-punctured torus $\mathbb{T}^2_*$, the morphism* Mod $(\mathbb{T}^2_*) \to GL_2(\mathbb{Z})$ *induces an isomorphism*

$$\mathrm{Mod}\left(\mathbb{T}^2_*\right) \cong SL_2\left(\mathbb{Z}\right).$$

Proof. We already know that the map Mod $(\mathbb{T}^2_*) \to GL_2(\mathbb{Z})$ is a group homomorphism. We need to show that it is injective and that its image is $SL_2(\mathbb{Z})$.

To show injectivity, let $\phi : \mathbb{T}^2_* \to \mathbb{T}^2_*$ be a diffeomorphism acting on $\pi_1\mathbb{T}^2$ as the identity. Let $\alpha : t \mapsto (e^{2i\pi t}, 1)$ and $\beta : t \mapsto (1, e^{2i\pi t})$ be the standard based loops in $\mathbb{T}^2$ that generate $\pi_1\mathbb{T}^2$. Let $\widetilde{\alpha}_0$ and $\widetilde{\beta}_0$ be the (unique) lifts of these paths at the origin. Consider also the lift $\widetilde{\phi}$ of ϕ that fixes the origin. Since $\widetilde{\phi}$ acts trivially on $\pi_1\mathbb{T}^2$, it fixes the endpoints of $\widetilde{\alpha}$ and $\widetilde{\beta}$. We may therefore apply Lemma 3.11 successively to find bigons and to isotopically modify ϕ until $\phi \circ \alpha = \alpha$ and $\phi \circ \beta = \beta$. The end of the proof of injectivity is now standard: ϕ descends to an isomorphism of the cut surface $\mathbb{T}^2_{\alpha,\beta}$ (which is a disc), fixing the boundary. Hence, ϕ is isotopic to $\mathrm{id}_{\mathbb{T}^2}$ by Lemma 4.1.

To see that the image is contained in $SL_2(\mathbb{Z})$, note that the determinant of the image of $[\phi] \in$ Mod $(\mathbb{T}^2_*)$ is an invertible integer, so it must be ± 1, but ϕ is orientation-preserving so $\widetilde{\phi} \circ \widetilde{\alpha}$ and $\widetilde{\phi} \circ \widetilde{\beta}$ form a left-handed basis of $\mathbb{Z}^2$ and the determinant must be $+1$.

For surjectivity, note that any matrix $A \in SL_2(\mathbb{Z})$ defines an orientation-preserving diffeomorphism of $\mathbb{R}^2$ which descends to an orientation-preserving diffeomorphism of $\mathbb{T}^2_*$ acting as A on the fundamental group. □

Corollary 4.13. Mod $(\mathbb{T}^2) \cong SL_2(\mathbb{Z})$.

Proof. Note that forgetting the puncture defines a group homomorphism

$$\mathrm{Mod}\left(\mathbb{T}^2_*\right) \to \mathrm{Mod}\left(\mathbb{T}^2\right).$$

We shall prove that this homomorphism is actually an isomorphism. The key ingredient will be the fact that $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ has a natural group structure which we shall denote multiplicatively, with identity element 1. Without loss of generality, we may assume that $\mathbb{T}^2_*$ is $\mathbb{T}^2$ punctured at 1.

Surjectivity. Let $\phi \in \mathrm{Homeo}^+(\mathbb{T}^2)$. Let α be a path in $\mathbb{T}^2$ from 1 to $\phi(1)$. Define

$$\phi_t = \alpha(t)^{-1}\phi.$$

Hence $\phi_\bullet$ is an isotopy from $\phi_0 = \phi$ to ϕ_1, which satisfies $\phi_1(1) = 1$, and is therefore in the image of Mod $(\mathbb{T}^2_*)$.

Injectivity. Let $\phi \in \mathrm{Homeo}^+(\mathbb{T}^2_*)$ such that there is an isotopy $\phi_\bullet$ in $\mathrm{Homeo}^+(\mathbb{T}^2)$ from ϕ to $\mathrm{id}_{\mathbb{T}^2}$. Define

$$\phi'_t = \phi_t(1)^{-1}\phi_t.$$

Then $\phi'_\bullet$ is an isotopy from ϕ to $\mathrm{id}_{\mathbb{T}^2}$ such that $\phi'_t(1) = 1$ for all t. Therefore, ϕ is isotopic to $\mathrm{id}_{\mathbb{T}^2}$ in $\mathrm{Homeo}^+(\mathbb{T}^2_*)$. □

4.5 The Alexander Method

Remark 4.14. *The previous computations of mapping class groups lead to the following idea: given a large enough collection of curves and arcs $(\alpha_i)_{i\in I}$ on a surface S s.t. $\phi\circ\alpha_i$ is homotopic to α_i for all i, we hope to conclude that ϕ is isotopic to id_S.*

Definition 4.15 (Filling a surface)**.** *A transverse collection of simple closed curves and simple proper arcs $(\alpha_i)_{i\in I}$ on a surface S is said to* fill *if each component of the cut surface $S_{(\alpha_i,\, i\in I)}$ is homeomorphic to either $\mathbb{D}^2$ or $\mathbb{D}^2_*$.*

This is analogous to spanning sets in vector spaces.

Lemma 4.16. *Let $(\alpha_i)_{1\leqslant i\leqslant n}$ and $(\beta_i)_{1\leqslant i\leqslant n}$ be two transverse collections of essential simple closed curves and simple proper arcs on S satisfying the following three conditions:*

(i) No bigons*: the $(\alpha_i)_{1\leqslant i\leqslant n}$ are pairwise in minimal position.*

(ii) No annuli*: the $(\alpha_i)_{1\leqslant i\leqslant n}$ are pairwise non-isotopic.*

(iii) No triangles*: for distinct i,j,k, at least one of $\alpha_i\cap\alpha_j$, $\alpha_j\cap\alpha_k$ and $\alpha_k\cap\alpha_i$ is empty.*

We also assume that the collection $(\beta_i)_{1\leqslant i\leqslant n}$ has no bigons, no annuli and no triangles.

If α_i is homotopic to β_i for all $1\leqslant i\leqslant n$, then there is an ambient isotopy $\phi_\bullet$ of S such that $\beta_i=\phi\circ\alpha_i$ for all $1\leqslant i\leqslant n$.

Proof. We use induction on n. If $n=1$, this is a mere restatement of Lemma 3.15. By induction, we may therefore assume that $\alpha_i=\beta_i$ for all $1\leqslant i<n$. We know that α_n and β_n are isotopic, so we need to show that we can find an isotopy between them that will preserve α_i for all $i<n$. Note that if α_n and β_n are not disjoint, then they form a bigon. By assumption, we see that the curves and arcs α_i have to cross the bigon transversely, which allows one to remove the bigon by performing an isotopy. After finitely many such bigon removals, we may assume that α_n and β_n are disjoint, so they bound an annulus. Hence, we can push β_n over the annulus, keeping α_i for $i<n$. □

Definition 4.17 (Structure graph)**.** *Let $(\alpha_i)_{i\in I}$ be a filling collection of transverse simple closed curves and proper arcs on a surface S. The* structure graph *$\Gamma_{(\alpha_i,\, i\in I)}$ is the graph $\bigcup_{i\in I}\alpha_i\cup\partial S$, with vertices at all intersection points and punctures.*

Proposition 4.18 (Alexander Method)**.** *Let $(\alpha_i)_{i\in I}$ be a finite filling collection of transverse simple closed curves and proper arcs without bigons, annuli or triangles on a surface S. Let $\phi\in$* Homeo$^+(S,\partial S)$*.*

(i) *If there exists $\sigma\in\mathfrak{S}_n$ s.t. for all $i\in I$, $\phi\circ\alpha_i=\alpha_{\sigma(i)}$, then ϕ induces an automorphism ϕ_Γ of $\Gamma_{(\alpha_i,\, i\in I)}$.*

(ii) *If ϕ_Γ is trivial, then ϕ is isotopic to id_S.*

In particular, under the hypotheses of (i)*, $[\phi]\in\mathrm{Mod}(S)$ has finite order (because $\mathrm{Aut}\left(\Gamma_{\{\alpha_i,\, i\in I\}}\right)$ is a finite group).*

Proof. (i) By Lemma 4.16, we may modify ϕ by an isotopy so that $\phi\left(\Gamma_{(\alpha_i,\, i\in I)}\right)=\Gamma_{(\alpha_i,\, i\in I)}$, so ϕ induces ϕ_Γ as claimed.

(ii) If ϕ_Γ is trivial, then ϕ fixes $\Gamma_{(\alpha_i,\, i\in I)}$ pointwise. Since ϕ is orientation-preserving, it induces a self-homeomorphism of the cut surface $S_{(\alpha_i,\, i\in I)}$ that acts trivially on $\pi_0 S_{(\alpha_i,\, i\in I)}$. By the Alexander Lemma, it follows that ϕ is isotopic to id_S. □

5 Dehn twists

5.1 Definition and action on curves

Definition 5.1 (Dehn twist for the annulus)**.** *Let $A = \mathbb{S}^1 \times I$ be an oriented annulus. In Proposition 4.9, we proved that* $\mathrm{Mod}(A) \cong \mathbb{Z}$*, with generator* $\delta : (z, x) \longmapsto (e^{2i\pi x} z, x)$*. Note that δ only depends on the orientation of A; it is called the* left Dehn twist in the core curve of the annulus*.*

Definition 5.2 (Dehn twist for any surface)**.** *Let α be an essential simple closed curve on S and let $N \subseteq S$ be a regular neighbourhood of α. Choose a homeomorphism $\iota : A \to N$, where $A = \mathbb{S}^1 \times I$, and pull the orientation of N back to A. Let δ be the associated left Dehn twist on A. We define*

$$\delta_\alpha(x) = \begin{cases} (\iota \circ \delta \circ \iota^{-1})(x) & \text{if } x \in N \\ x & \text{otherwise} \end{cases}.$$

We write $T_\alpha = [\delta_\alpha] \in \mathrm{Mod}(S)$. This is the (left) Dehn twist in α*.*

Lemma 5.3. *The Dehn twist T_α only depends on the isotopy class of α (and on the orientation of S).*

Proof. Suppose that α' is isotopic to α. Let N' be a regular neighbourhood of α'. Fix an orientation of α, which also induces an orientation of α'. Write $\partial N = \alpha_- \cup \alpha_+$ and $\partial N' = \alpha'_- \cup \alpha'_+$ (the curves $\alpha_\pm$ and $\alpha'_\pm$ are defined by the orientation of α and α'). Since α is isotopic to α', it follows that $\alpha_\pm$ is isotopic to $\alpha'_\pm$. Therefore, there is an ambient isotopy on S taking N to N', which allows us to assume without loss of generality that $N = N'$. Now δ_α and $\delta_{\alpha'}$ are both supported on N and define the canonical generator of $\mathrm{Mod}(N)$, so they are isotopic, i.e. $T_\alpha = T_{\alpha'}$. □

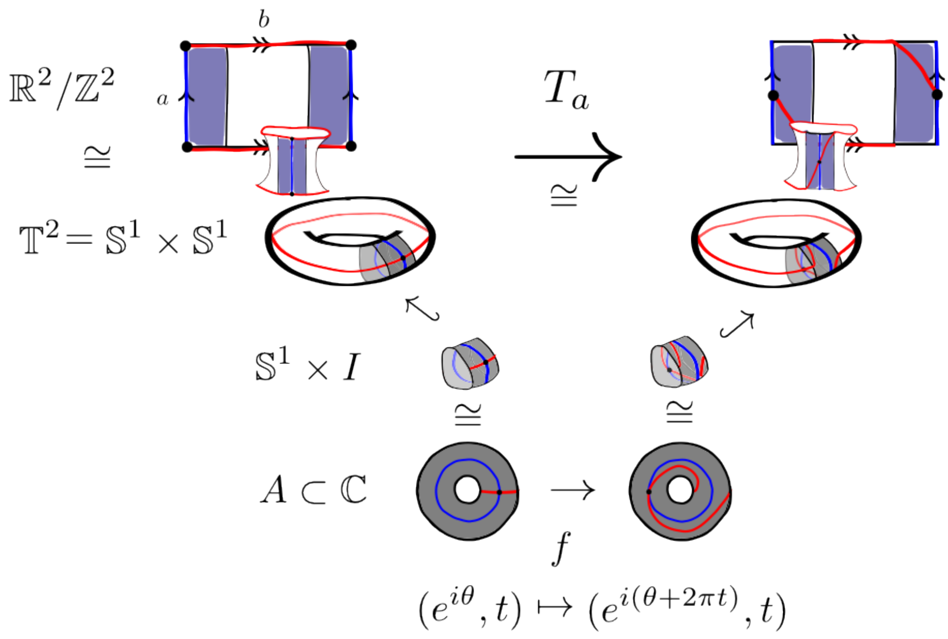

Figure 3: A Dehn twist on the torus

Remark 5.4. *Let α be an essential simple closed curve on S, let β be a simple closed curve or simple proper arc on S intersecting α transversely. We can draw $T_\alpha^k(\beta)$ as follows: draw $k \cdot |\alpha \cap \beta|$ parallel copies of α, push β slightly to the left and then modify the resulting picture by surgery: if T_α a left Dehn twist, the surgery turns left from β to α. Of course, there is no* a priori *guarantee that the resulting curve cannot be simplified.*

5.2 Order and intersection number

Lemma 5.5. *If α is an essential simple closed curve and β is a simple closed curve or proper arc, then*

$$i\left(T_\alpha^k(\beta), \beta\right) = |k| \cdot i(\alpha, \beta)^2.$$

Proof. We may assume that α and β are in minimal position. Apply the process of Remark 5.4 to produce $\beta' = T_\alpha^k(\beta)$. Since $|\beta \cap \beta'| = |k| \cdot i(\alpha, \beta)^2$, it suffices to prove that β and β' are in minimal position. Suppose β and β' form a bigon bounded by $b \subseteq \beta$ and $b' \subseteq \beta'$. Both orientations of intersections arise, so b' either leaves β on the left and returns on the left or leaves and returns on the right. If b' leaves and returns on the right, then b' is included in some copy of α, which contradicts the fact that α and β were in minimal position. If b' leaves and returns on the left, then we can push β slightly to the right instead when constructing β'. Now the previous argument applies, yielding a contradiction again. □

Proposition 5.6. *If α is an essential simple closed curve on S, then T_α has infinite order in* $\mathrm{Mod}(S)$.

Proof. Using Lemma 5.5, it is enough to find a simple closed curve or proper arc β such that $i(\alpha, \beta) > 0$.

- If α is nonseparating, then Corollary 3.20 gives the existence of a simple closed curve β such that $i(\alpha, \beta) = 1$.
- If α is a boundary component, then it can be taken to lie on a 3-holed sphere in S and it is easy to construct β such that $i(\alpha, \beta) = 2$.
- If α is separating but not a boundary component, then it can be taken to lie on a 4-punctured sphere, dividing it into twice-punctured discs. It is again easy to construct β with $i(\alpha, \beta) = 2$. □

5.3 Basic properties of Dehn twists

Lemma 5.7. *Two Dehn twists T_α and T_β are equal if and only if $\alpha \sim \beta^{\pm 1}$.*

Proof. ($\Leftarrow$) See Lemma 5.3. ($\Rightarrow$) Suppose $\alpha \not\sim \beta^{\pm 1}$. We claim that there exists a simple closed curve or proper arc γ on S such that $i(\beta, \gamma) = 0$ but $i(\alpha, \gamma) > 0$. Indeed, if $i(\alpha, \beta) > 0$, we may choose $\gamma = \beta$; otherwise, we may assume that α and β are disjoint. Therefore, we may consider the connected component Σ of S_β containing α, and use a change of coordinates (Corollary 3.20) to construct γ. Now by Lemma 5.5,

$$i(T_\beta(\gamma), \gamma) = i(\beta, \gamma)^2 = 0 \qquad \text{and} \qquad i(T_\alpha(\gamma), \gamma) = i(\alpha, \gamma)^2 > 0,$$

from which it follows that $T_\alpha \neq T_\beta$. □

Remark 5.8. *For $\phi \in \mathrm{Mod}(S)$, we have*

$$\phi T_\alpha \phi^{-1} = T_{\phi \circ \alpha}.$$

It follows that T_α is conjugate to T_β iff *α and β have the same topological type.*

Lemma 5.9. *Let $\phi \in \mathrm{Mod}(S)$ and let α, β be essential simple closed curves on S.*

(i) $[\phi, T_\alpha] = 1$ *if and only if $\phi \circ \alpha \sim \alpha^{\pm 1}$.*

(ii) $[T_\alpha, T_\beta] = 1$ *if and only if $i(\alpha, \beta) = 0$.*

Proof. (i) Use Lemma 5.7 together with Remark 5.8. (ii) Note that T_α and T_β commute iff $T_\beta(\alpha) \sim \alpha^{\pm 1}$ iff $i(\alpha, \beta) = 0$. □

5.4 Multitwists

Definition 5.10 (Multicurves and multitwists)**.** *A* multicurve $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_n$ *is a finite set of essential, pairwise disjoint, pairwise non-isotopic simple closed curves on S. A* multitwist *associated to α is a mapping class of the form $T_{\alpha_1}^{k_1} \cdots T_{\alpha_n}^{k_n}$.*

Proposition 5.11. *If $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_n$ is a multicurve, then the natural homomorphism*

$$\mathbb{Z}^n \to \mathrm{Mod}(S),$$

defined by $(k_1, \ldots, k_n) \mapsto T_{\alpha_1}^{k_1} \cdots T_{\alpha_n}^{k_n}$, is injective.

Proof. The above map is a homomorphism by Lemma 5.9. To prove the injectivity, suppose without loss of generality that $k_1 \neq 0$. Consider the cut surface $S_{\alpha_2,\ldots,\alpha_n}$ and let Σ be the component containing α_1. Thus α_1 is an essential simple closed curve on Σ not homotopic to one of the boundary components $\alpha_2, \ldots, \alpha_n$. Therefore there is a simple closed curve or proper arc β on Σ with endpoints not on $\alpha_2, \ldots, \alpha_n$ and such that $i(\alpha_1, \beta) > 0$. Since β does not meet any α_i with $i \geqslant 2$, it follows that

$$T_{\alpha_2}^{k_2} \cdots T_{\alpha_n}^{k_n}(\beta) = \beta.$$

Moreover, Lemma 5.5 implies that $T_{\alpha_1}^{k_1}(\beta) \not\sim \beta$, so $T_{\alpha_1}^{k_1} \cdots T_{\alpha_n}^{k_n} \neq 1$. □

Corollary 5.12. *The centre of* $\mathrm{Mod}\,(S_{g,n,b})$ *contains a copy of* $\mathbb{Z}^b$.

6 Further computations of mapping class groups

6.1 Pairs of pants

Remark 6.1. *The surface $S_{0,0,3}$ is called the* pair of pants*. It plays an important role, since if we cut up a closed surface maximally along pairwise non-isotopic curves,the resulting components will all be pairs of pants.*

Remark 6.2. *Using Remark 4.5 and Corollary 5.12, we have maps*

$$\mathbb{Z}^b \hookrightarrow \mathrm{Mod}\,(S_{0,n,b}) \twoheadrightarrow \mathfrak{S}_n.$$

Theorem 6.3. *If $n + b \leqslant 3$, then*

$$\mathrm{Mod}\,(S_{0,n,b}) \cong \mathbb{Z}^b \times \mathfrak{S}_n.$$

Proof. Let $S = S_{0,n,b}$. Following Remark 6.2, we shall show that the following sequence is exact:

$$1 \to \mathbb{Z}^b \to \mathrm{Mod}\,(S) \to \mathfrak{S}_n \to 1.$$

Let α_1, α_2 be simple proper arcs on S satisfying the hypotheses of the Alexander Method (Proposition 4.18). Let $\phi \in \mathrm{Ker}\,(\mathrm{Mod}(S) \to \mathfrak{S}_n)$. We can naturally embed S into $S_{0,3,0}$ (replacing each boundary component by a puncture), and then extend α_i to $\overline{\alpha}_i$ (so that those are arcs between punctures) and ϕ to $\overline{\phi}$ (by the identity on $S_{0,3,0} \backslash S$). Now $\overline{\phi} \circ \overline{\alpha}_i \sim \overline{\alpha}_i$ for all i, so $\phi \circ \alpha_i \sim \alpha_i$ by an isotopy that can move endpoints. We write $\hat{S} = S \cup_{\partial S} S$. We can double each α_i and ϕ to $\hat{\alpha}_i$ and $\hat{\phi}$. Now we have isotopies $\hat{\phi} \circ \hat{\alpha}_i \sim \hat{\alpha}_i$ in $\hat{S}$; therefore, after making them transverse by a small isotopy, $\hat{\phi} \circ \hat{\alpha}_i$ and $\hat{\alpha}_i$ are either disjoint or bound a bigon $D \hookrightarrow \hat{S}$. If $D \hookrightarrow S \subseteq \hat{S}$, then we may modify ϕ by an isotopy and reduce $|\alpha_i \cap (\phi \circ \alpha_i)|$ by two. Otherwise, we have a half-bigon, i.e. a bigon cut by a boundary component. We apply a Dehn twist δ in this boundary component in S. We will obtain $|(\delta \circ \alpha_i) \cap (\phi \circ \alpha_i)| = |\alpha_i \cap (\phi \circ \alpha_i)| + 1$, but this process also creates a new bigon; pushing over it reduces the number of intersections by 2. Therefore, after iterating, we eventually find $\psi \in \mathbb{Z}^d \leqslant \mathrm{Mod}(S)$ such that $\phi \circ \alpha_i \sim \psi \circ \alpha_i$. By Proposition 4.18, $\phi \sim \psi$. □

6.2 The inclusion homomorphism

Definition 6.4 (Essential subsurface)**.** *Let $\Sigma \subseteq S$ be a subsurface. We say that Σ is* essential *if one of the following three equivalent conditions is satisfied:*

(i) *The map $j_* : \pi_1\Sigma \to \pi_1 S$ induced by the inclusion $j : \Sigma \hookrightarrow S$ is injective.*

(ii) *$S\backslash\Sigma$ has no disc component.*

(iii) *Every simple closed curve in Σ bounding a disc in S also bounds a disc in Σ.*

Definition 6.5 (Inclusion homomorphism)**.** *Let $\Sigma \subseteq S$ be a closed, connected, essential subsurface. Then there is an obvious homomorphism* $\mathrm{Homeo}^+(\Sigma, \partial\Sigma) \to \mathrm{Homeo}^+(S, \partial S)$ *given by extension by the identity on $S\backslash\Sigma$. The induced homomorphism*

$$\iota : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$$

is called the inclusion homomorphism*.*

Lemma 6.6. *Let $\Sigma \subseteq S$ be an essential subsurface. Let α, β be essential simple closed curves on Σ that are not isotopic into boundary components of Σ. If $\alpha \simeq \beta$ in S, then $\alpha \simeq \beta$ in Σ.*

Proof. Make α, β transverse. If $\alpha \cap \beta \neq \varnothing$, then they bound a bigon in S. Since Σ is essential, α and β also bound a bigon in Σ. Hence, after finitely many bigon removals, we may assume that α and β are disjoint. Therefore, they bound an annulus A in S. Since α, β are not isotopic into boundary components of Σ, it follows that $A \subseteq \Sigma$. □

Theorem 6.7. *Let $\Sigma \subseteq S$ be a connected, closed (i.e. with open complement), essential subsurface. Let $\alpha_1, \ldots, \alpha_m \subseteq \partial\Sigma$ be components bounding punctured discs in S; let $\beta_1^\pm, \ldots, \beta_n^\pm \subseteq \partial\Sigma$ be pairs of components bounding annuli in S. Then the kernel of the inclusion homomorphism $\iota : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ is given by*

$$\mathrm{Ker}\,\iota = \left\langle (T_{\alpha_i})_{1\leqslant i\leqslant m}, \left(T_{\beta_j^+} T_{\beta_j^-}^{-1}\right)_{1\leqslant j\leqslant n} \right\rangle.$$

Proof. Define the *interior boundary* of Σ by $\partial_i\Sigma = \partial\Sigma\backslash\partial S$. Let $\phi \in \mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ such that $\phi \in \mathrm{Ker}\,\iota$. It is enough to prove that ϕ is isotopic to a homeomorphism of Σ supported on a regular neighbourhood of $\partial_i\Sigma$. This will imply that ϕ is a multitwist, and the result will follow from Proposition 5.11.

Write $\Sigma \cong S_{g,n,b}$.

- If $g = 0$ and $n + b \leqslant 3$, we know that every mapping class in $\mathrm{Mod}(\Sigma)$ fixing the punctures is a product of Dehn twists.
- If $g \geqslant 1$ or $n + b > 3$, then there exist essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on Σ without triangles, bigons or annuli, and such that every complementary component is a disc, a punctured disc or an annulus with one boundary component on $\partial\Sigma$. For each i, we have $\phi \circ \gamma_i \simeq \gamma_i$ in S (because $\phi \in \mathrm{Ker}\,\iota$), so $\phi \circ \gamma_i \simeq \gamma_i$ in Σ by Lemma 6.6. Reasoning as in the Alexander Method (c.f. Proposition 4.18), we show that $\phi \simeq \mathrm{id}$ away from a regular neighbourhood of $\partial\Sigma$. □

6.3 Capping

Definition 6.8 (Central extension)**.** *A* central extension *is a short exact sequence*

$$1 \to A \to G \to Q \to 1$$

of groups, such that $A \subseteq Z(G)$.

Corollary 6.9. *Let α be a boundary curve of S. We define a new surface $\overline{S}$ by glueing a punctured disk on α, i.e. $\overline{S} = S_\alpha \cup \mathbb{D}^2_*$. Then there is a central extension*

$$1 \to \langle T_\alpha \rangle \to \mathrm{PMod}(S) \to \mathrm{PMod}\left(\overline{S}\right) \to 1.$$

Corollary 6.10. *Let α be a multicurve on S with m components. Define*

$$\mathrm{Mod}_\alpha(S) = \{\phi \in \mathrm{Mod}(S),\ \phi \circ \alpha = \alpha\}.$$

Then there is a central extension

$$1 \to \mathbb{Z}^m \to \mathrm{Mod}\left(S_\alpha\right) \to \mathrm{Mod}_\alpha(S) \to 1.$$

Note that, if S_α is disconnected, we set $\mathrm{Mod}\left(S_\alpha\right) = \prod_{\Sigma \in \pi_0 S_\alpha} \mathrm{Mod}(\Sigma)$.

6.4 The Birman exact sequence

Notation 6.11. *We consider a surface of finite type S, and we denote by S_* the surface with an added puncture (or equivalently, with a marked point).*

Definition 6.12 (Outer automorphism group)**.** *Let G be a group. For $\gamma \in G$, define*

$$i_\gamma : g \in G \longmapsto \gamma g \gamma^{-1} \in G.$$

The automorphism i_γ is called an inner automorphism *of G. The set of inner automorphisms form a normal subgroup $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$; and we have an isomorphism $\mathrm{Inn}(G) \cong G/Z(G)$. The* outer automorphism group *of G is*

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G).$$

Remark 6.13. *There is a natural commutative diagram:*

$$\begin{array}{ccc} \mathrm{PMod}\left(S_*\right) & \longrightarrow & \mathrm{PMod}(S) \\ \downarrow & & \downarrow \\ \mathrm{Aut}\left(\pi_1 S\right) & \longrightarrow & \mathrm{Out}\left(\pi_1 S\right) \end{array}$$

The map $\mathrm{PMod}\left(S_*\right) \to \mathrm{Aut}\left(\pi_1 S\right)$ is given by action on loops based at $*$, and the map $\mathrm{PMod}\left(S\right) \to \mathrm{Out}\left(\pi_1 S\right)$ is given by action up to conjugation by an element of $\pi_1 S$.

Remark 6.14. *If $\chi(S) < 0$, then we know that $\pi_1 S$ has trivial centre; it follows that there is an exact sequence*

$$1 \to \pi_1 S \to \mathrm{Aut}\left(\pi_1 S\right) \to \mathrm{Out}\left(\pi_1 S\right) \to 1.$$

Lemma 6.15. *The map $\mathrm{PMod}\left(S_*\right) \twoheadrightarrow \mathrm{PMod}(S)$ is surjective.*

Proof. Let $\phi \in \mathrm{Homeo}^+\left(S, \partial S\right)$. Since S is connected, let α be a path from $*$ to $\phi(*)$. Extend α to an isotopy $\psi_\bullet$ from id_S with $\psi_1(*) = \phi(*)$. Now $\psi_\bullet^{-1} \circ \phi$ is an isotopy from ϕ to an element of $\mathrm{Homeo}^+\left(S_*, \partial S_*\right)$. □

Lemma 6.16. *If $\partial S = \varnothing$, then the map $\mathrm{PMod}\left(S_*\right) \hookrightarrow \mathrm{Aut}\left(\pi_1 S\right)$ is injective.*

Proof. There is a filling set of loops $\left(\alpha_i\right)_{i \in I}$ in S based at $*$, generating $\pi_1 S$, and satisfying the hypotheses of the Alexander Method (Proposition 4.18). Let $\phi \in \mathrm{PMod}\left(S_*\right)$ such that ϕ acts trivially on $\pi_1 S$. Then $\phi \circ \alpha_i \simeq \alpha_i$ for all i, so $\phi \simeq \mathrm{id}_S$ by the Alexander Method. □

Lemma 6.17. *If $\partial S = \varnothing$, then the map $\mathrm{PMod}(S) \hookrightarrow \mathrm{Out}\left(\pi_1 S\right)$ is injective.*

Proof. Same proof as for Lemma 6.16, noting that either $S = S_{0,n,0}$ (with $n \leqslant 3$) and $\mathrm{PMod}(S) = 1$, or there is indeed a filling set of loops in S satisfying the hypotheses of the Alexander Method. □

Lemma 6.18. *Let α be a simple closed curve on S based at $*$. Consider simple closed curves $\alpha_\pm$ bounding a regular neighbourhood of α (with signs determined by the orientation of S and α).*

Then the mapping class

$$T_{\alpha_+} \circ T_{\alpha_-}^{-1}$$

of S_* induces i_α on $\pi_1 S$. In particular if $\chi(S) < 0$, Remark 6.14 tells us that $\pi_1 S \cong \mathrm{Inn}(\pi_1 S) \leqslant \mathrm{Aut}(\pi_1 S)$ and Lemma 6.16 implies $\mathrm{PMod}(S_*) \hookrightarrow \mathrm{Aut}(\pi_1 S)$. Since $\pi_1 S$ is generated by simple closed curves, we have, as subgroups of $\mathrm{Aut}(\pi_1 S)$,

$$\pi_1 S \leqslant \mathrm{PMod}(S_*) \leqslant \mathrm{Aut}(\pi_1 S).$$

Proof. Extend $\{\alpha\}$ to a standard generating set B for $\pi_1 S$. It suffices to check that, for all $\beta \in B$, we have $\alpha \cdot \beta \cdot \alpha^{-1} \simeq \delta_{\alpha_+} \delta_{\alpha_-}^{-1} \beta$. If $\beta = \alpha$, this is trivial. Otherwise, separate the cases where β leaves α on one side and returns on the other, or β leaves and return on the same side, and draw the surgery diagrams for the Dehn twists as explained in Remark 5.4. □

Theorem 6.19 (Birman)**.** *If S is a surface such that $\chi(S) < 0$, then we have the following exact sequence:*

$$1 \to \pi_1 S \to \mathrm{PMod}(S_*) \to \mathrm{PMod}(S) \to 1.$$

Proof. If $\partial S = \varnothing$, Remark 6.13 and Lemmas 6.15, 6.16, 6.17 and 6.18 yield a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1 S & \longrightarrow & \mathrm{PMod}(S_*) & \longrightarrow & \mathrm{PMod}(S) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1 S & \longrightarrow & \mathrm{Aut}(\pi_1 S) & \longrightarrow & \mathrm{Out}(\pi_1 S) & \longrightarrow & 1
\end{array}$$

Note that the map $\pi_1 S \to \mathrm{PMod}(S_*)$ is the *point-pushing map* that is defined by the statement of Lemma 6.18. □

High-level proof. Consider the sequence $\mathrm{Diffeo}(S_*) \to \mathrm{Diffeo}(S) \xrightarrow{\mathrm{ev}_*} S$. This is a fibration and therefore there is a long exact sequence

$$\pi_1 \mathrm{Diffeo}(S) \to \pi_1 S \to \underbrace{\pi_0 \mathrm{Diffeo}(S_*)}_{=\mathrm{PMod}(S_*)} \to \underbrace{\pi_0 \mathrm{Diffeo}(S)}_{=\mathrm{PMod}(S)} \to \underbrace{\pi_0 S}_{=1}.$$

Since $\chi(S) < 0$, $\mathrm{Diffeo}(S)$ is contractible; thus $\pi_1 \mathrm{Diffeo}(S) = 1$ and the result follows. □

6.5 Generation by Dehn twists in genus zero

Corollary 6.20 (Dehn)**.** *Let $S = S_{0,n,b}$. Then there is a finite collection of simple closed curves A on S such that Dehn twists in the elements of A generate $\mathrm{PMod}(S)$.*

Moreover, $\mathrm{Mod}(S)$ is finitely generated.

Proof. We first do the case $b = 0$ by induction on n. When $n = 0, 1, 2, 3$, there is nothing to prove because $\mathrm{PMod}(S) = 1$ (c.f. Proposition 4.7 and Corollary 4.8). For the inductive step, consider the Birman exact sequence of $S_{0,n-1,0}$:

$$1 \to \pi_1 S_{0,n-1,0} \to \mathrm{PMod}(S_{0,n,0}) \to \mathrm{PMod}(S_{0,n-1,0}) \to 1.$$

We also note that any Dehn twist on $S_{0,n-1,0}$ lifts to a Dehn twist on $S_{0,n,0}$. Now Lemma 6.18 implies that $\pi_1 S_{0,n-1,0}$, seen as a subgroup of $\mathrm{PMod}(S_{0,n,0})$, is generated by products of Dehn twists.

Therefore, PMod $(S_{0,n,0})$ is generated by a finite number of Dehn twists. If $b \neq 0$, we apply Corollary 6.9 and use induction on b.

Hence PMod(S) is generated by finitely many Dehn twists. Since $[\mathrm{Mod}(S) : \mathrm{PMod}(S)] < +\infty$, it follows that Mod$(S)$ is finitely generated (it is generated by generators of PMod(S) and coset representatives of $\mathrm{Mod}(S)/\,\mathrm{PMod}(S)$). □

Corollary 6.21. *If* PMod (S_g) *is generated by finitely many Dehn twists, then so is* PMod $(S_{g,n,b})$ *for any* n, b.

Proof. Same proof as Corollary 6.20. □

6.6 The complex of curves

Definition 6.22 (Complex of curves)**.** *Let S be a surface of finite type. The* complex of curves $C(S)$ *is the simplicial complex defined as follows:*

- *Vertices are unoriented isotopy classes of essential simple closed curves on S that are not isotopic into ∂S.*
- *A set of vertices $\{[\alpha_0], \ldots, [\alpha_n]\}$ spans an n-simplex* iff $i(\alpha_i, \alpha_j) = 0$ *for all i, j.*

Note that $C(S)$ is a flag complex. Its 1*-skeleton is called the* curve graph.

Remark 6.23. *There is a natural action*

$$\mathrm{Mod}(S) \curvearrowright C(S).$$

Remark 6.24. *Note that the definition of the complex of curves does not distinguish boundary components from punctures; we shall henceforth assume that $S = S_{g,n} = S_{g,n,0}$.*

Example 6.25. (i) *If $g = 0$ and $n \leqslant 3$, then $S \in \{\mathbb{S}^2, \mathbb{C}, \mathbb{C}^*, S_{0,3}\}$ and $C(S) = \varnothing$.*

(ii) *If $S \in \{S_{1,0}, S_{1,1}, S_{0,4}\}$, then $C(S)$ has infinitely many vertices and no edge.*
Note that the cases above are all the surfaces $S_{g,n}$ satisfying $3g + n \leqslant 4$.

Theorem 6.26. *If $S = S_{g,n}$ with $3g + n \geqslant 5$, then $C(S)$ is connected.*

Proof. Let α, β be essential simple closed curves on S. Our goal is to find a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ such that $i(\alpha_i, \alpha_{i-1}) = 0$. We proceed by induction on $i(\alpha, \beta)$. If $i(\alpha, \beta) = 0$, there is nothing to prove; if $i(\alpha, \beta) = 1$, we use the change of coordinate principle (Proposition 3.19) to assume without loss of generality that α, β are, say, the two generators of the fundamental group of a torus, and γ is the boundary of the one-holed torus containing α, β. Since $3g + n \geqslant 5$, γ is essential, so we can choose $\alpha_1 = \gamma$. For the inductive step, we assume that α, β are in minimal position and $i(\alpha, \beta) \geqslant 2$. We choose orientations on α, β and we let $x \neq y$ be two points of $\alpha \cap \beta$ that are consecutive on β. There are two cases:

- The crossings at x and y have the same orientation. We then consider a curve γ following α until x, then β until y, then α again. We have $i(\alpha, \gamma) = 1$. This implies in particular that γ is essential. Moreover, $i(\beta, \gamma) < i(\alpha, \beta)$, so we may apply the induction hypothesis to (β, γ).
- The crossings at x and y have opposite orientations. We construct a curve γ_1 following α until y, then β in the reverse direction until x, then α again, and γ_2 following α until x, then β until y, then α again. We have $i(\gamma_1, \alpha) = i(\gamma_2, \alpha) = 0$; moreover, $i(\gamma_1, \beta), i(\gamma_2, \beta) < i(\alpha, \beta)$. The curves γ_1, γ_2 cannot bound discs, for otherwise α, β would not be in minimal position. They could bound punctured discs; in this case, consider a curve γ_1' following β until x, then α until y, then β again, and another curve γ_2' following β in the reverse direction until y, then α until x, then β in the reverse direction again. If both γ_1', γ_2' bound punctured discs, we show that $S = S_{0,4}$, which is impossible; otherwise we can argue as in the first case. □

Corollary 6.27. *Let $S = S_{g,n}$ with $g \geqslant 2$. If α, β are nonseparating simple closed curves then there exists a path in $C(S)$ from α to β, only traversing nonseparating curves.*

Proof. We first assume that $n \leqslant 1$. By Theorem 6.26, there is a shortest path $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ in $C(S)$. If α_{k-1} is nonseparating, we can conclude by induction on k. Let us therefore assume that α_{k-1} is separating. Note that, by minimality of k, α_{k-2} must be in the same component as β of the cut surface $S_{\alpha_{k-1}}$ (otherwise we could just remove α_{k-1}). Denote by Σ the component of $S_{\alpha_{k-1}}$ not containing α_{k-2} or β. Since $n \leqslant 1$, Σ has genus at least 1, so there is a nonseparating curve α' in Σ. Therefore we can replace α_{k-1} by α' and conclude as before.

Now suppose that $n > 1$. Arguing as above, the only problem arises if Σ has genus 0. In this case, the component $\Sigma' \subseteq S_{\alpha_{k-1}}$ containing α_{k-2} and β has at most $n-1$ punctures, so we can conclude by induction on n. □

6.7 Generation by Dehn twists

Remark 6.28. *We have constructed a complex $C(S)$ that is connected for most surfaces S. The idea is now that, given a group G acting on a space X, connectivity results for X yield generating sets for G, as illustrated by the following lemma.*

Lemma 6.29. *Let G be a group acting by homeomorphisms on a path-connected space X. If Y is an open subset of X such that $G \cdot Y = X$, then*

$$G = \langle \{g \in G,\ gY \cap Y \neq \varnothing\} \rangle .$$

Lemma 6.30. *Let α be a nonseparating curve on a surface S. Consider all nonseparating simple closed curves β on S that are disjoint from α. There are only finitely many* $\mathrm{Mod}\,(S_\alpha)$*-orbits of such curves in the cut surfaces (by Proposition 3.19); let $\beta_1, \ldots, \beta_k$ be orbit representatives. By Proposition 3.19, we can choose homeomorphisms $\phi_1, \ldots, \phi_k$ such that $\phi_j \circ \alpha = \beta_j$.*

If S has genus at least 2, then

$$\mathrm{Mod}(S) = \left\langle \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \cup \{\phi_1, \ldots, \phi_k\} \right\rangle .$$

Note that in the stabiliser, α is considered as a vertex of $C(S)$, i.e. we forget its orientation.

Proof. Let $g \in \mathrm{Mod}(S)$. We have a vertex $g\alpha \in C(S)$; consider a path $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell = g\alpha$ of nonseparating simple closed curves in $C(S)$. We can write $\alpha_i = g_i \alpha$ for $0 \leqslant i \leqslant \ell$. By using induction on ℓ, we may assume that $g_{\ell-1} \in \left\langle \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \cup \{\phi_1, \ldots, \phi_k\} \right\rangle$. Now consider $\beta = g_{\ell-1}^{-1} g\alpha$; it is a nonseparating curve on S, disjoint from α (because $g\alpha$ is disjoint from $g_{\ell-1}\alpha$). Therefore, there exists $h \in \mathrm{Mod}\,(S_\alpha)$ and $1 \leqslant j \leqslant k$ such that $h\beta = \beta_j = \phi_j \alpha$. It follows that

$$h g_{\ell-1}^{-1} g\alpha = \phi_j \alpha,$$

which implies that $g \in g_{\ell-1} h^{-1} \phi_j \,\mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha)$. But $h \in \mathrm{Mod}\,(S_\alpha) \subseteq \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha)$, and therefore $g \in \left\langle \mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \cup \{\phi_1, \ldots, \phi_k\} \right\rangle$ as required. □

Lemma 6.31. *Let $S = S_g$. If α, β are disjoint nonseparating simple closed curves on S, then there exists a sequence of Dehn twists taking α to β.*

Proof. By Proposition 3.19, there exists α_1 on S such that $i\,(\alpha_1, \alpha) = i\,(\alpha_1, \beta) = 1$. In other words, we have a path $\alpha = \alpha_0, \alpha_1, \alpha_2 = \beta$, pairwise intersecting once. It follows that

$$T_{\alpha_i} T_{\alpha_{i+1}}\,(\alpha_i) = \alpha_{i+1},$$

so that $T_{\alpha_1} T_\beta T_\alpha T_{\alpha_1}\,(\alpha) = \beta$. □

Lemma 6.32. *If α, β are simple closed curves with $i(\alpha, \beta) = 1$, then*

$$T_\beta T_\alpha^2 T_\beta(\alpha) = \alpha^{-1},$$

where α^{-1} is the curve α with orientation reversed.

Proof. Using Proposition 3.19, we may assume that α, β live on a once-punctured torus. We can then conclude using either the surgery description of Dehn twists, or the fact that $\mathrm{Mod}(\mathbb{T}^2_*) \cong SL_2(\mathbb{Z})$. □

Theorem 6.33. *Let S be a connected, oriented surface of finite type. Then there is a finite collection of simple closed curves on S such that Dehn twists in this collection generate* $\mathrm{PMod}(S)$*.*

In particular, $\mathrm{Mod}(S)$ *is finitely generated.*

Proof. By Corollaries 6.20 and 6.21, we may assume that $g \geqslant 1$ and $n = b = 0$. If $g = 1$, then $S = \mathbb{T}^2$, so $\mathrm{Mod}(S) \cong SL_2(\mathbb{Z})$, which is generated by the following elementary matrices:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

For $g \geqslant 2$, fix α a nonseparating curve on S. By Lemma 6.30, $\mathrm{Mod}(S)$ is generated by $\mathrm{Stab}_{\mathrm{Mod}(S)}(\alpha) \cup \{\phi_1, \ldots, \phi_k\}$. Lemma 6.31 implies that each ϕ_j is generated by Dehn twists. Lemma 6.32 implies that the stabiliser of α is generated by $\mathrm{Mod}_\alpha(S)$ (and hence by $\mathrm{Mod}(S_\alpha)$ by Corollary 6.10) and Dehn twists. Since $g(S_\alpha) < g(S)$ because α is nonseparating, we can conclude by induction on the genus. □

7 Further topics

7.1 Nielsen-Thurston classification

Notation 7.1. *In this section, the surface S is assumed to be hyperbolic and without boundary.*

Definition 7.2 (Periodic, reducible mapping classes)**.** *A mapping class $\phi \in \mathrm{Mod}(S)$ is said to be:*

- Periodic *if it has finite order,*
- Reducible *if there exists a multicurve α on S such that $\phi \circ \alpha \simeq \alpha^{\pm 1}$.*

Remark 7.3. *A mapping class $\phi \in \mathrm{Mod}(S)$ is periodic* iff *it is an isometry for some hyperbolic structure on S.*

Definition 7.4 (Singular foliation)**.** *A* singular foliation *$\mathcal{F}$ on S is a maximal atlas of charts such that*

(i) *Away from some finite subset $P \subseteq S$, the local model is $(0,1)^2 \subseteq \mathbb{R}^2$, with horizontal* leaves.

(ii) *At P, the local model is a k-*pronged singularity *for some $k \geqslant 3$.*

Moreover, the transition maps are required to send leaves to leaves.

A transverse measure *μ on $\mathcal{F}$ assigns a length to each path transverse to $\mathcal{F}$ in a way that only depends on the leaves crossed.*

Definition 7.5 (Pseudo-Anosov mapping class)**.** *An element $\phi \in \mathrm{Mod}(S)$ is said to be* pseudo-Anosov *if there exists a transverse pair of singular foliations equipped with transverse measure $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ and a $\lambda > 1$ such that*

$$\phi(\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda\mu_u) \qquad \text{and} \qquad \phi(\mathcal{F}_s, \mu_s) = \left(\mathcal{F}_s, \frac{1}{\lambda}\mu_s\right).$$

The index u stands for unstable *and s stands for* stable*.*

Theorem 7.6 (Nielsen-Thurston classification)**.** *Each* $\phi \in \text{Mod}(S)$ *is one of the following:*

(i) *Periodic,*

(ii) *Reducible,*

(iii) *Pseudo-Anosov.*

Note that ϕ *can be both periodic and reducible; however, if it is pseudo-Anosov then it is none of the others.*

This classification is analogous to the Jordan normal form in linear algebra.

7.2 Teichmüller space

Notation 7.7. *In this section, the surface* S *is (again) assumed to be hyperbolic and without boundary.*

Definition 7.8 (Teichmüller space)**.** *Let* $\text{HypMet}(S)$ *be the set of all hyperbolic metrics on* S*. Note that we have an action* $\text{Diffeo}(S) \curvearrowright \text{HypMet}(S)$*, which induces an action of* $\text{Mod}(S) = \text{Diffeo}(S)/\text{Diffeo}_0(S)$ *on* $\text{Diffeo}_0(S)\backslash \text{HypMet}(S)$*. The* Teichmüller space *of* S *is*

$$\mathcal{T}(S) = \text{Diffeo}_0(S)\backslash \text{HypMet}(S).$$

Hence there is an action $\text{Mod}(S) \curvearrowright \mathcal{T}(S)$*.*

Theorem 7.9. *There is a natural topology on* $\mathcal{T}(S)$*, and we have*

$$\mathcal{T}(S) \cong \mathbb{R}^{6g-6}.$$

Remark 7.10. *On the one-holed torus* $S_{1,0,1}$*, hyperbolic structures are determined by cuff lengths. Hence, for any surface* S*, the coordinates on* $\mathcal{T}(S) \cong \mathbb{R}^{6g-6}$ *are the lengths of the* $3g-3$ *curves in a pants decomposition, together with* $3g-3$ *turning parameters.*

Theorem 7.11 (Frecke)**.** $\text{Mod}(S) \curvearrowright \mathcal{T}(S)$ *properly discontinuously.*

Definition 7.12 (Moduli space)**.** *The* moduli space *of* S *is*

$$\mathcal{M}(S) = \text{Mod}(S)\backslash\mathcal{T}(S).$$

Theorem 7.13. *If* $\mathcal{PMF}(S)$ *is the projectivised space of measured foliations, then*

(i) $\mathcal{PMF}(S) \cong \mathbb{S}^{6g-7}$*,*

(ii) $\mathcal{T}(S) \cup \mathcal{PMF}(S) \cong \mathbb{D}^{6g-6}$*.*

Remark 7.14. *The key idea of Thurston's proof of Theorem 7.6 was to apply Brouwer's Fixed Point Theorem to the action of a mapping class* ϕ *on* $\mathcal{T}(S) \cup \mathcal{PMF}(S) \cong \mathbb{D}^{6g-6}$*. This is similar to the classification of hyperbolic isometries in Proposition 2.2.*

7.3 Open questions

Remark 7.15. *Here are three open questions on mapping class groups:*

(i) *Is* $\text{Mod}(S)$ *linear, i.e. is there an embedding* $\text{Mod}(S) \hookrightarrow GL_n(\mathbb{C})$ *for some* n*?*

(ii) *Is there a finite-index subgroup of* $\text{Mod}(S)$ *that surjects onto* $\mathbb{Z}$*?*

(iii) *If* $\Gamma \leqslant \text{Mod}(S)$*, is there a finite-sheeted cover* $S_0 \twoheadrightarrow S$ *such that the set of mapping classes lifting to* S_0 *is a subgroup of* Γ*?*

References

[1] B. Farb and D. Margalit. *A primer on mapping class groups.*