Percolation and Related Topics

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1.1 Percolation

Definition 1.1 (Bond percolation)**.** *Let $d \geqslant 2$ and $p \in [0,1]$. Consider the lattice $\mathbb{Z}^d$ (with edge set $\mathbb{E}^d$). Each edge $e \in \mathbb{E}^d$ is declared* open *with probability p and* closed *otherwise; states of different edges are independent.*

In other words, the configuration space is $\Omega = \{0,1\}^{\mathbb{E}^d}$, equipped with the product σ-algebra and the product $\mathbb{P}_p$ of Bernoulli measures of parameter p. For $e \in \mathbb{E}^d$, e is open in the configuration ω if $\omega(e) = 1$. The set of open edges of is $\eta(\omega) = \left\{e \in \mathbb{E}^d,\ \omega(e) = 1\right\}$.

Our aim will be to study the geometry of $\eta(\omega)$ as p varies.

Definition 1.2 (Connectivity and open clusters)**.** *Let $x, y \in \mathbb{Z}^d$. We say that x is* connected to *y, and we write $x \leftrightarrow y$ (in ω) if there is an open path from x to y in the configuration ω. We also write $x \leftrightarrow \infty$ if x lies in some infinite open path.*

The relation $\leftrightarrow$ is an equivalence relation on $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, the equivalence class of x is denoted by C_x and called the open cluster at *x. In particular, we write $C = C_0$, where 0 is the origin of $\mathbb{Z}^d$.*

Definition 1.3 (Percolation probability)**.** *The* percolation probability *is the function $\theta : [0,1] \to [0,1]$ defined by*

$$\theta(p) = \mathbb{P}_p\left(|C| = +\infty\right) = \mathbb{P}_p\left(0 \leftrightarrow \infty\right).$$

Proposition 1.4. *The percolation probability is a nondecreasing function.*

Proof. The idea is to couple percolation processes corresponding to different values of p by considering independent and identically distributed random variables $(U_e)_{e \in \mathbb{E}^d}$ with uniform law on $[0,1]$. For more details, see Theorem 1.20. □

Definition 1.5 (Critical probability)**.** *The* critical probability *is defined by*

$$p_c = \sup\left\{p \in [0,1],\ \theta(p) = 0\right\}.$$

By monotonicity, $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$.

Conjecture 1.6. $\theta\left(p_c\right) = 0$.

The result is known for $d = 2$ and $d \geqslant 11$.

Theorem 1.7. *If $d \geqslant 2$, then $0 < p_c < 1$. Values of p with $0 < p < p_c$ (resp. $p_c < p < 1$) are called* subcritical *(resp.* supercritical*).*

Proof. We first show that $p_c > 0$. To do this, denote by σ_n the number of *self-avoiding walks* (i.e. paths visiting no vertex more than once) of length n in the lattice $\mathbb{Z}^d$ and starting at 0. A basic question will be to understand the asymptotic behaviour of $(\sigma_n)_{n \in \mathbb{N}}$. We will also denote by N_n the random variable giving the number of open self-avoiding walks of length n in the percolation process. Note that we have:

$$\begin{aligned}
\theta(p) = \mathbb{P}_p\left(0 \leftrightarrow \infty\right) &\leqslant \mathbb{P}_p\left(\bigcap_{n \in \mathbb{N}} \left(N_n \geqslant 1\right)\right) \\
&\leqslant \limsup_{n \to +\infty} \mathbb{E}_p N_n = \limsup_{n \to +\infty} \sum_{\substack{\pi \text{ self-avoiding walk} \\ \text{of length } n}} \mathbb{P}_p\left(\pi \text{ is open}\right) \\
&= \limsup_{n \to +\infty} \sum_{\substack{\pi \text{ self-avoiding walk} \\ \text{of length } n}} p^n = \limsup_{n \to +\infty} \sigma_n p^n.
\end{aligned}$$

Now, we can give a crude upper-bound for σ_n by noticing that $\sigma_n \leqslant (2d)(2d-1)^{n-1}$. Therefore:

$$\theta(p) \leqslant \limsup_{n \to +\infty} \frac{2d}{2d-1}\left((2d-1)p\right)^n.$$

This proves that $\theta(p) = 0$ if $p < \frac{1}{2d-1}$, so $p_c \geqslant \frac{1}{2d-1} > 0$.

We now show that $p_c < 1$. Note first that $\mathbb{Z}^d \subseteq \mathbb{Z}^{d+1}$, so $\theta(p,d) \leqslant \theta\left(p, d+1\right)$ and $p_c(d) \geqslant p_c\left(d+1\right)$. It is therefore sufficient to prove the result for $d = 2$, and so we shall assume that $d = 2$. We denote by Γ_n the random variable giving the number of dual cycles of length n in the lattice $\mathbb{Z}^2$, containing 0 in their interior, and only traversing closed edges of $\mathbb{Z}^2$. We shall also write γ_n for the total number of such cycles. We have:

$$1-\theta(p)=\mathbb{P}_p\left(|C|<+\infty\right)\leqslant\mathbb{P}_p\left(\bigcup_{n\in\mathbb{N}}\left(\Gamma_n\geqslant 1\right)\right)$$
$$\leqslant\sum_{n\in\mathbb{N}}\mathbb{E}_p\Gamma_n=\sum_{n\in\mathbb{N}}\gamma_n(1-p)^n.$$

But to each dual cycle containing 0, we may associate a self-avoiding walk of length $(n-1)$ starting at one of the n vertices $(0,-n),\ldots,(0,-1)$. Thus $\gamma_n\leqslant n\sigma_{n-1}$, which gives:

$$1-\theta(p)\leqslant\frac{4}{9}\sum_{n\in\mathbb{N}}n\left(3\left(1-p\right)\right)^n\xrightarrow[p\to 1]{}0.$$

Hence, there exists $p'<1$ such that $1-\theta(p)<1$ for $p\geqslant p'$. This implies that $p_c\leqslant p'<1$. □

Remark 1.8. *The duality argument used in the above proof is called* Peierls' argument *and comes from statistical mechanics.*

1.2 Self-avoiding walks

Notation 1.9. *Let $\mathbb{L}$ be a lattice, i.e. a vertex-transitive graph: the group of graph automorphisms of $\mathbb{L}$ acts transitively on the set of vertices of $\mathbb{L}$. We denote by σ_n the number of self-avoiding walks of length n starting at a point $0\in\mathbb{L}$.*

Our question will be to understand the asymptotic behaviour of $(\sigma_n)_{n\in\mathbb{N}}$.

Lemma 1.10. *For all $m,n\in\mathbb{N}$, we have $\sigma_{m+n}\leqslant\sigma_m\sigma_n$.*

The sequence $(\log\sigma_n)_{n\in\mathbb{N}}$ is therefore subadditive.

Proof. Note that $\sigma_m\sigma_n$ is the number of (not necessarily self-avoiding) walks of length $m+n$ formed of an m-step self-avoiding walk followed by an n-step self-avoiding walk. Since all self-avoiding walks of length $m+n$ are of that type, it follows that $\sigma_{m+n}\leqslant\sigma_m\sigma_n$.

Note that we have used the fact that $\mathbb{L}$ is transitive. □

Theorem 1.11 (Subadditive inequality theorem)**.** *Assume that $f:\mathbb{N}\to\mathbb{N}$ is* subadditive*: $f(m+n)\leqslant f(m)+f(n)$ for all $m,n\in\mathbb{N}$. Then the sequence $\left(\frac{f(n)}{n}\right)_{n\geqslant 1}$ has a limit given by:*

$$\lim_{n\to+\infty}\frac{f(n)}{n}=\inf_{n\geqslant 1}\frac{f(n)}{n}\in\left[-\infty,+\infty\right).$$

Proof. We let $\ell=\inf_{n\geqslant 1}\frac{f(n)}{n}\in[-\infty,+\infty)$ and we want to show that $\frac{f(n)}{n}\xrightarrow[n\to+\infty]{}\ell$. We shall do the proof in the case where $\ell>-\infty$. Let $\varepsilon>0$ and pick $n_0\geqslant 1$ s.t.

$$\ell\leqslant\frac{f\left(n_0\right)}{n_0}\leqslant\ell+\varepsilon.$$

Now, let $M=\sup_{0\leqslant r<n_0}|f(r)|$ and choose $n_1\geqslant n_0$ such that $0\leqslant\frac{M}{n_1}\leqslant\varepsilon$. For $n\geqslant n_1$, we can write $n=qn_0+r$ with $q\geqslant 0$ and $0\leqslant r<n_0$, so that

$$\ell\leqslant\frac{f(n)}{n}\leqslant\frac{qf\left(n_0\right)+f(r)}{n}\leqslant f\left(n_0\right)+\frac{f(r)}{n_1}\leqslant\ell+2\varepsilon.$$ □

Corollary 1.12. *There exists a constant $\kappa=\kappa\left(\mathbb{L}\right)\geqslant 1$ such that $\log\sigma_n=\left(\log\kappa\right)n\left(1+o(1)\right)$, or in other words:*

$$\sigma_n=\kappa^{n(1+o(1))}.$$

The constant $\kappa(\mathbb{L})$ is called the connective constant *of $\mathbb{L}$.*

Our aim will now be to determine $\kappa\left(\mathbb{L}\right)$ for $\mathbb{L}=\mathbb{Z}^d$ and for other lattices.

Example 1.13. *For $\mathbb{L} = \mathbb{Z}$, we have $\sigma_n = 2$ for $n \geqslant 1$, so $\kappa = 1$.*

Conjecture 1.14. *It is believed that $\sigma_n \sim A\kappa^n n^{11/32}$ for $\mathbb{L} = \mathbb{Z}^2$. The exponent $\frac{11}{32}$ is called the* critical exponent.

It is known that $\sigma_n \sim A\kappa^n$ for $\mathbb{L} = \mathbb{Z}^d$ with $d \geqslant 5$.

1.3 Connective constant of the hexagonal lattice

Notation 1.15. *We now want to determine the connective constant of the* hexagonal lattice $\mathbb{H}$.

We embed $\mathbb{H}$ in the complex plane as in Figure 1. We shall change slightly our notation for the purpose of the proof and write σ_n for the number of self-avoiding walks between midpoints of edges (rather than between vertices). Note that this is equal to the former σ_{n+1}, so the asymptotic behaviour remains unchanged. We consider the generating function

$$Z(x) = \sum_{n \in \mathbb{N}} \sigma_n x^n = \sum_{\gamma \text{ s.a.w. from } a} x^{|\gamma|}.$$

Our aim is to show that Z has radius of convergence $\chi = \frac{1}{\sqrt{2+\sqrt{2}}}$. Given a self-avoiding walk γ, we shall denote by $T(\gamma)$ the turning angle *of γ, i.e. the angle between the initial and the final directions of γ.*

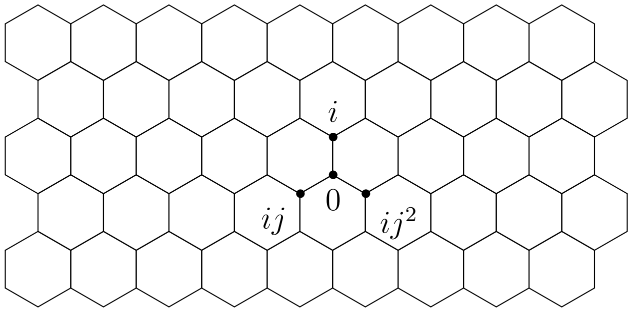

Figure 1: The hexagonal lattice $\mathbb{H} \subseteq \mathbb{C}$

Lemma 1.16. *Fix a bounded and simply-connected region $\mathcal{M}$ of $\mathbb{C}$. Given a midpoint z of $\mathbb{H}$, define*

$$F(z) = F^{x,\sigma}(z) = \sum_{\gamma \text{ s.a.w. } a \to z \text{ in } \mathcal{M}} x^{|\gamma|} \exp\left(-i\sigma T(\gamma)\right).$$

Let v be a vertex of $\mathbb{H}$ and let p, q, r be the three neighbouring midpoints. If $\sigma = \frac{5}{8}$ and $x = \chi = \frac{1}{\sqrt{2+\sqrt{2}}}$, then

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0. \qquad (*)$$

This is a discrete analyticity *result.*

Proof. For $k \in \{1, 2, 3\}$, let $\mathcal{P}_k$ be the set of all self-avoiding walks in $\mathcal{M}$ visiting exactly k points of $\{p, q, r\}$.

Consider the set $\mathcal{P}_3$. Given $\gamma \in \mathcal{P}_3$, we may assume that p is the first point of $\{p, q, r\}$ met by γ, and we denote by ρ the subwalk of γ stopped at p. After p, the walk crosses the vertex v and can either continue to the left (say, to r) or to the right (to q). If it continues to r, it then follows a self-avoiding walk τ from r to q and must necessarily stop at q. To this walk γ corresponds another walk $\overline{\gamma}$ which continues to q after v and then follows the walk τ in the reverse direction; denote that walk by $\overline{\gamma}$. This defines an involution $\bar{\cdot} : \mathcal{P}_3 \to \mathcal{P}_3$ without fixed point, and note that the aggregate contribution of γ and $\overline{\gamma}$ to the left-hand side of Equation $(*)$ is given by:

$$c\left(\overline{\theta} e^{-i\sigma\frac{4\pi}{3}} + \theta e^{i\sigma\frac{4\pi}{3}}\right) = 2c\cos\left(\frac{2\pi}{3}(2\sigma + 1)\right),$$

where $c = (p-v)x^{|\rho|+|\tau|+1}e^{-i\sigma T(\rho)}$ and $\theta = \frac{q-v}{p-v} = e^{i\frac{2\pi}{3}}$. If $\sigma = \frac{5}{8}$, then $\cos\left(\frac{2\pi}{3}(2\sigma+1)\right) = 0$, so the contributions of γ and $\overline{\gamma}$ cancel out, which implies that the contribution of $\mathcal{P}_3$ is 0.

Now consider $\mathcal{P}_1 \cup \mathcal{P}_2$. Let $\gamma \in \mathcal{P}_1$, assume that p is the first point of $\{p,q,r\}$ met by γ, let ρ be the subwalk of γ stopped at p, and consider the two walks of $\mathcal{P}_2$ obtained from γ by either continuing one step to q or one step to r. This defines a partition of $\mathcal{P}_1 \cup \mathcal{P}_2$ into subsets of cardinal 3, and the contribution of each such subset to the left-hand side of $(*)$ is

$$c\left(1 + \theta x e^{i\sigma\frac{\pi}{3}} + \overline{\theta} x e^{-i\sigma\frac{\pi}{3}}\right),$$

where $c = (p-v)x^{|\rho|}e^{-i\sigma T(\rho)}$ and $\theta = \frac{q-v}{p-v} = e^{i\frac{2\pi}{3}}$. We check that this contribution cancels out when $x = \frac{1}{2\cos\left(\frac{\pi}{8}\right)} = \chi$, which implies that the contribution of $\mathcal{P}_1 \cup \mathcal{P}_2$ is also zero, and therefore Equation $(*)$ holds. $\square$

Theorem 1.17. *The hexagonal lattice* $\mathbb{H}$ *has connective constant* $\kappa(\mathbb{H}) = \sqrt{2+\sqrt{2}}$.

Proof. We will show that Z has radius of convergence $\chi = \frac{1}{\sqrt{2+\sqrt{2}}}$.

First step: $Z(\chi) = \infty$. We shall work in a region $\mathcal{M} = \mathcal{M}_{m,n}$ of the complex plane which is a trapezium with a lower basis containing $2m+1$ midpoints of edges (the set of these points will be denoted by L_m), two edges to the left and right making respective angles of $\frac{\pi}{6}$ and $-\frac{\pi}{6}$ from the vertical, containing each n midpoints of edges (the sets of these points will be denoted by $T^-_{m,n}$ and $T^+_{m,n}$ respectively) and a horizontal upper basis (whose set of midpoints will be denoted by $U_{m,n}$). We assume moreover that a lies in the middle of L_m. Summing Equation $(*)$ of Lemma 1.16 over all vertices v in $\mathcal{M}$, we see that only terms corresponding to the boundary remain. We write

$$\tau^{\pm}_{m,n} = \sum_{\gamma: a\to T^{\pm}_{m,n}} x^{|\gamma|}, \qquad \lambda_{m,n} = \sum_{\gamma: a\to L_m} x^{|\gamma|}, \qquad \nu_{m,n} = \sum_{\gamma: a\to U_{m,n}} x^{|\gamma|}.$$

Hence, the sum of Equation $(*)$ over $\mathcal{M}$ yields (for $\sigma = \frac{5}{8}$ and $x = \chi$)

$$-iF(a) - i\Re\left(e^{i\sigma\pi}\right)\lambda_{m,n} + i\theta e^{-i\sigma\frac{2\pi}{3}}\tau^-_{m,n} + i\nu_{m,n} + i\overline{\theta}e^{i\sigma\frac{2\pi}{3}}\tau^+_{m,n} = 0.$$

Since $F(a) = 1$, we deduce that

$$\alpha\lambda_{m,n} + \beta\underbrace{\left(\tau^+_{m,n} + \tau^-_{m,n}\right)}_{\tau_{m,n}} + \nu_{m,n} = 1, \tag{$\blacklozenge$}$$

with $\alpha = \cos\left(\frac{3\pi}{8}\right)$ and $\beta = \cos\left(\frac{\pi}{4}\right)$. Now, note that $(\lambda_{m,n})$ and $(\nu_{m,n})$ are nondecreasing sequences of m. By Equation $(\blacklozenge)$, it follows that $(\tau_{m,n})$ is a nonincreasing sequence of m. Therefore, we have $\lambda_{m,n} \xrightarrow[m\to+\infty]{\nearrow} \lambda_n$, $\nu_{m,n} \xrightarrow[m\to+\infty]{\nearrow} \nu_n$ and $\tau_{m,n} \xrightarrow[m\to+\infty]{\searrow} \tau_n$, with

$$\alpha\lambda_n + \beta\tau_n + \nu_n = 1. \tag{$\lozenge$}$$

Assume first that $\tau_n > 0$ for some $n \geqslant 0$. Then $\tau_{m,n} \geqslant \tau_n > 0$ for all m, so $Z(\chi) \geqslant \sum_{m\in\mathbb{N}} \tau_{m,n} = +\infty$. If on the other hand $\tau_n = 0$ for all $n \geqslant 0$, consider the quantity $\lambda_{n+1} - \lambda_n$. This is the number of paths that start at a and reach the horizontal strip comprised between heights n and $n+1$ before returning to height 0. Such a path can be decomposed into two paths starting at the top and ending at the bottom, with one edge counted twice. Therefore

$$\lambda_{n+1} - \lambda_n \leqslant \frac{1}{\chi}\nu^2_{n+1}.$$

Using the fact that $\alpha\lambda_n + \nu_n = 1$ (by $(\lozenge)$), we obtain

$$\nu_n \leqslant \nu_{n+1} + \frac{\alpha}{\chi}\nu^2_{n+1}.$$

It follows by induction that $\nu_n \geqslant \frac{C}{n}$ with $C = \min\left\{\nu_1, \frac{\chi}{\alpha}\right\}$, and therefore

$$Z(\chi) \geqslant \sum_{n\in\mathbb{N}} \nu_n \geqslant \sum_{n\geqslant 1} \frac{C}{n} = +\infty.$$

This proves that $Z(\chi) = +\infty$.

Second step: $Z(x) < +\infty$ if $0 < x < \chi$. We will call *bridge* any self-avoiding walk (between midpoints) starting at its lowest height and finishing at its highest height. Note that every half-space self-avoiding walk can be decomposed into a sequence of bridges of decreasing heights $T_0 > T_1 > \cdots > T_i$ (by choosing successive minima and maxima). Moreover, every full-space walk can be decomposed into two half-space walks (by cutting at the maximum), and therefore into two sequences of bridges with associated heights $T_0 > T_1 > \cdots > T_i$ and $S_0 > S_1 > \cdots > S_j$. Therefore

$$Z(x) \leqslant 2 \sum_{\substack{T_0 > \cdots > T_i \\ S_0 > \cdots > S_j}} \left(\nu_{T_0} \cdots \nu_{T_i}\right)\left(\nu_{S_0} \cdots \nu_{S_j}\right) = 2\left(\prod_{n\in\mathbb{N}} (1+\nu_n)\right)^2.$$

It is therefore sufficient to prove that the family $(\nu_n)_{n\in\mathbb{N}}$ is summable. But note that

$$\nu_n(x) \leqslant \left(\frac{x}{\chi}\right)^n \nu_n(\chi) \leqslant \left(\frac{x}{\chi}\right)^n,$$

because $\nu_n(\chi) \leqslant 1$ by Equation ($\lozenge$). Hence $\sum_{n\in\mathbb{N}} \nu_n \leqslant \sum_{n\in\mathbb{N}} \left(\frac{x}{\chi}\right)^n < +\infty$ for $0 < x < \chi$, so $Z(x) < +\infty$. ☐

1.4 Back to percolation

Proposition 1.18. *The critical probability and the connective constant of* $\mathbb{Z}^d$ *satisfy*

$$\frac{1}{\kappa(\mathbb{Z}^d)} \leqslant p_c(\mathbb{Z}^d) \leqslant 1 - \frac{1}{\kappa(\mathbb{Z}^d)}.$$

Proof. In the proof of Theorem 1.7, we have seen that

$$\theta(p) \leqslant \limsup_{n\to+\infty} \sigma_n p^n.$$

But note that $\log(\sigma_n p^n) \sim n(\log\kappa + \log p)$. Hence, if $p < \frac{1}{\kappa}$, then $\log\kappa + \log p < 0$ and $\sigma_n p^n \xrightarrow[n\to+\infty]{} 0$, which implies that $\theta(p) = 0$. This shows that $p_c \geqslant \frac{1}{\kappa}$.

For the upper-bound in the case $d = 2$, we need to elaborate on the proof of Theorem 1.7. We denote by F_m the event that there exists a closed cycle of the dual lattice of $\mathbb{Z}^2$ containing the box $\Lambda(m) = [-m, m]^d$ in its interior. We have, as in the proof of Theorem 1.7,

$$1 - \theta(p) \leqslant \mathbb{P}_p(F_m) \leqslant \sum_{n=4m}^{\infty} n\sigma_{n-1}(1-p)^n.$$

If $p > 1 - \frac{1}{\kappa}$, then the above sum converges, and therefore one may find a value of m such that $\mathbb{P}_p(F_m) \leqslant \frac{1}{2}$. Thus, $\theta(p) > 0$, which proves that $p_c \leqslant 1 - \frac{1}{\kappa}$.

For other values of d, note that $p_c(d) \leqslant p_c(2)$ and $\kappa(d) \geqslant \kappa(2)$. As a consequence, $1 - \frac{1}{\kappa(d)} \geqslant 1 - \frac{1}{\kappa(2)} \geqslant p_c(2) \geqslant p_c(d)$. ☐

Notation 1.19. *Recall that the configuration space we use to model percolation is* $\Omega = \{0,1\}^E$*, where* E *is the set of edges. The set* Ω *is partially ordered by* $\omega \leqslant \omega' \iff \forall e \in E,\ \omega(e) \leqslant \omega'(e)$.

Theorem 1.20. *Let* $f : \Omega \to \mathbb{R}$ *be a nondecreasing integrable function. Then the function* $p \mapsto \mathbb{E}_p(f)$ *is nondecreasing.*

Proof. Model the percolation process as follows: let $(U_e)_{e\in E}$ be a family of independent and identically distributed random variables following a uniform law on $[0,1]$. For each edge e, set $\eta_p(e) = \mathbb{1}(U_e < p)$. For a given p, $(\eta_p(e))_{e\in E}$ is a family of independent random variables following a Bernoulli law with parameter p. Note moreover that $p \leqslant p' \Rightarrow \eta_p(e) \leqslant \eta_{p'}(e)$ for all e. Therefore:

$$\mathbb{E}_p(f) = \mathbb{E}(f(\eta_p)) \leqslant \mathbb{E}(f(\eta_{p'})) = \mathbb{E}_{p'}(f). \qquad \square$$

Remark 1.21. *Theorem 1.20 implies Proposition 1.4.*

Definition 1.22 (Oriented percolation)**.** *Let $d \geqslant 2$ and $p \in [0,1]$. Consider the lattice $\mathbb{Z}^d$ (with edge set $\mathbb{E}^d$). Each edge $e \in \mathbb{E}^d$ is declared open with probability p and closed otherwise; states of different edges are independent. As opposed to standard bond percolation, each edge is oriented to the North or to the East. We define*

$$\vec{\theta}(p) = \mathbb{P}_p(0 \text{ lies in an infinite oriented path}),$$

and $\vec{p_c} = \sup\left\{p \in [0,1],\ \vec{\theta}(p) = 0\right\}$.

Theorem 1.23. $0 < \vec{p_c} < 1$.

Proof. Clearly $\vec{p_c} \geqslant p_c > 0$. For the other inequality, we use the same idea as in Theorem 1.7: we count dual cycles which block oriented paths from 0 to ∞ (therefore, only edges going right or downwards matter); this yields:

$$1 - \vec{\theta}(p) \leqslant \sum_{n\geqslant 4} 4^{n-1}(1-p)^{n/2} \xrightarrow[p\to 1]{} 0. \qquad \square$$

2 Association and influence

2.1 The Holley and FKG inequalities

Definition 2.1 (Increasing sets and functions)**.** *Recall that the configuration space we use to model percolation is $\Omega = \{0,1\}^E$, where E is the set of edges. The set Ω is partially ordered by $\omega \leqslant \omega' \iff \forall e \in E,\ \omega(e) \leqslant \omega'(e)$.*

- *A subset $A \subseteq \Omega$ is called* increasing *if $\omega \in A$* and *$\omega \leqslant \omega' \Longrightarrow \omega' \in A$.*
- *A subset $A \subseteq \Omega$ is called* decreasing *if $\Omega \backslash A$ is increasing.*
- *A function $f : \Omega \to \mathbb{R}$ is called* increasing *if $\omega \leqslant \omega' \Longrightarrow f(\omega) \leqslant f(\omega')$.*

Note that a subset A is increasing iff *the function $\mathbb{1}_A$ is increasing.*

Definition 2.2 (Stochastic ordering)**.** *Let $\mathcal{P}$ be the set of probability measures on Ω, let $\mu, \mu' \in \mathcal{P}$. We say that $\mu \leqslant_{st} \mu'$ if one of the following two equivalent conditions is satisfied:*

(i) *For all increasing subsets $A \subseteq \Omega$, $\mu(A) \leqslant \mu'(A)$.*

(ii) *For all increasing functions $f : \Omega \to \mathbb{R}$, $\mu(f) \leqslant \mu'(f)$ (where $\mu(f)$ is the integral of f relative to μ, i.e. the expectation of f).*

The partial order $\leqslant_{st}$ is called the stochastic ordering*.*

Theorem 2.3 (Baby Strassen)**.** *For $\mu_1, \mu_2 \in \mathcal{P}$, the following assertions are equivalent:*

(i) $\mu_1 \leqslant_{st} \mu_2$.

(ii) *There exists a probability measure κ on Ω^2 s.t.*

(a) *The first marginal of* κ *is* μ_1 *and the second one is* μ_2*,*

(b) $\kappa(S) = 1$ *where* $S = \{(\omega_1, \omega_2) \in \Omega^2,\ \omega_1 \leqslant \omega_2\}$*.*

Proof. (ii) $\Rightarrow$ (i) Let $A \subseteq \Omega$ be an increasing event. Then

$$\mu_1(A) = \kappa\,(A \times \Omega) = \kappa\,((A \times \Omega) \cap S) \leqslant \kappa\,(A \times A) \leqslant \kappa\,(\Omega \times A) = \mu_2(A). \qquad \square$$

Notation 2.4. *For* $\omega_1, \omega_2 \in \Omega$*, we define*

$$(\omega_1 \vee \omega_2)\,(e) = \max\{\omega_1(e), \omega_2(e)\} \qquad \text{and} \qquad (\omega_1 \wedge \omega_2)\,(e) = \min\{\omega_1(e), \omega_2(e)\}\,.$$

Given $\omega \in \Omega$ *and* $e \in E$*, define* $\omega^e, \omega_e \in \Omega$ *by* $\omega^e = \omega \vee \mathbb{1}_{\{e\}}$ *and* $\omega_e = \omega \wedge \mathbb{1}_{\Omega\setminus\{e\}}$*.*

Theorem 2.5 (Holley)**.** *Let* μ_1, μ_2 *be two positive probability measures on* $\Omega = \{0,1\}^E$ *(i.e.* $\mu_i(\omega) > 0$ *for all* $\omega \in \Omega$*), with* E *finite. Assume that the following inequality is satisfied for all* $\omega_1, \omega_2 \in \Omega$*:*

$$\mu_2\,(\omega_1 \vee \omega_2)\,\mu_1\,(\omega_1 \wedge \omega_2) \geqslant \mu_1\,(\omega_1)\,\mu_2\,(\omega_2)\,.$$

Then $\mu_1 \leqslant_{st} \mu_2$*.*

Proof. First choose a positive probability measure μ on Ω and consider a Markov chain $(X_t)_{t\geqslant 0}$ in continuous time on Ω with single edge-flips, i.e. with generator G defined by

$$G\,(\omega_e, \omega^e) = 1, \qquad G\,(\omega^e, \omega_e) = \frac{\mu\,(\omega_e)}{\mu\,(\omega^e)},$$

and $G\,(\omega, \omega') = 0$ for all other pairs $\omega \neq \omega'$, and $G(\omega, \omega)$ is such that $\sum_{\omega' \in \Omega} G\,(\omega, \omega') = 0$ for all $\omega \in \Omega$. Therefore

$$\mu(\omega) G\,(\omega, \omega') = \mu\,(\omega')\,G\,(\omega', \omega)\,.$$

It follows that the Markov chain $(X_t)_{t\geqslant 0}$ with generator G is reversible, irreducible, and has invariant probability measure μ.

Now do the same thing with pairs: let μ_1, μ_2 be as in the statement of the theorem, let $S = \{(\pi, \omega) \in \Omega^2,\ \pi \leqslant \omega\}$. Consider a Markov chain $(X_t, Y_t)_{t\geqslant 0}$ on $S \subseteq \Omega^2$ s.t. $(X_0, Y_0) = (0, 1)$ and with generator H defined by

$$\begin{aligned} H\,((\pi_e, \omega)\,, (\pi^e, \omega^e)) &= 1, \\ H\,((\pi, \omega^e)\,, (\pi_e, \omega_e)) &= \frac{\mu_2\,(\omega_e)}{\mu_2\,(\omega^e)}, \\ H\,((\pi^e, \omega^e)\,, (\pi_e, \omega^e)) &= \frac{\mu_1\,(\pi_e)}{\mu_1\,(\pi^e)} - \frac{\mu_2\,(\omega_e)}{\mu_2\,(\omega^e)}. \end{aligned}$$

Note that the positivity of $H\,((\pi^e, \omega^e)\,, (\pi_e, \omega^e))$ follows from the fact that $\mu_2\,(\pi^e \vee \omega_e)\,\mu_1\,(\pi^e \wedge \omega_e) \geqslant \mu_1\,(\pi^e)\,\mu_2\,(\omega_e)$, which is true by assumption. Also note that $(X_t)_{t\geqslant 0}$ is now a Markov chain with invariant probability measure μ_1, and $(Y_t)_{t\geqslant 0}$ is a Markov chain with invariant probability measure μ_2. Therefore, the unique invariant probability measure of $(X_t, Y_t)_{t\geqslant 0}$ is some κ which has μ_1 as first marginal, μ_2 as second, and $\kappa(S) = 1$. Theorem 2.3 implies that $\mu_1 \leqslant_{st} \mu_2$. $\square$

Theorem 2.6 (FKG)**.** *Let* μ *be a positive probability measure on* $\Omega = \{0,1\}^E$*, with* E *finite. Assume that the following inequality holds for all* $\omega_1, \omega_2 \in \Omega$*:*

$$\mu\,(\omega_1 \vee \omega_2)\,\mu\,(\omega_1 \wedge \omega_2) \geqslant \mu\,(\omega_1)\,\mu\,(\omega_2)\,.$$

Then $\mu\,(fg) \geqslant \mu(f)\mu(g)$ *for all increasing functions* $f, g : \Omega \to \mathbb{R}$ *(or equivalently,* $\mu\,(A \cap B) \geqslant \mu\,(A)\,\mu\,(B)$ *for all increasing events* $A, B \subseteq \Omega$*).*

Proof. Let $\mu_1 = \mu$ and μ_2 be the probability measure defined by

$$\mu_2(\omega) = \frac{g(\omega)\mu(\omega)}{\sum_{\omega' \in \Omega} g\,(\omega')\,\mu\,(\omega')}.$$

We may assume that $g > 0$ by replacing it by $g + n$ for n large enough. Then μ_1, μ_2 satisfy the hypotheses of Holley's Theorem (Theorem 2.5), so $\mu_1 \leqslant_{st} \mu_2$, which yields $\mu(fg) \geqslant \mu(f)\mu(g)$. $\square$

2.2 Disjoint occurence and the BK inequality

Remark 2.7. *The product measure $\mathbb{P}_p$ on $\Omega = \{0,1\}^E$ (with E finite) satisfies the FKG condition. It follows from Theorem 2.6 that*

$$\mathbb{P}_p(A \cap B) \geqslant \mathbb{P}_p(A)\mathbb{P}_p(B),$$

for all increasing events A, B.

Notation 2.8. *Let $\Omega = \{0,1\}^E$ with E finite. For $\omega \in \Omega$ and $F \subseteq E$, define the* cylinder event

$$C(\omega, F) = \left\{\omega' \in \Omega,\ \omega_{|F} = \omega'_{|F}\right\}.$$

Moreover, denote $\omega_F = \omega_{|F} \times 0^{E \backslash F} \in \Omega$.

Definition 2.9 (Disjoint occurrence)**.** *Let $\Omega = \{0,1\}^E$ with E finite. Given $A, B \subseteq \Omega$, define:*

(i) $A \,\Box\, B = \{\omega \in \Omega,\ \exists F \subseteq E,\ C(\omega, F) \subseteq A \text{ and } C(\omega, E \backslash F) \subseteq B\}$,

(ii) $A \circ B = \left\{\omega \in \Omega,\ \exists F \subseteq E,\ \omega_F \in A \text{ and } \omega_{E\backslash F} \in B\right\}$.

Hence $A \circ B = A \,\Box\, B$ if A and B are increasing.

Theorem 2.10 (BK)**.** *Let $\Omega = \{0,1\}^E$ with E finite. If $A, B \subseteq \Omega$ are increasing events, then*

$$\mathbb{P}_p(A \circ B) \leqslant \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Proof. Write $E = \{e_1, \ldots, e_N\}$. Consider the duplicate sample space $\Omega \times \Omega'$, where $\Omega' = \{0,1\}^E = \Omega$; we equip $\Omega \times \Omega'$ with the product measure $\hat{\mathbb{P}} = \mathbb{P}_p \times \mathbb{P}_p$. For $(\omega, \omega') \in \Omega \times \Omega'$ and $1 \leqslant j \leqslant N+1$, define

$$\omega_j = (\omega'(e_1), \ldots, \omega'(j-1), \omega(j), \ldots, \omega(N)) \in \Omega.$$

Define in addition $\hat{A}_j = \{(\omega, \omega') \in \Omega \times \Omega',\ \omega_j \in A\} \subseteq \Omega \times \Omega'$ and $\hat{B} = B \times \Omega' \subseteq \Omega \times \Omega'$. Note that

- $\hat{A}_1 = A \times \Omega'$ and $\hat{B} = B \times \Omega'$, so $\hat{\mathbb{P}}\left(\hat{A}_1 \circ \hat{B}\right) = \mathbb{P}_p(A \circ B)$,
- $\hat{A}_{N+1} = \Omega \times A$ and $\hat{B} = B \times \Omega'$, so

$$\hat{\mathbb{P}}\left(\hat{A}_{N+1} \circ \hat{B}\right) = \hat{\mathbb{P}}\left(\bigcup_{F_1, F_2 \subseteq E} \left\{(\omega, \omega'),\ \omega'_{F_1} \in A \text{ and } \omega_{E\backslash F_2} \in B\right\}\right) = \hat{\mathbb{P}}(A \times B) = \mathbb{P}_p(A)\mathbb{P}_p(B).$$

It is therefore enough to prove that $\hat{\mathbb{P}}\left(\hat{A}_j \circ \hat{B}\right) \leqslant \hat{\mathbb{P}}\left(\hat{A}_{j+1} \circ \hat{B}\right)$ for all $1 \leqslant j \leqslant N$. To do this, fix $1 \leqslant j \leqslant N$ and condition on the event $\text{Ш} = \{(\omega, \omega'),\ \forall i \neq j,\ \omega(e_i) = \mu_i \text{ and } \omega'(e_i) = \nu_i\}$. Given $(\omega, \omega') \in \text{Ш}$, there are three cases:

(i) $\hat{A}_j \circ \hat{B}$ does not occur when $\omega(e_j) = \omega'(e_j) = 1$, so $\hat{\mathbb{P}}\left(\hat{A}_j \circ \hat{B} \mid \text{Ш}\right) = 0 \leqslant \hat{\mathbb{P}}\left(\hat{A}_{j+1} \circ \hat{B} \mid \text{Ш}\right)$.

(ii) $\hat{A}_j \circ \hat{B}$ occurs when $\omega(e_j) = \omega'(e_j) = 0$. In that case, so does $\hat{A}_{j+1} \circ \hat{B}$, which implies that $\hat{\mathbb{P}}\left(\hat{A}_j \circ \hat{B} \mid \text{Ш}\right) \leqslant \hat{\mathbb{P}}\left(\hat{A}_{j+1} \circ \hat{B} \mid \text{Ш}\right)$.

(iii) Neither of the two cases above hold. Since $\hat{A}_j \circ \hat{B}$ does not depend on the value of $\omega'(e_j)$ and since we assume that we are in none of the above cases, it follows that

$$\hat{\mathbb{P}}\left(\hat{A}_j \circ \hat{B} \mid \text{Ш}\right) = \hat{\mathbb{P}}(\omega(e_j) = 1 \mid \text{Ш}) = p.$$

Likewise, since $\hat{A}_{j+1} \circ \hat{B}$ must occur when $\omega'(e_j) = 1$, we have

$$\hat{\mathbb{P}}\left(\hat{A}_{j+1} \circ \hat{B} \mid \text{Ш}\right) \geqslant \hat{\mathbb{P}}(\omega'(e_j) = 1 \mid \text{Ш}) = p.$$

□

Theorem 2.11 (Reimer)**.** *Let $\Omega = \{0,1\}^E$ with E finite. If $A, B \subseteq \Omega$ are any events, then*

$$\mathbb{P}_p(A \,\Box\, B) \leqslant \mathbb{P}_p(A)\mathbb{P}_p(B).$$

2.3 Influence

Definition 2.12 (Influence)**.** *Let $\Omega = \{0,1\}^E$ with E finite. Given $A \subseteq \Omega$ and $e \in E$, the* (absolute) influence *of e is defined by*

$$I_A(e) = \mathbb{P}_p\left(\mathbb{1}_A(\omega_e) \neq \mathbb{1}_A(\omega^e)\right).$$

If A is an increasing event, then

$$I_A(e) = \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e),$$

where $A^e = \{\omega \in \Omega,\ \omega^e \in A\}$ and $A_e = \{\omega \in \Omega,\ \omega_e \in A\}$.

Theorem 2.13. *There exists an absolute constant $c \in (0,+\infty)$ s.t. for any finite set E, and for any $A \subseteq \Omega = \{0,1\}^E$, we have*

$$\sum_{e\in E} I_A(e) \geqslant c\mathbb{P}_{1/2}(A)\,\mathbb{P}_{1/2}\left(\overline{A}\right)\log\left(\frac{1}{\max_{e\in E} I_A(e)}\right).$$

Remark 2.14. *Let $\Omega = \{0,1\}^E$ with $|E| = N < +\infty$. If $m = \max_{e\in E} I_A(e)$, we have $Nm \geqslant \sum_{e\in E} I_A(e)$, and therefore Theorem 2.13 implies that*

$$-\frac{m}{\log m} \geqslant \frac{c}{N}\mathbb{P}_{1/2}(A)\mathbb{P}_{1/2}\left(\overline{A}\right).$$

From this we can deduce that

$$\max_{e\in E} I_A(e) \geqslant c\mathbb{P}_{1/2}(A)\mathbb{P}_{1/2}\left(\overline{A}\right)\frac{\log N}{N}.$$

Remark 2.15. *Theorem 2.13 remains valid if $\mathbb{P}_{1/2}$ is replaced by any product measure on any finite product (in particular by $\mathbb{P}_p$ on $\Omega = \{0,1\}^E$).*

Theorem 2.16 (Russo)**.** *Let $\Omega = \{0,1\}^E$ with E finite. For $A \subseteq \Omega$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \sum_{e\in E}\left(\mathbb{P}_p(A^e) - \mathbb{P}_p(A_e)\right).$$

Proof. Write $\mathbb{P}_p(A) = \sum_{\omega\in\Omega}\mathbb{1}_A(\omega)p^{|\eta(\omega)|}(1-p)^{N-|\eta(\omega)|}$ with $N = |E|$ and $\eta(\omega) = \{e \in E,\ \omega(e) = 1\}$. It follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) &= \frac{1}{p(1-p)}\sum_{\omega\in\Omega}\mathbb{1}_A(\omega)\left((1-p)\left|\eta(\omega)\right| - p\left(N - \left|\eta(\omega)\right|\right)\right)p^{|\eta(\omega)|}\left(1-p\right)^{N-|\eta(\omega)|}\\
&= \frac{1}{p(1-p)}\mathbb{E}_p\left[(1-p)\left|\eta\right|\mathbb{1}_A - p\left(N - |\eta|\right)\mathbb{1}_A\right]\\
&= \frac{1}{p(1-p)}\mathbb{E}_p\left[|\eta|\,\mathbb{1}_A - pN\mathbb{1}_A\right]\\
&= \frac{1}{p(1-p)}\sum_{e\in E}\mathbb{E}_p\left(\mathbb{1}_{\{e \text{ open}\}}\mathbb{1}_A - p\mathbb{1}_A\right).
\end{aligned}$$

But note that

$$\mathbb{E}_p\left(\mathbb{1}_{\{e \text{ open}\}}\mathbb{1}_A\right) = \mathbb{P}_p\left(A \mid e \text{ open}\right)\mathbb{P}_p\left(e \text{ open}\right) = p\mathbb{P}_p(A^e),$$

and

$$\mathbb{E}_p\left(p\mathbb{1}_A\right) = p\left(p\mathbb{P}_p(A^e) + (1-p)\mathbb{P}_p(A_e)\right).$$

Therefore $\mathbb{E}_p\left(\mathbb{1}_{\{e \text{ open}\}}\mathbb{1}_A - p\mathbb{1}_A\right) = p(1-p)\left(\mathbb{P}_p(A^e) - \mathbb{P}_p(A_e)\right)$, from which the result follows. □

Corollary 2.17. *Let $\Omega = \{0,1\}^E$ with E finite. If $A \subseteq \Omega$ is an increasing event, then*

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) \geqslant c\mathbb{P}_p(A)\mathbb{P}_p\left(\overline{A}\right)\log\left(\frac{1}{\max_{e\in E} I_A(e)}\right).$$

It follows that if $I_A(e)$ does not depend on e, then $\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) \geqslant c\mathbb{P}_p(A)\mathbb{P}_p\left(\overline{A}\right)\log N$, with $N = |E|$. This means that the function $p \mapsto \mathbb{P}_p(A)$ has a sharp threshold*: it stays close to 0, then increases very quickly and stays close to 1 (at least for large values of N).*

3 Further percolation

Notation 3.1. *We return to bond percolation on $\mathbb{Z}^d$ with $d \geqslant 2$.*

Remark 3.2. *Let Ӝ be the event that there exists an infinite open cluster. Note that the Kolmogorov Zero-One Law implies that $\mathbb{P}_p(\text{Ӝ}) \in \{0,1\}$ for all p. Moreover*

$$\theta(p) = \mathbb{P}_p(|C_0| = +\infty) \leqslant \mathbb{P}_p(\text{Ӝ}) = \mathbb{P}_p\left(\bigcup_{x\in\mathbb{Z}^d} (|C_x| = +\infty)\right) \leqslant \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(|C_x| = +\infty) = \sum_{x\in\mathbb{Z}^d} \theta(p).$$

It follows that:

- *In the* subcritical phase *($0 \leqslant p < p_c$), $\theta(p) = 0$ and almost surely there is no infinite open cluster.*
- *In the* supercritical phase *($p_c < p \leqslant 1$), $\theta(p) > 0$ and almost surely there exists an infinite open cluster.*

3.1 Subcritical phase

Notation 3.3. *For $n \in \mathbb{N}$, we shall write $\Lambda(n) = [-n,+n]^d \subseteq \mathbb{Z}^d$ and $\partial\Lambda(n) = \Lambda(n)\backslash\Lambda(n-1)$. Thus*

$$\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) = \lim_{n\to+\infty} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n)).$$

Theorem 3.4. (i) *For $0 \leqslant p < p_c$, there exists $\psi(p) > 0$ s.t.*

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n)) \leqslant e^{-n\psi(p)}.$$

(ii) *For $p_c < p \leqslant 1$, we have*

$$\theta(p) \geqslant \frac{p - p_c}{p(1-p_c)}.$$

Proof. Given $0 \in S \subseteq \mathbb{Z}^d$, $|S| < +\infty$, we define the *external edge boundary* of S by

$$\Delta S = \{e = \langle x,y\rangle,\ x \in S,\ y \notin S\}.$$

For $e \in \Delta S$, we shall always write $e = \langle x,y\rangle$ with $x \in S$. For $y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, define

$$E_n(y) = (y \leftrightarrow \partial\Lambda(n)) \subseteq \Omega,$$

and $E_n = E_n(0)$, $g_p(n) = \mathbb{P}_p(E_n)$. Also set

$$\varphi_p(S) = p \sum_{\langle x,y\rangle\in\Delta S} \mathbb{P}_p(0 \leftrightarrow x \text{ in } S).$$

Now, choose $L \in \mathbb{N}$ in such a way that $S \subseteq \Lambda(L)$. Then for every k, we have, using the BK Inequality (Theorem 2.10),

$$\begin{aligned} g_p(kL) &\leqslant \sum_{e=\langle x,y\rangle\in\Delta S} \mathbb{P}_p((0 \leftrightarrow x \text{ in } S) \circ (e \text{ open}) \circ (y \leftrightarrow \partial\Lambda(kL))) \\ &\overset{\text{(BK)}}{\leqslant} \sum_{e=\langle x,y\rangle\in\Delta S} p\mathbb{P}_p(0 \leftrightarrow x \text{ in } S)\underbrace{\mathbb{P}_p(E_{kL}(y))}_{\leqslant g_p((k-1)L)} \\ &\leqslant \varphi_p(S) g_p((k-1)L). \end{aligned}$$

By induction, it follows that $g_p(kL) \leqslant \varphi_p(S)^k$. Let

$$\widetilde{p}_c = \sup\{p \in [0,1],\ \text{there exists a finite set } S \ni 0 \text{ with } \varphi_p(S) < 1\}.$$

If $p < \widetilde{p}_c$, pick S with $\varphi_p(S) < 1$. We have $g_p(n) \leqslant \varphi_p(S)^{\lfloor n/L \rfloor}$, and since $g_p(n) < 1$ for $n \geqslant 1$ and $p < 1$, we have $g_p(n) \leqslant e^{-n\psi(p)}$ for some $\psi(p) > 0$.

Proving that $p_c = \widetilde{p}_c$ will imply (i). We shall actually prove that for $p > \widetilde{p}_c$, $\theta(p) \geqslant \frac{p-p_c}{p(1-p_c)}$. This will imply that $\theta(p) > 0$ for $p > \widetilde{p}_c$, and therefore $\widetilde{p}_c \geqslant p_c$. But we also know that if $p > p_c$, then there cannot exist a set S as in the definition of $\widetilde{p}_c$ (otherwise we would have $\theta(p) = 0$), so that $p_c \geqslant \widetilde{p}_c$ and therefore $\widetilde{p}_c = p_c$, which will prove both (i) and (ii).

So it suffices to prove that for $p > \widetilde{p}_c$, $\theta(p) \geqslant \frac{p-p_c}{p(1-p_c)}$. We define a random variable $\underline{S} = \{x \in \Lambda(n),\ x \not\leftrightarrow \partial\Lambda(n)\}$. We shall now estimate $g_p(n) = \mathbb{P}_p(0 \leftrightarrow \partial\Lambda(n))$ using Russo's Formula (Theorem 2.16):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p} g_p(n) &= \sum_{e \in E} \mathbb{P}_p\left(e \text{ is pivotal for } \{0 \leftrightarrow \partial\Lambda(n)\}\right) \\
&= \frac{1}{1-p} \sum_{e \in E} \mathbb{P}_p\left(e \text{ is pivotal for } \{0 \leftrightarrow \partial\Lambda(n)\},\ e \text{ is closed}\right) \\
&= \frac{1}{1-p} \sum_{e \in E} \sum_{\substack{S \ni 0 \\ \Delta S \ni e}} \mathbb{P}_p\left(e \text{ is pivotal for } \{0 \leftrightarrow \partial\Lambda(n)\},\ e \text{ is closed},\ \overline{S} = S\right) \\
&= \frac{1}{1-p} \sum_{S \ni 0} \sum_{e = \langle x,y \rangle \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\leftrightarrow} x,\ \overline{S} = S\right) \\
&= \frac{1}{1-p} \sum_{S \ni 0} \sum_{e = \langle x,y \rangle \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\leftrightarrow} x\right) \mathbb{P}_p\left(\overline{S} = S\right) \\
&= \frac{1}{p(1-p)} \sum_{S \ni 0} \mathbb{P}_p\left(\overline{S} = S\right) \sum_{e = \langle x,y \rangle \in \Delta S} p\mathbb{P}_p\left(0 \overset{S}{\leftrightarrow} x\right) \\
&= \frac{1}{p(1-p)} \sum_{S \ni 0} \mathbb{P}_p\left(\overline{S} = S\right) \varphi_p(S).
\end{aligned}$$

Now if $p > \widetilde{p}_c$, then $\varphi_p(S) \geqslant 1$ for all S, so that

$$\frac{\mathrm{d}}{\mathrm{d}p} g_p(n) \geqslant \frac{1}{p(1-p)} \sum_{S \ni 0} \mathbb{P}_p\left(\overline{S} = S\right) = \frac{1 - g_p(n)}{p(1-p)}.$$

Integrating this differential inequality yields

$$\log\left(\frac{1 - g_{\widetilde{p}_c}(n)}{1 - g_p(n)}\right) \geqslant \log\left(\frac{p}{1-p} \cdot \frac{1 - \widetilde{p}_c}{\widetilde{p}_c}\right),$$

from which it follows that $\frac{1}{1-g_p(n)} \geqslant \frac{1-g_{\widetilde{p}_c}(n)}{1-g_p(n)} \geqslant \frac{p}{1-p} \cdot \frac{1-\widetilde{p}_c}{\widetilde{p}_c}$ and therefore, for $p > \widetilde{p}_c$,

$$g_p(n) \geqslant \frac{p - \widetilde{p}_c}{p(1 - \widetilde{p}_c)}.$$

Making $n \to +\infty$ gives the claimed inequality. □

3.2 Supercritical phase

Remark 3.5. *We have seen (in Remark 3.2) that in the supercritical phase there is almost surely an infinite open cluster. The next question is: how many infinite open clusters are there?*

Lemma 3.6. *If A is a translation-invariant event, then $\mathbb{P}(A) \in \{0,1\}$, where $\mathbb{P}$ is any product measure on Ω.*

Proof. For $\varepsilon > 0$, a measure-theoretic argument shows that there is a finite set $S \subseteq \mathbb{Z}^d$ and an event A_S defined on S only such that $\mathbb{P}(A \triangle A_S) < \varepsilon$. Now choose a translation τ such that $\tau S \cap S = \varnothing$. Then A_S is independent of τA_S. But A_S approximates A and τA_S approximates $\tau A = A$, so we can deduce that $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2$, and therefore $\mathbb{P}(A) \in \{0,1\}$. □

Theorem 3.7. *Let N be the number of infinite open clusters. Then for all $p \in [0,1]$, we have either $\mathbb{P}_p(N=0) = 1$ or $\mathbb{P}_p(N=1) = 1$.*

Proof. If $\theta(p) = 0$, then $\mathbb{P}_p(N=0) = 1$, so we henceforth assume that $\theta(p) > 0$ and we wish to prove that $\mathbb{P}_p(N=1) = 1$.

First step: there exists $k_p \in \mathbb{N} \cup \{\infty\}$ s.t. $\mathbb{P}_p(N = k_p) = 1$. Note that N is invariant w.r.t. translations of the configuration. Moreover, $\mathbb{P}_p$ is a product measure; it follows from Lemma 3.6 that $\mathbb{P}_p(N \geqslant n) \in \{0,1\}$ for all n, so it suffices to set $k_p = \sup\{n \in \mathbb{N},\ \mathbb{P}_p(N \geqslant n) = 1\}$.

Second step: $k_p \notin \mathbb{N}_{\geqslant 2}$. Suppose for contradiction that $2 \leqslant k_p < +\infty$. For $n \in \mathbb{N}$, let C_n be the event that Λ_n intersects at least two distinct infinite open clusters. Since $\lim_{n\to+\infty} \mathbb{P}_p(C_n) = 1$, there exists an n such that $\mathbb{P}_p(C_n) \geqslant \frac{1}{2}$. By making all the edges inside Λ_n open, we have

$$\mathbb{P}_p(N \leqslant k_p - 1) \geqslant \frac{1}{2} p^{|E(\Lambda_n)|} > 0,$$

a contradiction.

Third step: $k_p \neq \infty$. Suppose for contradiction that $3 \leqslant k_p \leqslant \infty$. Consider the box $L_n = \left\{x \in \mathbb{Z}^d,\ \|x\|_1 \leqslant n\right\}$. As before, there exists an n such that the probability that L_n intersects at least three distinct infinite open clusters is at least $\frac{1}{2}$. We now say that a point $x \in \mathbb{Z}^d$ is a *trifurcation* if $x \leftrightarrow \infty$ and if the removal of x and its adjacent edges breaks C_x into three distinct infinite open clusters and no finite cluster. Let T_x be the event that x is a trifurcation. Pick points $x, y, z \in \partial L_n$ such that x, y, z lie in distinct infinite open clusters off L_n. Given x, y, z, there exists a configuration $\omega_{x,y,z}$ inside L_n such that 0 is a trifurcation when $\omega_{x,y,z}$ occurs. Therefore

$$\mathbb{P}_p(T_0) \geqslant \frac{1}{2} \left(\min\{p, 1-p\}\right)^{|E(L_n)|} > 0.$$

Now, in a situation where 0 is a trifurcation, we can produce a graph of trifurcations; this graph is a forest of degree 3. A graph-theoretic argument then shows that there exists an $\alpha > 0$ such that

$$\frac{\sharp\text{trifurcations in } \partial L_n}{\sharp\text{trifurcations in } L_n} \geqslant \alpha > 0.$$

Thus

$$|S_n| \geqslant \mathbb{E}\left(\sharp\text{trifurcations in } \partial L_n\right) \geqslant \alpha \mathbb{E}\left(\sharp\text{trifurcations in } L_n\right) = \alpha |L_n| \mathbb{P}_p(T_0).$$

We deduce the existence of a constant $C > 0$ such that $n^{d-1} \geqslant C n^d$, which gives a contradiction for large values of n. □

Corollary 3.8. *If $p > p_c$, then for all vertices x, y,*

$$\mathbb{P}_p(x \leftrightarrow y) \geqslant \theta(p)^2 > 0.$$

Proof. By Theorem 3.7 and the FKG inequality (Theorem 2.6), we have

$$\mathbb{P}_p(x \leftrightarrow y) \geqslant \mathbb{P}_p(x \leftrightarrow y,\ x \leftrightarrow \infty,\ y \leftrightarrow \infty) = \mathbb{P}_p(x \leftrightarrow \infty,\ y \leftrightarrow \infty) \overset{\text{(FKG)}}{\geqslant} \theta(p)^2 > 0.$$ □

Theorem 3.9 (Slab Critical Point Theorem)**.** *When $d \geqslant 3$, define a* slab of thickness $k+1$ *by*

$$S_k = \{0, 1, \dots, k\}^{d-2} \times \mathbb{Z}^2 \subseteq \mathbb{Z}^d.$$

We have $p_c(S_k) \geqslant p_c$, so $p_c(S_k) \xrightarrow[k\to+\infty]{} \hat{p}_c \geqslant p_c$.

In fact, $\hat{p}_c = p_c$.

3.3 Exact critical probabilities

Lemma 3.10. *For bond percolation on $\mathbb{Z}^2$, $\theta\left(\frac{1}{2}\right)=0$.*

Proof. We assume for contradiction that $\theta\left(\frac{1}{2}\right)>0$. By Theorem 3.7, there is $\mathbb{P}_{1/2}$-almost surely a unique infinite open cluster. We denote by $T(n)$ the box $[0,n]^2$, with edges labelled ℓ (left), r (right), b (bottom) and t (top). Choose n_0 large enough so that, for $n\geqslant n_0$,

$$\mathbb{P}_{1/2}\left(\partial T(n)\leftrightarrow\infty\right)\geqslant 1-\left(\frac{1}{8}\right)^4.$$

Let $n=n_0+1$. Let A^g be the event that the edge labelled g is joined to ∞ off $T(n)$. We have, using the FKG inequality (Theorem 2.6),

$$\left(\frac{1}{8}\right)^4\geqslant\mathbb{P}_{1/2}\left(\partial T(n)\not\leftrightarrow\infty\right)=\mathbb{P}_{1/2}\left(\overline{A}^\ell\cap\overline{A}^r\cap\overline{A}^b\cap\overline{A}^t\right)\overset{\text{(FKG)}}{\geqslant}\mathbb{P}_{1/2}\left(\overline{A}^g\right)^4.$$

It follows that $\mathbb{P}_{1/2}\left(A^g\right)\geqslant\frac{7}{8}$ for all g. Now consider the dual box $T(n)_\vee\simeq[0,n-1]^2$ with $n-1\geqslant n_0$, and let $A^g_\vee$ be the event that the edge labelled g is joined to ∞ by a dual open path off $T(n)_\vee$. As before, we have $\mathbb{P}_{1/2}\left(A^g_\vee\right)\geqslant\frac{7}{8}$. Therefore

$$1-\mathbb{P}_{1/2}\left(A^\ell\cap A^r\cap A^b_\vee\cap A^t_\vee\right)\leqslant 4\cdot\frac{1}{8}=\frac{1}{2}.$$

But the event $A^\ell\cap A^r\cap A^b_\vee\cap A^t_\vee$ has probability zero because it contradicts the uniqueness of infinite open clusters in both the primal and the dual lattice. This is a contradiction. □

Theorem 3.11. *For bond percolation on $\mathbb{Z}^2$, $p_c=\frac{1}{2}$.*

Proof. ($\geqslant$) Follows from Lemma 3.10. ($\leqslant$) Assume for contradiction that $p_c>\frac{1}{2}$. Consider the box $B_n=[0,n+1]\times[0,n]\subseteq\mathbb{Z}^2$ and let A_n be the event that B_n has a left-to-right open crossing (i.e. an open path connecting the left boundary of B_n to its right boundary). Consider the dual box $B_n^\vee$ of B_n. We take the convention that an open edge in $\mathbb{Z}^2$ is always crossed by a dual closed edge, and *vice versa*. Let $A_n^\vee$ be the event that $B_n^\vee$ has a bottom-to-top open crossing. Note that exactly one of A_n and $A_n^\vee$ must occur; moreover, $B_n^\vee$ has the same geometry as B_n, so $\mathbb{P}_{1/2}\left(A_n\right)=\mathbb{P}_{1/2}\left(A_n^\vee\right)$. It follows that $\mathbb{P}_{1/2}\left(A_n\right)=\frac{1}{2}$. But if $p_c>\frac{1}{2}$, then $\frac{1}{2}$ is subcritical, so by Theorem 3.4 $\mathbb{P}_{1/2}\left(A_n\right)\leqslant(n+1)\,e^{-\gamma n}$ for some $\gamma>0$, which gives a contradiction for large n. □

3.4 RSW theory

Notation 3.12. *Let $\mathbb{T}$ be the triangular lattice, which we embed in the plane by*

$$\mathbb{T}=\left\{m\mathbf{i}+n\mathbf{j},\ (m,n)\in\mathbb{Z}^2\right\},$$

where $\mathbf{i}=(1,0)$ and $\mathbf{j}=\frac{1}{2}\left(1,\sqrt{3}\right)$.

In this section, we shall study site percolation on $\mathbb{T}$, i.e. each vertex is coloured black with probability p, white otherwise.

We also introduce the following notations:

- *$R_{a,b}$ is the subgraph of $\mathbb{T}$ induced by vertices in $[0,a]\times[0,b]$, $L\left(R_{a,b}\right)$ (resp. $R\left(R_{a,b}\right)$) is the set of vertices of $\mathbb{T}$ at distance at most $\frac{1}{2}$ from the left (resp. right) edge of $[0,a]\times[0,b]$.*
- *$H_{a,b}$ is the event that $L\left(R_{a,b}\right)$ is connected to $R\left(R_{a,b}\right)$ by a black path in $R_{a,b}$.*

We fix $p=\frac{1}{2}$ and $\mathbb{P}=\mathbb{P}_{1/2}$.

Lemma 3.13. $\mathbb{P}\left(H_{2a,b}\right)\geqslant\frac{1}{4}\mathbb{P}\left(H_{a,b}\right)$.

Proof. Consider the box $[0,a]\times[0,b]$ and the reflection ρ whose axis is the vertical line at a. Given a path g from the left to the right edge of $[0,a]\times[0,b]$, we define U_g to be the part of $[0,a]\times[0,b]$ that lies under g and let

$$J_g = U_g \cap \partial\left([0,a]\times[0,b]\right).$$

We denote by B_g (resp. $W_{\rho g}$) the event that g (resp. ρg) is connected to ρJ_g (resp. J_g) by a path of $U_g \cap \rho U_g$ that intersects $g \cup \rho g$ only once and every vertex (except possibly the endvertex on g) is black (resp. white). We observe that $B_g \cup W_{\rho g}$ must occur (by a duality argument). But by symmetry, $\mathbb{P}(B_g) = \mathbb{P}(W_g)$, which implies that

$$\mathbb{P}(B_g) = \mathbb{P}(W_g) \geqslant \frac{1}{2}.$$

Moreover, if L (resp. R) is the left (resp. right) edge of the box $[0,2a]\times[0,b]$, and J is the union of the left and bottom edges of the box $[0,a]\times[0,b]$, then

$$\mathbb{P}(H_{2a,b}) \geqslant \mathbb{P}(L \leftrightarrow \rho J,\ R \leftrightarrow J) \overset{\text{(FKG)}}{\geqslant} \mathbb{P}(L \leftrightarrow \rho J)^2.$$

Now let γ be the random variable denoting the highest left-right black crossing in he rectangle $R_{a,b}$. We have

$$\mathbb{P}(L \leftrightarrow \rho J) \geqslant \sum_g \mathbb{P}(\gamma = g, B_g) = \sum_g \mathbb{P}(\gamma = g)\,\mathbb{P}(B_g) \geqslant \frac{1}{2}\sum_g \mathbb{P}(\gamma = g) = \frac{1}{2}\mathbb{P}(H_{a,b}).$$

It follows that $\mathbb{P}(H_{2a,b}) \geqslant \mathbb{P}(L \leftrightarrow \rho J)^2 \geqslant \frac{1}{4}\mathbb{P}(H_{a,b})$. ☐

Corollary 3.14. $\mathbb{P}\left(H_{2^k a,b}\right) \geqslant \left(\frac{1}{4}\right)^{2^k-1}\mathbb{P}(H_{a,b})$.

Lemma 3.15. $\mathbb{P}\left(H_{a,a/\sqrt{3}}\right) \geqslant \frac{1}{2}$ *for* $a \geqslant 1$.

Proof. Use a self-duality argument to show that there exists a left-right crossing in the rhombus of dimensions $\left(a, \frac{a}{\sqrt{3}}\right)$ with probability $\frac{1}{2}$. ☐

3.5 Cardy's formula

Theorem 3.16 (Cardy's formula)**.** *Consider a Jordan curve bounding a domain D in the plane with four points b,a,c,x on the boundary. Assume the plane is covered by a triangular lattice with mesh δ. By Riemann's Theorem, there exists a conformal map from $\mathring{D}$ to an equilateral triangle such that a,b,c are sent to vertices A,B,C of that triangle. Let X be the image of x under that map (X lies on the boundary of the triangle). Then*

$$\mathbb{P}(ac \leftrightarrow bx \text{ in } D) \xrightarrow[\delta\to 0]{} |BX|.$$

Sketch of proof. We set $\delta = \frac{1}{n}$ and we shall make $n \to +\infty$. Let $\tau = e^{2i\pi/3}$, let $A_1 = A = 0$, $A_\tau = B = 1$, $A_{\tau^2} = C = e^{i\pi/3}$. For $z \in T$ (T is the triangle ABC), let $E_i^n(z)$ be the event that there exists a black path from $A_{\tau^{i-1}}A_{\tau^{i+1}}$ to $A_{\tau^{i-1}}A_{\tau^i}$ separating z from $A_{\tau^i}A_{\tau^{i+1}}$. Let $H_i^n(z) = \mathbb{P}(E_i^n(z))$, extended to T by interpolation. Then there exist C,α such that

$$\left|H_j^n(z) - H_j^n(z')\right| \leqslant C\,|z-z'|^\alpha,$$

for all z,z',j,n. By the Arzelà-Ascoli Theorem, any sequence of functions in $\left(H_j^n\right)_{n\in\mathbb{N}}$ has a convergent subsequence (for uniform convergence). Now we want to show that there is only one possible limit of convergent subsequence, and this will imply convergence. We define

$$G_1 = H_1 + H_2 + H_3,$$
$$G_2 = H_1 + \tau H_2 + \tau^2 H_3.$$

Then a theorem says that G_1, G_2 are analytic functions of z. Since G_1 is real-valued, it follows that it is constant. And $\Re(G_2) = \frac{1}{2}(3H_1 - 1)$, so H_1 is harmonic and may be derived explicitly.

The rest of the proof uses the so-called *exploration process.* ☐

4 The Ising, Potts and random cluster models

4.1 The models

Definition 4.1 (Ising model)**.** *Let $G = (V, E)$ be a finite connected graph. Define $\Sigma = \{\pm 1\}^V$; a* spin vector *is an element $\sigma = (\sigma_x)_{x\in V} \in \Sigma$. The* hamiltonian *of a spin vector is defined by*

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in E} \sigma_x \sigma_y - h \sum_{x\in V} \sigma_x,$$

where $J, h \in \mathbb{R}$ are parameters. The (Lenz) Ising model *is the probability measure $\lambda = \lambda_\beta$ on Σ defined by*

$$\lambda(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)},$$

where $\beta \geqslant 0$ is a parameter (corresponding to the inverse temperature) and $Z = \sum_{\sigma\in\Sigma} e^{-\beta H(\sigma)}$ is the partition function.

We normally take $h = 0$. The case $J > 0$ is called the ferromagnet *while the case $J < 0$ is called the* antiferromagnet*. In this course, we will take $J > 0$ (i.e. adjacent vertices tend to be in the same state) and even $J = 1$ for simplicity. Therefore*

$$\lambda(\sigma) \propto \exp\left(\beta \sum_{\langle x,y\rangle\in E} \sigma_x\sigma_y\right).$$

Definition 4.2 (Potts model)**.** *The* Potts model *is the generalisation of the Ising model obtained by replacing $\{\pm 1\}$ by $\{1, 2, \ldots, q\}$. Thus the state space is $\Sigma = \{1, 2, \ldots, q\}^V$ and the probability measure satisfies*

$$\pi(\sigma) \propto \exp\left(\beta \sum_{\langle x,y\rangle\in E} \mathbb{1}\left(\sigma_x = \sigma_y\right)\right).$$

Note that, when $q = 2$, $\pi_\beta = \lambda_{\beta/2}$.

Definition 4.3 (Random cluster model)**.** *Consider as before a finite graph $G = (V, E)$ and let $\Omega = \{0, 1\}^E$. Let $p \in [0, 1]$, $q > 0$. The* random cluster model *is the probability measure $\varphi_{p,q}$ on Ω defined by*

$$\varphi_{p,q}(\omega) \propto q^{k(\omega)} \prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)},$$

where $k(\omega)$ is the number of open components (including isolated vertices) of the configuration ω (again, the edge e is called open *if $\omega(e) = 1$, closed otherwise).*

For $q = 1$, the random cluster model is simply bond percolation on G.

4.2 Link with percolation

Notation 4.4. *We are going to construct a coupling of the Potts model and the random cluster model on a finite connected graph $G = (V, E)$ when $q \in \mathbb{N}_{\geqslant 2}$. We define a probability measure μ on $\Sigma\times\Omega$ by*

$$\mu(\sigma, \omega) \propto \mathbb{P}_p(\omega)\mathbb{1}_F(\sigma, \omega),$$

where $\mathbb{P}_p$ is the probability measure on Ω for standard edge percolation, and

$$F = \left\{(\sigma, \omega) \in \Sigma\times\Omega,\ \forall e = \langle x, y\rangle \in E,\ \omega(e) = 1 \Rightarrow \sigma_x = \sigma_y\right\}.$$

In other words, we are adding to bond percolation the constraint that whenever an edge is open, its endpoints have the same state.

Proposition 4.5. *Properties of the measure μ on $\Sigma\times\Omega$:*

(i) *The marginal on* Σ *is the Potts model with parameter* $\beta = -\log(1-p)$.

(ii) *The marginal on* Ω *is the random cluster model.*

(iii) *The conditional law given* ω *is the model where each cluster receives a uniform spin independently.*

(iv) *The conditional law given* σ *is the model where, for* $e = \langle x, y \rangle$, *if* $\sigma_x \neq \sigma_y$ *then* $\omega(e) = 0$, *otherwise* $\omega(e) = 1$ *with probability* p, *independently of other edges.*

Definition 4.6 (Correlation and connection functions). (i) *The* correlation function *of the Potts model is given by*

$$\tau(x,y) = \pi(\sigma_x = \sigma_y) - \frac{1}{q}.$$

(ii) *The* connection function *of the random cluster model is given by*

$$\varphi(x \leftrightarrow y).$$

Theorem 4.7. *Assume that* $q \in \mathbb{N}_{\geqslant 2}$, *let* $\beta \geqslant 0$ *and* $p = 1 - e^{-\beta}$. *Then*

$$\tau_{\beta,q}(x,y) = \left(1 - \frac{1}{q}\right) \varphi_{p,q}(x \leftrightarrow y).$$

This gives a strong link between correlation in the Potts model and connection in the random cluster model.

Proof. We have

$$\begin{aligned}
\tau(x,y) &= \pi(\sigma_x = \sigma_y) - \frac{1}{q} \\
&= \sum_{\omega \in \Omega} \mu(\sigma, \omega) \left(\mathbb{1}(\sigma_x = \sigma_y) - \frac{1}{q} \right) \\
&= \sum_{\omega \in \Omega} \varphi(\omega) \sum_{\sigma \in \Sigma} \mu(\sigma \mid \omega) \left(\mathbb{1}(\sigma_x = \sigma_y) - \frac{1}{q} \right) \\
&= \sum_{\omega \in \Omega} \varphi(\omega) \left(\mathbb{1}\left(x \overset{\omega}{\leftrightarrow} y\right) \left(1 - \frac{1}{q}\right) + \mathbb{1}\left(x \overset{\omega}{\not\leftrightarrow} y\right) \cdot 0 \right) \\
&= \left(1 - \frac{1}{q}\right) \varphi(x \leftrightarrow y).
\end{aligned}$$

□

Proposition 4.8. *The random cluster model* $\varphi_{p,q}$ *has the following properties:*

(i) FKG inequality. *If* $q \geqslant 1$, *then* $\varphi_{p,q}$ *is positively associated.*

(ii) Comparison inequalities.

(a) *If* $q' \geqslant \max\{q, 1\}$ *and* $p' \leqslant p$, *then* $\varphi_{p',q'} \leqslant_{st} \varphi_{p,q}$.

(b) *If* $q' \geqslant \max\{q, 1\}$ *and* $\frac{p'}{q'(1-p')} \geqslant \frac{p}{q(1-p)}$, *then* $\varphi_{p',q'} \geqslant_{st} \varphi_{p,q}$.

Proof. (i) Use the FKG inequality (Theorem 2.6). (ii) Use the Holley inequality (Theorem 2.5). □

4.3 Negative association

Definition 4.9 (Edge-negative association)**.** *A probability measure φ on $\{0,1\}^E$ is said to be* edge-negatively associated *if for all edges e, f, we have*

$$\varphi\left(\omega(e)=1, \omega(f)=1\right) \leqslant \varphi\left(\omega(e)=1\right)\varphi\left(\omega(f)=1\right).$$

Remark 4.10. *Proposition 4.8 leads to the following question: is $\varphi_{p,q}$ edge-negatively associated for $q<1$?*

Theorem 4.11. *Let G be a finite connected graph. Then the measure $\varphi_{p,q}$ converges weakly to*

- *The* uniform connected subgraph *measure $\mathcal{UCS}$ if $p=\frac{1}{2}$ and $q \to 0$,*
- *The* uniform spanning tree *measure $\mathcal{UST}$ if $p, q, \frac{q}{p} \to 0$,*
- *The* uniform forest *measure $\mathcal{UF}$ if $p=q \to 0$.*

Proof. We prove the result for the uniform forest. We write $\eta(\omega)=\{e \in E,\ \omega(e)=1\}$ and we assume that $p=q$. Then

$$\varphi_{p,q}(\omega) \propto p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|}q^{k(\omega)} \propto \frac{p^{|\eta(\omega)|+k(\omega)}}{(1-p)^{|\eta(\omega)|}}.$$

Note that $|\eta(\omega)|+k(\omega) \geqslant |V|$ with equality iff there are no cycles. the result follows. □

Theorem 4.12. *$\mathcal{UST}$ is edge-negatively associated.*

Conjecture 4.13. *$\mathcal{UCS}$ and $\mathcal{UF}$ are edge-negatively associated.*

4.4 Infinite volume limits for the random cluster model

Remark 4.14. *The random cluster model is well-defined for finite graphs, but we want to extend the definition to infinite graphs, for instance $\mathbb{Z}^d$.*

Notation 4.15. *We work on $\mathbb{Z}^d$, with $d \geqslant 2$. Given a bounded region $\Lambda \subseteq \mathbb{Z}^d$, we have a random cluster measure $\varphi_{\Lambda,p,q}$ on Λ. We add a* boundary condition*: either $b=0$ and all edges outside Λ are closed, or $b=1$ and all edges outside Λ are open. We now define the measure $\varphi^b_{\Lambda,p,q}$ in the same manner as $\varphi_{\Lambda,p,q}$, but by taking into account connectivity through the boundary when counting open clusters.*

Theorem 4.16. *For $q \geqslant 1$ and $b \in \{0,1\}$, the measures $\left(\varphi^b_{\Lambda,p,q}\right)_{\Lambda\subseteq\mathbb{Z}^d}$ converge weakly to a measure $\varphi^b_{p,q}$ as $\Lambda \to \mathbb{Z}^d$.*

The measure $\varphi^b_{p,q}$ is called the infinite volume measure*.*

Proof. We assume that $b=1$ (the proof is similar if $b=0$). To prove weak convergence, it suffices to prove that $\left(\varphi^1_{\Lambda,p,q}(A)\right)_{\Lambda\subseteq\mathbb{Z}^d}$ converges for all increasing cylinder events A. But, if $\Lambda \subseteq \Lambda' \subseteq \mathbb{Z}^d$, then we have, using Proposition 4.8,

$$\varphi^1_{\Lambda,p,q}(A) = \varphi^1_{\Lambda',p,q}\left(A \mid \text{every edge of } \Lambda'\setminus\Lambda \text{ is open}\right) \overset{\text{(FKG)}}{\geqslant} \varphi^1_{\Lambda',p,q}(A).$$

Therefore the limit exists by monotonicity. □

Remark 4.17. *An infinite volume measure can also be defined using the so-called* DLR method*.*

Definition 4.18 (Percolation probability for the random cluster model)**.** *Given $b \in \{0,1\}$, $q \geqslant 1$ and $p \in [0,1]$, we define*

$$\theta^b(p,q) = \varphi^b_{p,q}\left(0 \leftrightarrow \infty\right).$$

By Proposition 4.8, $\theta^b(p,q)$ is nondecreasing in p, and we define

$$p^b_c(q) = \sup\left\{p \in [0,1],\ \theta^b(p,q)=0\right\}.$$

Theorem 4.19. *There exists a countable subset $\mathcal{D}_q \subseteq [0,1]$ such that*

$$\forall p \in [0,1] \backslash \mathcal{D}_q,\ \varphi^0_{p,q} = \varphi^1_{p,q}.$$

Corollary 4.20. $p^1_c(q) = p^0_c(q)$.

Proof. Assume for contradiction that $p^1_c(q) \neq p^0_c(q)$ with, say, $p^1_c(q) < p^0_c(q)$. Then, in the open interval $(p^1_c(q), p^0_c(q))$, we would have $\theta^1(q) > 0 = \theta^0(q)$, and therefore $\varphi^1_{p,q} \neq \varphi^0_{p,q}$, contradicting Theorem 4.19. □

Definition 4.21 (Order parameter for the Potts model)**.** *For the Potts model with q states, we define the* order parameter *by*

$$\mathcal{M}(\beta, q) = \lim_{\Lambda \to \mathbb{Z}^d} \left(\pi^1_{\Lambda,q}(\sigma_0 = 1) - \frac{1}{q} \right) = \left(1 - \frac{1}{q} \right) \theta^1(p,q),$$

where $\pi^1_{\Lambda,q}$ is the probability measure conditioned by the event that all vertices off Λ have state 1.

There is a critical parameter $\beta_c = -\log(1 - p_c(q))$.

Theorem 4.22. *For $q \geqslant 1$, $0 < p_c(q) < 1$.*

Proof. The comparison inequalities (Proposition 4.8) imply that

$$\varphi^1_{p',1} \leqslant_{st} \varphi^1_{p,q} \leqslant_{st} \varphi_{p,1},$$

where $p' = \frac{p}{p+q(1-p)}$. It follows that $0 < p_c(1) \leqslant p_c(q) \leqslant \frac{qp_c(1)}{1+(q-1)p_c(1)} < 1$, using the fact that $0 < p_c(1) < 1$ by Theorem 1.7. □

Theorem 4.23. *When $d = 2$ and $q \geqslant 1$,*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Proof. Define a dual random cluster measure on the square lattice, with dual parameter $p^\vee$ satisfying $\frac{p^\vee}{1-p^\vee} = q\frac{1-p}{p}$, and show that this mapping $p \mapsto p^\vee$ has the unique value $p = \frac{\sqrt{q}}{1+\sqrt{q}}$ as a fixed point. □

References

[1] G.R. Grimmett. *Probability on Graphs.*