Rigidity Theorems for Hyperbolic Groups

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Abstract

The purpose of this essay is to study actions of hyperbolic groups on real trees. After developing the basic theory of hyperbolic metric spaces and groups, including an elementary study of the isometries of real trees, we focus on the technical tool of Hausdorff-Gromov convergence, for which we prove a compactness criterion. We then proceed to apply these ideas to non-rigid hyperbolic groups, showing that Hausdorff-Gromov convergence can be used to make such groups act on real trees. Using Rips' classification of actions on real trees, we deduce Paulin's characterisation of rigidity, and the fact that rigid hyperbolic groups are co-Hopfian.

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Introduction

If group theory is the study of symmetry, then the group of automorphisms of a group must capture its algebraic symmetries. A **non-rigid** group is one which is very symmetric, or in other words, which has a large automorphism group. Hence a non-rigid group is, intuitively, a very flexible group. Can we say something more about such groups? Will their symmetry allow us to understand them better?

This is where geometry comes in: if a group is **hyperbolic**, that is to say, if it has geometric properties similar to that of negatively-curved manifolds, then it will be possible to use non-rigidity, play with negative curvature and distort the geometry of our group until we make it act on a tree. The tree in question will not be one in the sense of graph theory, but a more general geometric object — a **real tree**. Such a tree, though not as well-behaved as a usual one, will turn out to have nice geometric properties. In particular, deep work of Rips will give us a classification of groups acting on real trees, and hence a classification of non-rigid hyperbolic groups.

This foundational idea of Paulin consisting in using non-rigidity to distort geometry was applied extensively by Rips and Sela, yielding a vast array of algebraic results related to rigidity, and we hope to show here how fruitful this method can be.

It will be our aim throughout this text to understand the geometric ideas of hyperbolicity, together with the technical tools that will allow us to construct a real tree as a limit of hyperbolic spaces, and to apply those methods to the context of rigidity.

Contents

Chapter 1

Hyperbolic metric spaces and groups

Hyperbolicity of metric spaces was first introduced by Mikhail Gromov in his 1987 article [Gro87], with the aim of generalising results related to negatively curved Riemannian manifolds. We start by giving two equivalent definitions of hyperbolicity and develop the foundations of the theory.

1.1 Hyperbolic metric spaces

How best to define what it means for a metric space to be hyperbolic, or in other words to have a geometry similar to that of the hyperbolic plane? One characteristic feature of hyperbolic geometry is that triangles are, in some sense, thin; for instance, the sum of their interior angles is strictly less than π. Of course, there is no elementary notion of angle in metric geometry, so we need to find another way to express thinness of triangles, and this is what the following definition does.

Consider a metric space X that is **geodesic**: given any two points $x, y \in X$, there exists an isometric embedding $c : [a, b] \to X$ with $c(a) = x$ and $c(b) = y$; this isometric embedding (or its image) is then called a **geodesic segment** from x to y, and can be denoted by $[x, y]$ when there is no risk of confusion with another geodesic segment between the same endpoints. A **geodesic triangle** Δ in X consists of three vertices $x_1, x_2, x_3 \in X$ together with geodesic segments $[x_1, x_2], [x_2, x_3], [x_3, x_1]$ between them. Given $\delta \geq 0$, we say that the triangle Δ is **δ-thin** if, for all $1 \leq i < j \leq 3$ and for all $p \in [x_i, x_j]$, we have

$$d\left(p, [x_i, x_k] \cup [x_k, x_j]\right) \leq \delta,$$

where x_k is the third vertex of Δ. We say that the geodesic metric space X is **Rips-δ-hyperbolic** if all its geodesic triangles are δ-thin. We also say that X is Rips-hyperbolic if it is Rips-δ-hyperbolic for some δ.

We first prove that the hyperbolic plane is Rips-hyperbolic.

Proposition 1.1. (i) *In the hyperbolic plane $\mathbb{H}^2$, the area of any geodesic triangle is less than π.*

(ii) *The hyperbolic plane $\mathbb{H}^2$ is Rips-hyperbolic.*

Proof. (i) If $\alpha_1, \alpha_2, \alpha_3$ are the interior angles of a geodesic triangle Δ with area $\mathcal{A}$, then the Gauß-Bonnet Formula implies that $\mathcal{A} = \pi - \sum_{i=1}^{3} \alpha_i < \pi$.

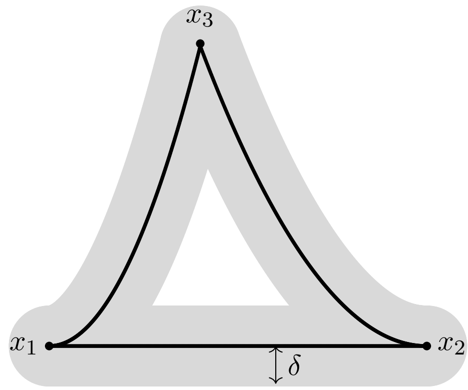

Figure 1: The thinness condition for geodesic triangles: every point of an edge should lie in the δ-neighbourhood of one of the other two edges

(ii) Let δ be the radius of a disk of area 2π in $\mathbb{H}^2$; we show that $\mathbb{H}^2$ is Rips-δ-hyperbolic. Consider a geodesic triangle Δ with vertices x, y, z in $\mathbb{H}^2$ and let $p \in [x, y]$. If p is at a distance less than δ from x or y, then it is clearly at a distance less than δ from $[x, z] \cup [z, y]$, so we may exclude that case. We consider the closed ball $B = \overline{B}_{\mathbb{H}^2}(p, \delta)$ with centre p and radius δ. Note that the geodesic segment $[x, y]$ cuts B along a diameter. Therefore, if $B \cap ([x, z] \cup [z, y]) = \varnothing$, then exactly half of the area of B is contained within Δ. But by choice of δ, B has area 2π, so Δ has area greater than π, which contradicts (i). Therefore, $B \cap ([x, z] \cup [z, y]) \neq \varnothing$, which means that $d(p, [x, z] \cup [z, y]) \leq \delta$. $\square$

This definition of hyperbolicity is not the one originally introduced by Gromov. We also give Gromov's definition, which is less visual but will turn out to be more convenient in some cases. Let X be a metric space. Given a basepoint $\omega \in X$, we define the **Gromov product** $(\cdot \mid \cdot)_\omega : X \times X \to \mathbb{R}$ by

$$(x \mid y)_\omega = \frac{1}{2}\left(d(x, \omega) + d(y, \omega) - d(x, y)\right) \geq 0,$$

for all $x, y \in X$. Note that, in a Euclidean space E, we would have the identity

$$\langle x, y \rangle = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x - y\|^2\right),$$

for all $x, y \in E$. Therefore, the Gromov product can be understood as a metric inner product, and the basepoint ω plays the role of the origin. For $\delta \geq 0$, we say that the metric space X is **Gromov-δ-hyperbolic** if, for all $x, y, z, \omega \in X$,

$$(x \mid y)_\omega \geq \min\left\{(x \mid z)_\omega, (y \mid z)_\omega\right\} - \delta.$$

Our first aim is to show that, for geodesic spaces, Gromov-hyperbolicity is equivalent to Rips-hyperbolicity. Following [DK18], we start by relating the Gromov product to projection on geodesic segments.

Lemma 1.2. *Let X be a geodesic metric space and ω, x, y in X. Then*

(i) *For all $p \in [x, y]$, we have $(\omega \mid x)_p + (\omega \mid y)_p = d(\omega, p) - (x \mid y)_\omega$.*

(ii) *We have the upper bound $(x \mid y)_\omega \leq d(\omega, [x, y])$.*

(iii) *If in addition X is Rips-δ-hyperbolic, then*

$$(x \mid y)_\omega \leq d\left(\omega, [x, y]\right) \leq (x \mid y)_\omega + 2\delta.$$

Proof. (i) For $p \in [x, y]$, the desired equality follows from $d(x, y) = d(x, p) + d(p, y)$.

(ii) This is a direct consequence of (i), together with the fact that the Gromov product is non-negative.

(iii) The first inequality is (ii), so it suffices to prove the second one. Consider a geodesic triangle Δ with vertices ω, x, y. It is δ-thin by assumption, so for all $p \in [x, y]$, we have

$$\min\left\{(\omega \mid x)_p, (\omega \mid y)_p\right\} \leq \min\left\{d\left(p, [\omega, x]\right), d\left(p, [\omega, y]\right)\right\} \leq \delta.$$

By continuity of $p \in [x, y] \longmapsto (\omega \mid x)_p - (\omega \mid y)_p \in \mathbb{R}$, there must exist $p \in [x, y]$ such that

$$(\omega \mid x)_p = (\omega \mid y)_p \leq \delta.$$

By (i), it follows that

$$d\left(\omega, [x, y]\right) - (x \mid y)_\omega \leq d(\omega, p) - (x \mid y)_\omega = (\omega \mid x)_p + (\omega \mid y)_p \leq 2\delta. \qquad \square$$

Proposition 1.3. *Let X be a geodesic metric space.*

(i) *If X is Rips-δ-hyperbolic, then it is Gromov-3δ-hyperbolic.*

(ii) *If X is Gromov-δ-hyperbolic, then it is Rips-4δ-hyperbolic.*

Proof. (i) Let $\omega, x, y, z \in X$. Choose $p \in [x, y]$ such that $d(\omega, p) = d\left(\omega, [x, y]\right)$. By Lemma 1.2,

$$(x \mid y)_\omega \geq d\left(\omega, [x, y]\right) - 2\delta = d(\omega, p) - 2\delta.$$

But note that

$$\begin{aligned}\min\left\{(x \mid z)_\omega, (y \mid z)_\omega\right\} &\leq \min\left\{d\left(\omega, [x, z]\right), d\left(\omega, [y, z]\right)\right\} \\ &\leq d(\omega, p) + \min\left\{d\left(p, [x, z]\right), d\left(p, [y, z]\right)\right\} \\ &\leq d(\omega, p) + \delta,\end{aligned}$$

where the last inequality comes from the δ-thinness of the geodesic triangle with vertices x, y, z. It follows that $(x \mid y)_\omega \geq \min\left\{(x \mid z)_\omega, (y \mid z)_\omega\right\} - 3\delta$.

(ii) We claim that, given $y, z \in X$ with a geodesic segment $[y, z]$ between them, if $p \in X$ satisfies

$$d(p, y) + d(p, z) \leq d(y, z) + 2\delta, \tag{$*$}$$

then $d\left(p, [y, z]\right) \leq 4\delta$. We assume without loss of generality that $d(p, y) \leq d(p, z)$ and we separate two cases:

- If $d(p, y) \geq d(y, z)$, then $(*)$ implies that $d\left(p, [y, z]\right) \leq d(p, z) \leq 2\delta$.

- If $d(p,y) < d(y,z)$, then by continuity of $q \in [y,z] \longmapsto d(q,y) - d(p,y)$, we may choose $q \in [y,z]$ such that $d(q,y) = d(p,y)$. By Gromov-hyperbolicity,

$$(y \mid z)_p \geq \min \left\{ (y \mid q)_p, (z \mid q)_p \right\} - \delta.$$

If $(y \mid q)_p \leq (y \mid z)_p + \delta$, then this yields

$$\begin{aligned} d(p,[y,z]) \leq d(p,q) &\leq d(q,y) + d(p,z) - d(y,z) + 2\delta \\ &= d(p,y) + d(p,z) - d(y,z) + 2\delta \leq 4\delta. \end{aligned}$$

Otherwise $(z \mid q)_p \leq (y \mid z)_p + \delta$ and

$$\begin{aligned} d(p,[y,z]) \leq d(p,q) &\leq d(q,z) + d(p,y) - d(y,z) + 2\delta \\ &= d(p,y) - d(q,y) + 2\delta = 2\delta. \end{aligned}$$

This proves the claim.

Consider a geodesic triangle Δ with vertices x, y, z and let $p \in [x,y]$. By Gromov-hyperbolicity,

$$(x \mid y)_z \geq \min \{ (x \mid p)_z, (y \mid p)_z \} - \delta.$$

Assume without loss of generality that $(x \mid p)_z \leq (y \mid p)_z$, so that the inequality can be rewritten as $d(p,z) - d(x,p) \leq d(y,z) - d(x,y) + 2\delta$, and therefore

$$d(p,y) + d(p,z) = (d(x,y) - d(x,p)) + d(p,z) \leq d(y,z) + 2\delta.$$

It follows from the claim that $d(p,[y,z]) \leq 4\delta$. ☐

Corollary 1.4. *A geodesic metric space X is Rips-hyperbolic (resp. Rips-0-hyperbolic) if and only if it is Gromov-hyperbolic (resp. Gromov-0-hyperbolic). Such a geodesic space is said to be* ***hyperbolic*** *(resp.* ***0-hyperbolic****).*

1.2 Stability of geodesics and quasi-isometry invariance

The one major result we will need about hyperbolic metric spaces is one that says that geodesics are stable: if a path is, in some sense, an 'approximate geodesic', then it is close to an actual geodesic. We follow [BH99] for this proof. The stability of geodesics will then lead us to the quasi-isometry invariance of hyperbolicity.

Given a path $c : [a,b] \to X$ in a metric space X, the **length** of c is defined by

$$\ell(c) = \sup_{a=t_0<t_1<\cdots<t_k=b} \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) \in [0,\infty].$$

Lemma 1.5. *If $c : [a,b] \to X$ is a path of finite length in a metric space X, then there exists a path $c' : [0,1] \to X$ such that $\operatorname{Im} c' = \operatorname{Im} c$, $c'(0) = c(0)$, $c'(1) = c(1)$ and for all $s,t \in [0,1]$,*

$$\ell\left(c'_{|[s,t]} \right) = \ell(c) \left| s - t \right|.$$

We say that c' is parametrised ***proportional to arc length****.*

Proof. Consider the function $\psi : t \in [a,b] \longmapsto \ell\left(c_{|[a,t]}\right) \in \mathbb{R}_+$. We may assume after a first reparametrisation that ψ is strictly increasing, so it defines a homeomorphism $[a,b] \to [0,\ell(c)]$. Now define

$$c' : t \in [0,1] \longmapsto c\left(\psi^{-1}\left(\ell(c)t\right)\right) \in X.$$

It is clear that $\operatorname{Im} c' = \operatorname{Im} c$, $c'(0) = c(0)$ and $c'(1) = c(1)$. Moreover, for $0 \leq s \leq t \leq 1$, we have $\ell\left(c'_{|[s,t]}\right) = \ell\left(\left(c \circ \psi^{-1}\right)_{|[\ell(c)s,\ell(c)t]}\right) = \ell(c)\,|s-t|$. $\square$

Following [BH99], the first step towards the stability of geodesics is to show that, in a hyperbolic space, geodesics never stray too far from continuous paths of finite length between the same endpoints.

Lemma 1.6. *Let X be a Rips-δ-hyperbolic space. Let c be a continuous path of finite length in X, with endpoints x, y. Then for any geodesic segment $[x,y]$, we have*

$$\sup_{p\in[x,y]} d\left(p, \operatorname{Im} c\right) \leq \delta\left|\log_2 \ell(c)\right| + 1.$$

Proof. Let $p \in [x,y]$. If $\ell(c) \leq 1$, then $d\left(p, \operatorname{Im} c\right) \leq d(p,x) \leq \ell\left([x,y]\right) \leq \ell(c) \leq 1$, so we henceforth assume that $\ell(c) > 1$. We also assume without loss of generality that c is parametrised proportional to arc length.

Consider a geodesic triangle Δ_0 with vertices $x = c(0)$, $y = c(1)$ and $c\left(\frac{1}{2}\right)$. By hyperbolicity, there exists $p_1 \in \left[c\left(\frac{k_1}{2}\right), c\left(\frac{k_1+1}{2}\right)\right]$ (with $k_1 \in \{0,1\}$) such that $d\left(p,p_1\right) \leq \delta$. We then consider a geodesic triangle Δ_1 with vertices $c\left(\frac{k_1}{2}\right)$, $c\left(\frac{k_1+1}{2}\right)$ and $c\left(\frac{2k_1+1}{4}\right)$; by hyperbolicity, there exists $p_2 \in \left[c\left(\frac{k_2}{2^2}\right), c\left(\frac{k_2+1}{2^2}\right)\right]$ (with $k_2 \in \{0,1,\ldots,2^2-1\}$) such that $d\left(p_1,p_2\right) \leq \delta$.

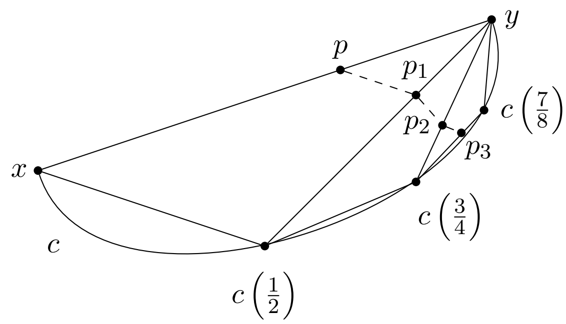

Figure 2: Geodesics don't stray away from paths of finite length in hyperbolic spaces

Inductively, we obtain a sequence of points $p_j \in \left[c\left(\frac{k_j}{2^j}\right), c\left(\frac{k_j+1}{2^j}\right)\right]$ (with $0 \leq k_j < 2^j$) such that $d\left(p_{j-1},p_j\right) \leq \delta$.

Now choose n such that $2^n \leq \ell(c) < 2^{n+1}$ and consider $p_n \in \left[c\left(\frac{k_n}{2^n}\right), c\left(\frac{k_n+1}{2^n}\right)\right]$. If q is the endpoint of $\left[c\left(\frac{k_n}{2^n}\right), c\left(\frac{k_n+1}{2^n}\right)\right]$ closest to p_n, then we have (denoting $p_0 = p$)

$$\begin{aligned} d\left(p, \operatorname{Im} c\right) \leq d(p,q) = d\left(p_0,q\right) &\leq \sum_{j=1}^{n} d\left(p_{j-1},p_j\right) + d\left(p_n,q\right) \\ &\leq n\delta + \frac{1}{2}\ell\left(c_{|\left[\frac{k_n}{2^n},\frac{k_n+1}{2^n}\right]}\right) = n\delta + \frac{\ell(c)}{2^{n+1}} \leq \delta\left|\log_2 \ell(c)\right| + 1. \end{aligned}$$ $\square$

We now make precise what we mean by an 'approximate geodesic' and use this occasion to recall the concept of quasi-isometry. Let $f : X \to Y$ be a map between metric spaces. Given $\lambda \geq 1$ and $\varepsilon \geq 0$, we say that f is a (λ, ε)**-quasi-isometric embedding** if the inequality

$$\frac{1}{\lambda} d_X(x, x') - \varepsilon \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \varepsilon$$

holds for all $x, x' \in X$. If in addition f is **quasi-surjective**, i.e. there is a constant $c \geq 0$ such that $d(y, \operatorname{Im} f) \leq c$ for all $y \in Y$, we say that f is a **quasi-isometry**, and that X is quasi-isometric to Y. Intuitively, this means that X looks roughly like Y, up to some bounded distortion.

A (λ, ε)**-quasi-geodesic** in a metric space X is a map $c : [a, b] \to X$ that is a (λ, ε)-quasi-isometric embedding: hence, a quasi-geodesic is a (not necessarily continuous) path that does not distort distances too much. The following lemma allows one to make quasi-geodesics continuous while controlling their length, in a somewhat similar way to reparametrising paths proportional to arc length in Lemma 1.5.

Lemma 1.7. *Let X be a geodesic space. If $c : [a, b] \to X$ is a (λ, ε)-quasi-geodesic, then there exists a continuous (λ, ε')-quasi-geodesic $c' : [a, b] \to X$ (with $\varepsilon' = 2(\lambda + \varepsilon)$) such that*

(i) $c'(a) = c(a)$ *and* $c'(b) = c(b)$,

(ii) *For all* $t, t' \in [a, b]$,

$$\ell\left(c'_{|[t,t']}\right) \leq k_1 |c'(t) - c'(t')| + k_2,$$

with $k_1 = \lambda(\lambda + \varepsilon)$ *and* $k_2 = (\lambda\varepsilon' + 3)(\lambda + \varepsilon)$,

(iii) $\mathscr{D}_H(\operatorname{Im} c, \operatorname{Im} c') \leq \lambda + \varepsilon$,

*where $\mathscr{D}_H$ is the **Hausdorff distance**:* $\mathscr{D}_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$.

Proof. Let $\Sigma = \{a, b\} \cup (\mathbb{Z} \cap (a, b)) = \{a = a_0 < \cdots < a_k = b\} \subseteq [a, b]$. Choose geodesic segments $[c(a_i), c(a_{i+1})]$ for all $i \in \{0, \ldots, k-1\}$ and define c' by concatenating (linear reparametrisations of) those geodesic segments:

$$c'_{|[a_i, a_{i+1}]} = [c(a_i), c(a_{i+1})].$$

It is clear that c' is continuous and that (i) and (iii) are satisfied.

For $t \in [a, b]$, we denote by $[t]_0$ the point of Σ minimising the distance to t and $[t]_1$ such that $t \in [[t]_0, [t]_1]$. We have

$$\begin{aligned} d(c'(t), c'(t')) &\leq d(c'(t), c'([t]_0)) + d(c'([t]_0), c'([t']_0)) + d(c'([t']_0), c'(t')) \\ &= |t - [t]_0| \cdot d(c([t]_0), c([t]_1)) + |t' - [t']_0| \cdot d(c([t']_0), c([t']_1)) \\ &\qquad + d(c([t]_0), c([t']_0)) \\ &\leq \lambda + \varepsilon + \lambda |[t]_0 - [t']_0| + \varepsilon \\ &\leq \lambda |t - t'| + 2(\lambda + \varepsilon), \end{aligned}$$

and similarly $d(c'(t), c'(t')) \geq \frac{1}{\lambda}|t - t'| - 2(\lambda + \varepsilon)$. Hence, c' is a (λ, ε')-quasi geodesic.

Now if $m, n \in \mathbb{Z} \cap [a, b]$ with $m \leq n$, then

$$\ell\left(c'_{|[m,n]}\right) = \sum_{j=m}^{n-1} \ell\left(c'_{[j,j+1]}\right) = \sum_{j=m}^{n-1} d(c(j+1), c(j)) \leq (\lambda + \varepsilon)|n - m|.$$

Likewise, $\ell\left(c'_{|[a,m]}\right) \leq (\lambda+\varepsilon)(m-a+1)$ and $\ell\left(c'_{|[n,b]}\right) \leq (\lambda+\varepsilon)(b-n+1)$. This shows that, for all $u, v \in \Sigma$,

$$\ell\left(c'_{|[u,v]}\right) \leq (\lambda+\varepsilon)(|u-v|+1).$$

It follows that, for all $t, t' \in [a,b]$,

$$\begin{aligned}\ell\left(c'_{|[t,t']}\right) &\leq d\left(c'(t), c'([t]_0)\right) + d\left(c'(t'), c'([t']_0)\right) + \ell\left(c'_{|[[t]_0,[t']_0]}\right)\\ &\leq (\lambda+\varepsilon)\left(\left|[t]_0 - [t']_0\right| + 2\right)\\ &\leq (\lambda+\varepsilon)\left(|t-t'|+3\right).\end{aligned}$$

But we also have

$$d\left(c'(t), c'(t')\right) \geq \frac{1}{\lambda}|t-t'| - \varepsilon',$$

from which it follows that

$$\ell\left(c'_{|[t,t']}\right) \leq (\lambda+\varepsilon)\left(\lambda d\left(c'(t), c'(t')\right) + \lambda\varepsilon' + 3\right). \qquad \square$$

We can now prove that geodesics are stable in hyperbolic spaces.

Theorem 1.8 (Stability of geodesics). *Given $\delta \geq 0$, $\lambda \geq 1$ and $\varepsilon \geq 0$, there exists $\rho = \rho(\delta, \lambda, \varepsilon) \geq 0$ such that, if a space X is Rips-δ-hyperbolic, $c : [a,b] \to X$ is a (λ, ε)-quasi-geodesic from x to y in X and $[x,y]$ is a geodesic from x to y, then we have the following bound on the Hausdorff distance:*

$$\mathscr{D}_H(\operatorname{Im} c, [x,y]) \leq \rho.$$

Proof. Let c' be the quasi-geodesic obtained after applying Lemma 1.7 to c. It suffices to prove the result for c' because $\mathscr{D}_H(\operatorname{Im} c, \operatorname{Im} c') \leq \lambda + \varepsilon$. We may therefore assume that c already satisfies all the conditions of Lemma 1.7.

Given $r \geq 0$ and $A \subseteq X$, we write $V_r(A) = \{x \in X,\ d(x,A) \leq r\}$. We thus need to show that $[x,y] \subseteq V_\rho(\operatorname{Im} c)$ and $\operatorname{Im} c \subseteq V_\rho([x,y])$.

First step: $[x,y] \subseteq V_\rho(\operatorname{Im} c)$. Let $p_0 \in [x,y]$ such that

$$d(p_0, \operatorname{Im} c) = \max_{p \in [x,y]} d(p, \operatorname{Im} c) = D.$$

Note that $d(x,p_0), d(y,p_0) \geq D$. We may therefore choose $q_0 \in [x,p_0]$ and $r_0 \in [y,p_0]$ such that

$$d(q_0,p_0) = d(r_0,p_0) = D.$$

We have $d(q_0, \operatorname{Im} c), d(r_0, \operatorname{Im} c) \leq D$ by definition, so there exist $q_1, r_1 \in \operatorname{Im} c$ such that $d(q_0,q_1), d(r_0,r_1) \leq D$. Now choose geodesics $[q_0,q_1]$ and $[r_0,r_1]$, and consider the path γ from q_0 to r_0 obtained by concatenating $[q_0,q_1]$, $c_{|[q_1,r_1]}$ and $[r_1,r_0]$. Note that

$$d(q_1,r_1) \leq d(q_1,q_0) + d(q_0,p_0) + d(p_0,r_0) + d(r_0,r_1) \leq 6D.$$

It follows from Lemma 1.7.(ii) that

$$\ell(\gamma) = \ell([q_0,q_1]) + \ell\left(c_{|[q_1,r_1]}\right) + \ell([r_1,r_0]) \leq 6Dk_1 + k_2 + 2D.$$

Using Lemma 1.6, we have

$$D = d(p_0, \operatorname{Im} c) \leq d(p_0, \operatorname{Im}\gamma) \leq \delta\left|\log_2 \ell(\gamma)\right| + 1 \leq \delta\left|\log_2(6Dk_1 + k_2 + 2D)\right| + 1.$$

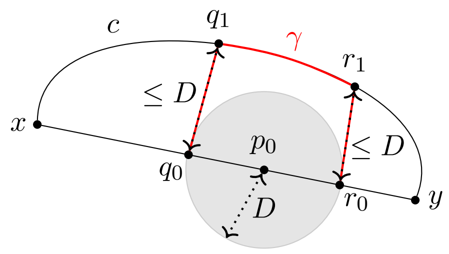

Figure 3: Finding a point of $\operatorname{Im} c$ close to $p_0 \in [x, y]$

This yields an upper-bound D_0 on D depending only on $\delta, \lambda, \varepsilon$. Hence, $[x, y] \subseteq V_{D_0}(\operatorname{Im} c)$.

Second step: $\operatorname{Im} c \subseteq V_\rho([x, y])$. Let $\rho = D_0(1 + k_1) + \frac{k_2}{2} \geq D_0$. We already have $[x, y] \subseteq V_{D_0}(\operatorname{Im} c) \subseteq V_\rho(\operatorname{Im} c)$, we shall now prove that $\operatorname{Im} c \subseteq V_\rho([x, y])$. Consider a subinterval $[a', b']$ of $c^{-1}(X \smallsetminus V_{D_0}([x, y])) \subseteq [a, b]$. Since $[x, y] \subseteq V_{D_0}(\operatorname{Im} c)$, we know that, for all $p \in [x, y]$,

$$\min\left\{d\left(p, \operatorname{Im} c_{|[a,a']}\right), d\left(p, \operatorname{Im} c_{|[b',b]}\right)\right\} \leq D_0.$$

By continuity of $p \in [x, y] \longmapsto d\left(p, \operatorname{Im} c_{|[a,a']}\right) - d\left(p, \operatorname{Im} c_{|[b',b]}\right)$, there exists $p \in [x, y]$ such that

$$d\left(p, \operatorname{Im} c_{|[a,a']}\right) = d\left(p, \operatorname{Im} c_{|[b',b]}\right) \leq D_0.$$

Therefore there exist $s \in [a, a']$ and $t \in [b', b]$ such that $d(p, c(s)), d(p, c(t)) \leq D_0$. Hence

$$\ell\left(c_{|[s,t]}\right) \leq k_1 d(c(s), c(t)) + k_2 \leq 2D_0 k_1 + k_2 = 2(\rho - D_0).$$

This implies that all points of $\operatorname{Im} c_{|[s,t]}$ lie at a distance at most $(\rho - D_0) + D_0 = \rho$ from $[x, y]$, or in other words

$$\operatorname{Im} c_{|[a',b']} \subseteq \operatorname{Im} c_{|[s,t]} \subseteq V_\rho([x, y]).$$

Since this is true for every subinterval of $c^{-1}(X \smallsetminus V_{D_0}([x, y]))$, we get $\operatorname{Im} c \subseteq V_\rho([x, y])$. $\square$

The reason why we need the stability of geodesics is the following strengthening of the definition of hyperbolicity, which will lead us to quasi-isometry invariance.

Corollary 1.9. *Let X be a geodesic metric space. Then X is hyperbolic if and only if for every $\lambda \geq 1$ and $\varepsilon \geq 0$, there exists a constant $\eta \geq 0$ such that every (λ, ε)-quasi-geodesic triangle in X is η-thin.*

Proof. ($\Leftarrow$) The given condition implies Rips-hyperbolicity because a geodesic triangle is $(1, 0)$-quasi-geodesic.

($\Rightarrow$) Assume that X is Rips-δ-hyperbolic for some $\delta \geq 0$ and let $\rho = \rho(\delta, \lambda, \varepsilon)$ as in Theorem 1.8. Let $x, y, z \in X$, together with (λ, ε)-quasi-geodesics $\gamma_{x,y}$ between x and y, $\gamma_{y,z}$ between y and z and $\gamma_{x,z}$ between x and z. If $p \in \operatorname{Im} \gamma_{x,y}$, then there exists $p' \in [x, y]$ such that $d(p, p') \leq \mathscr{D}_H(\operatorname{Im} \gamma_{x,y}, [x, y])$. By hyperbolicity, there exists $q' \in [x, z] \cup [z, y]$ (and we may assume without loss of generality that $q' \in [x, z]$) such that

$d(p', q') \leq \delta$. Again, there exists $q \in \operatorname{Im} \gamma_{x,z}$ such that $d(q, q') \leq \mathscr{D}_H(\operatorname{Im} \gamma_{x,z}, [x, z])$. Applying Theorem 1.8, we have

$$\begin{aligned} d(p,q) &\leq d(p,p') + d(p',q') + d(q',q) \\ &\leq \mathscr{D}_H(\operatorname{Im} \gamma_{x,y}, [x,y]) + \delta + \mathscr{D}_H(\operatorname{Im} \gamma_{x,z}, [x,z]) \leq \delta + 2\rho, \end{aligned}$$

which shows that the quasi-geodesic triangle under consideration is $(\delta + 2\rho)$-thin. □

Corollary 1.10 (Quasi-isometry invariance of hyperbolicity)**.** *Let X, Y be two geodesic metric spaces and let $f : X \to Y$ be a (λ, ε)-quasi-isometric embedding. If Y is Rips-δ-hyperbolic, then X is Rip-η-hyperbolic, where η is a constant depending only on $\delta, \lambda, \varepsilon$.*

In particular, hyperbolicity is invariant under quasi-isometry (for geodesic spaces).

Proof. If Δ is a geodesic triangle in X, then $f(\Delta)$ is a (λ, ε)-quasi-geodesic triangle in Y, so it must be η_0-slim for some constant $\eta_0 = \eta_0(\delta, \lambda, \varepsilon)$ by Corollary 1.9. It follows that Δ itself is η-slim for some constant $\eta = \eta(\delta, \lambda, \varepsilon)$. □

1.3 Real trees and their isometries

For the purpose of this essay, we will be particularly interested in the case of 0-hyperbolic geodesic spaces — such spaces are called **real trees**. The main idea of Paulin's Theorem will be that, given a group acting nicely on a hyperbolic space and with enough automorphisms, we can somehow shrink the hyperbolicity constant until we get a 0-hyperbolic space, or in other words a real tree. To understand why we might want to do that, the aim of this section will be to show that real trees have nice geometric properties that are reminiscent of those of the hyperbolic plane. We will not need any of the results we prove in this section, because we shall use instead a much stronger theorem due to Rips, but they should give some motivation for studying real trees.

We first give an equivalent (and more common) definition of real trees, taken from [Bes01].

Proposition 1.11. *Let X be a geodesic metric space. Then the following two assertions are equivalent:*

(i) *X is a real tree, i.e. a 0-hyperbolic geodesic space.*

(ii) *For any two points $x, y \in X$, there is a unique **arc** between x and y (i.e. the image of a topological embedding $c : [a, b] \to X$ with $c(a) = x$ and $c(b) = y$), and this arc is (the image of) a geodesic segment.*

Proof. (i) ⇒ (ii) The fact that X is 0-hyperbolic means that any geodesic triangle in X is degenerate. From this, it is clear that there must be a unique geodesic segment between any two points of X. And if there were an arc c between two points x, y such that $\operatorname{Im} c \neq [x, y]$, then considering geodesic segments from x and y to a point of $\operatorname{Im} c \setminus [x, y]$ would yield a non-degenerate geodesic triangle, which is impossible.

(ii) ⇒ (i) If there is a unique arc between any two points of X, then any geodesic triangle in X is degenerate, so X is 0-hyperbolic. □

Example. *Simplicial trees are real trees.*

In fact, it is also true that a (not necessarily geodesic) Gromov-0-hyperbolic space embeds isometrically into a real tree, but we will never need to consider non-geodesic hyperbolic spaces in this essay.

We now proceed to study isometries of real trees. The following few results give a classification of isometries appearing without proof in [Bes01]; there appears a strong analogy with the standard classification of hyperbolic isometries. We first need a technical lemma that will be useful to find the axis of an isometry.

Lemma 1.12. *Let T be a real tree and let ϕ belong to the group* $\mathrm{Isom}(T)$ *of isometries of T.*

(i) *If $x \in T$, then, for all $z \in [x, \phi(x)]$, we have $d(z, \phi(z)) \leq d(x, \phi(x))$.*

(ii) *If $x, y \in T$ with $\phi(x) = x$ and $\phi(y) = y$, then, for all $z \in [x, y]$, we have $\phi(z) = z$.*

Proof. (i) For $z \in [x, \phi(x)]$, we have

$$d(z, \phi(z)) \leq d(z, \phi(x)) + d(\phi(x), \phi(z)) = d(z, \phi(x)) + d(x, z) = d(x, \phi(x)).$$

(ii) Let $z \in [x, y]$. Then

$$d(x, \phi(z)) = d(\phi(x), \phi(z)) = d(x, z) \quad \text{and similarly} \quad d(y, \phi(z)) = d(y, z).$$

Therefore, if $\phi(z) \neq z$, then z cannot lie on $[x, \phi(z)]$ for otherwise we would have $d(x, \phi(z)) = d(x, z) + d(z, \phi(z)) > d(x, z)$. Likewise $z \notin [y, \phi(z)]$. This would yield two different paths from x to y, one going through z and the other going through $\phi(z)$; this is a contradiction and therefore $\phi(z) = z$. □

Proposition 1.13. *Let T be a real tree and let $\phi \in \mathrm{Isom}(T)$. Then the infimum*

$$\ell(\phi) = \inf_{x \in T} d(x, \phi(x))$$

is attained. More precisely, for any $x_0 \in T$, there exists $x_m \in [x_0, \phi(x_0)]$ such that

$$d(x_m, \phi(x_m)) = \min_{x \in T} d(x, \phi(x)).$$

Proof. Let $x_0 \in T$. The segment $[x_0, \phi(x_0)]$ being compact, there exists $x_m \in [x_0, \phi(x_0)|$ such that

$$d(x_m, \phi(x_m)) = \min_{x \in [x_0, \phi(x_0)]} d(x, \phi(x)).$$

Note that $\phi(x_m) \in \phi([x_0, \phi(x_0)]) = [\phi(x_0), \phi^2(x_0)]$. Therefore, the uniqueness of paths from x_m to $\phi(x_m)$ implies that $[x_m, \phi(x_m)] \subseteq [x_0, \phi(x_0)] \cup [\phi(x_0), \phi^2(x_0)]$. Hence

$$d(x_m, \phi(x_m)) = \min_{z \in [x_m, \phi(x_m)]} d(z, \phi(z)). \tag{$*$}$$

Now assume for contradiction that there exists $x_1 \in T$ such that $d(x_1, \phi(x_1)) < d(x_m, \phi(x_m))$. We may choose x_1 minimal in the sense that

$$d(x_1, \phi(x_1)) = \min_{z \in [x_1, \phi(x_1)]} d(z, \phi(z)).$$

Note that the segments $[x_m, \phi(x_m)]$ and $[x_1, \phi(x_1)]$ are disjoint, for otherwise Lemma 1.12.(i) would imply the existence of $z \in [x_m, \phi(x_m)]$ with $d(z, \phi(z)) \leq d(x_1, \phi(x_1)) < d(x_m, \phi(x_m))$, in contradiction with $(*)$.

Hence, there exist points $y \in [x_m, \phi(x_m)]$ and $z \in [x_1, \phi(x_1)]$ such that the interior of $[y, z]$ is disjoint from $[x_m, \phi(x_m)] \cup [x_1, \phi(x_1)]$. We assume that $\phi(x_1) \neq x_1$ (the proof is similar if $\phi(x_1) = x_1$) and we write $d = d(y, z)$. It follows from minimality that the points

$x_m, \phi(x_m), \phi^2(x_m), \ldots$ are aligned, and similarly for $x_1, \phi(x_1), \phi^2(x_1), \ldots$. Hence, for all $k \geq 1$,

$$\begin{aligned} d(x_m, x_1) &= d\left(\phi^k(x_m), \phi^k(x_1)\right) \\ &= d\left(\phi^k(x_m), y\right) + d(y,z) + d\left(z, \phi^k(x_1)\right) \\ &= kd(x_m, \phi(x_m)) - d(x_m, y) + kd(x_1, \phi(x_1)) - d(x_1, z) + d. \end{aligned}$$

This yields a contradiction when $k \to \infty$. $\square$

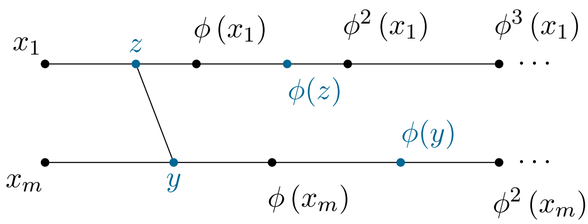

Figure 4: An impossible picture in a real tree

In analogy with hyperbolic geometry, the real number $\ell(\phi) = \min_{x\in T} d(x, \phi(x))$ will be called the **translation length** of ϕ and the set $\mathcal{A}(\phi) = \{x \in T,\ d(x, \phi(x)) = \ell(\phi)\}$ will be called the **axis** of ϕ. These notions give us a classification of isometries of real trees.

Theorem 1.14. *Let T be a real tree and let $\phi \in \mathrm{Isom}(T)$.*

(i) *If $\ell(\phi) = 0$ then ϕ fixes a non-empty subtree of T. We say that ϕ is **elliptic**.*

(ii) *If $\ell(\phi) > 0$, then the axis $\mathcal{A}(\phi)$ is isometric to $\mathbb{R}$, stable by ϕ, and ϕ acts on it by translation by $\ell(\phi)$. We say that ϕ is **hyperbolic**.*

Proof. (i) By Lemma 1.12.(ii), the set of fixed points of ϕ is convex; it is therefore a subtree of T, and it is non-empty by Proposition 1.13.

(ii) We assume for simplicity that $\ell(\phi) = 1$. By Proposition 1.13, $\mathcal{A}(\phi) \neq \varnothing$; take $x_0 \in \mathcal{A}(\phi)$. By Lemma 1.12.(i), the segment $[x_0, \phi(x_0)]$ is included in $\mathcal{A}(\phi)$ and therefore $\phi^k([x_0, \phi(x_0)]) \subseteq \mathcal{A}(\phi)$ for all $k \in \mathbb{Z}$. There exists an isometry $j_0 : [0,1] \to [x_0, \phi(x_0)]$ and we define

$$j : t \in \mathbb{R} \longmapsto \phi^{\lfloor t \rfloor} \circ j_0(t - \lfloor t \rfloor) \in \mathcal{A}(\phi).$$

To show that j is an isometric embedding, let $s < t$ be two real numbers, and let

$$n_0 \leq s < n_0 + 1 < \cdots < n_0 + r \leq t < n_0 + r + 1$$

be all the integers between them. Using the fact that $\phi^k(x) \in \left[\phi^{k-1}(x), \phi^{k+1}(x)\right]$ for all $x \in \mathcal{A}(\phi)$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} d(j(t), j(s)) &= d\left(\phi^{n_0+r} \circ j_0(t - n_0 - r), \phi^{n_0} \circ j_0(s - n_0)\right) \\ &= (t - n_0 - r) + d\left(\phi^{n_0+r}(j_0(0)), \phi^{n_0}(j_0(1))\right) + (1 - (s - n_0)) \\ &= \sum_{k=1}^{r-1} d\left(\phi^{n_0+k+1}(x_0), \phi^{n_0+k}(x_0)\right) + 1 + t - s - r \\ &= t - s. \end{aligned}$$

It is also clear that $\phi(j(t)) = j(t+1)$, i.e. ϕ acts by translation. It remains to show that j is onto. Assume for contradiction that there exists a point $x_1 \in \mathcal{A}(\phi) \smallsetminus j(\mathbb{R})$. Then x_1 also lies on the image of an isometric embedding $j' : \mathbb{R} \to \mathcal{A}$ on which ϕ acts by translation. If there existed $s, t \in \mathbb{R}$ such that $j(s) = j'(t)$, then we would have $j(s+k) = \phi^k(j(s)) = \phi^k(j'(t)) = j'(t+k)$ for all $k \in \mathbb{Z}$ and we would conclude that $j(\mathbb{R}) = j'(\mathbb{R})$, which is excluded. Therefore, $j(\mathbb{R}) \cap j'(\mathbb{R}) = \varnothing$, but there must still be a path from a point $y_0 \in j(\mathbb{R})$ to a point $y_1 \in j'(\mathbb{R})$ such that the interior of $[y_0, y_1]$ is disjoint from $j(\mathbb{R}) \cup j'(\mathbb{R})$. This would give

$$|\phi(y_0) - \phi(y_1)| = |\phi(y_0) - y_0| + |y_0 - y_1| + |y_1 - \phi(y_1)| = 2\ell + |y_0 - y_1|,$$

a contradiction. Therefore, j is onto, so it is an isometry $\mathbb{R} \to \mathcal{A}(\phi)$. $\square$

1.4 Hyperbolic groups

Now that we have developped some elements of the theory of hyperbolic metric spaces, we are ready to introduce our main object of interest: hyperbolic groups. Recall that we may associate to each group Γ with generating set S the **Cayley graph** $\mathrm{Cay}(\Gamma, S)$, which is a connected metric graph and therefore a geodesic space. The induced metric d_S on Γ is called the **word metric** with respect to S, and we also have a notion of **word length** defined by $|g|_S = d_S(e, g)$ for $g \in G$. If S and S' are two finite generating sets for Γ, then $\mathrm{Cay}(\Gamma, S)$ is quasi-isometric to $\mathrm{Cay}(\Gamma, S')$, and their common quasi-isometry class defines a geometric structure for Γ.

We say that a finitely generated group Γ is **hyperbolic** if any of its Cayley graphs is hyperbolic. This definition makes sense because we know from Corollary 1.10 that hyperbolicity is a quasi-isometry invariant for geodesic spaces. When we defined hyperbolicity, our initial aim was to find a class of metric spaces whose geometry would be somehow similar to that of the hyperbolic plane. The following examples confirm that hyperbolic groups generalise negatively curved geometry to a wider setting.

Examples. (i) *Finitely generated free groups are* 0*-hyperbolic.*

(ii) *If S is a connected surface of finite type with a hyperbolic structure, then $\pi_1 S$ is hyperbolic.*

(iii) *The Coxeter group $W = \langle s_1, \ldots, s_n \mid \forall k,\ s_k^2 = (s_k s_{k+1})^{p_k} = 1 \rangle$ (with cyclic notation $s_{n+1} = s_1$) is hyperbolic if $\frac{1}{p_1} + \cdots + \frac{1}{p_n} < n - 2$.*

Proof. (i) Any Cayley graph of a finitely generated free group is a (simplicial) tree, so it is 0-hyperbolic.

(ii) The universal cover $\widetilde{S}$ of S is a convex subset of the hyperbolic plane $\mathbb{H}^2$. Moreover, we have a proper action by isometries of $\pi_1 S$ on $\widetilde{S}$, and this action is cocompact because $\widetilde{S}/\pi_1 S \cong S$ is compact, so the Švarc-Milnor Lemma implies that $\pi_1 S$ is finitely generated and quasi-isometric to $\widetilde{S}$. But the latter is hyperbolic by Proposition 1.1, so the invariance of hyperbolicity under quasi-isometry (Corollary 1.10) implies that $\pi_1 S$ is hyperbolic.

(iii) The condition on $p_1, \ldots, p_n$ implies that there is a polygon in $\mathbb{H}^2$ with interior angles $\frac{\pi}{p_1}, \ldots, \frac{\pi}{p_n}$. Poincaré's Theorem on fundamental polygons (see [dlH00, V.B]) implies that W acts properly cocompactly by isometries on $\mathbb{H}^2$, from which it follows by the Švarc-Milnor Lemma that W is quasi-isometric to $\mathbb{H}^2$ and hence hyperbolic. $\square$

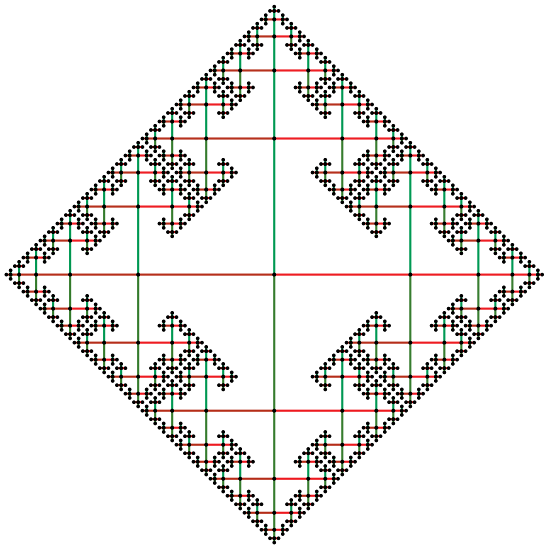

Figure 5: The Cayley graph of the free group of rank 2 with its standard generating set

1.5 Centralisers and quasi-convexity in hyperbolic groups

As an example of the various properties of hyperbolic groups, and also because this will come useful later, we now investigate abelian subgroups and centralisers in hyperbolic groups. More precisely, our goal is to prove that, in hyperbolic groups, an infinite cyclic subgroup has finite index in its centraliser. In particular, hyperbolic groups cannot contain large abelian subgroups.

To do this, we follow [BH99] and introduce the notion of quasi-convexity: given a metric space X, a subset $C \subseteq X$ is said to be **quasi-convex** if there is a constant $k \geq 0$ such that for all $x, y \in C$, any geodesic segment $[x, y]$ remains at a distance at most k from C. We want to prove that infinite cyclic subgroups are quasi-convex in hyperbolic groups. It will follow from the following lemma that they are, in some sense, not distorted.

Lemma 1.15. *Let Γ be a group with finite generating set S. Let H be a subgroup of Γ that is quasi-convex as a subset of $\mathrm{Cay}(\Gamma, S)$. Then H is finitely generated and the inclusion $H \hookrightarrow \Gamma$ is a quasi-isometric embedding.*

Proof. Let $h \in H$ and consider a geodesic segment $[e, h]$ in $\mathrm{Cay}(\Gamma, S)$; this corresponds to a word $h = s_1 \cdots s_\ell$ with $s_i \in S$ and $\ell = |h|_S$. For $0 \leq i \leq \ell$, quasi-convexity implies the existence of $a_i \in H$ and $u_i \in \Gamma$ with $|u_i|_S \leq k$ such that

$$s_1 \cdots s_i = a_i u_i.$$

Note that we can choose $a_0 = e$, $a_\ell = h$ and $u_0 = u_\ell = e$. Hence, if we set $t_i = a_{i-1}^{-1} a_i \in H$ for $i \geq 1$, then $s_i = u_{i-1}^{-1} t_i u_i$. In particular, t_i belongs to the closed ball $T = \overline{B}\,(e, 2k+1)$. Moreover,

$$h = s_1 \cdots s_\ell = \left(u_0^{-1} t_1 u_1\right)\left(u_1^{-1} t_2 u_2\right) \cdots \left(u_{\ell-1}^{-1} t_\ell u_\ell\right) = t_1 \cdots t_\ell,$$

This proves that the finite set T generates H, and we have

$$|h|_T \leq |h|_S \leq \left(\max_{t \in T} |t|_S\right) |h|_T$$

for all $h \in H$, proving that the inclusion $(H, d_T) \hookrightarrow (\Gamma, d_S)$ is quasi-isometric. $\square$

It will be necessary to use the following technical lemma from [BH99], saying that in a hyperbolic space, two geodesic segments starting from the same point remain close to each other.

Lemma 1.16. *Let X be a Rips-δ-hyperbolic space. Let $c_i : [0, T_i] \to X$ (for $i \in \{1,2\}$) be two geodesic segments with $c_1(0) = c_2(0)$, and $d = d\left(c_1\left(T_1\right), c_2\left(T_2\right)\right)$. Extend c_1, c_2 to $[0,T] \to X$, with $T = \max\{T_1, T_2\}$ by setting $c_{i|[T_i,T]} = c_i\left(T_i\right)$. Then, for all $t \in [0,T]$,*

$$d\left(c_1(t), c_2(t)\right) \leq 2\left(\delta + d\right).$$

Proof. Let $t \in [0,T]$. Since the geodesic triangle with vertices $c_1(0), c_1(T), c_2(T)$ is δ-thin, there exists $y \in c_2\left([0,T]\right) \cup \left[c_2(T), c_1(T)\right]$ such that $d\left(c_1(t), y\right) \leq \delta$.

- If $y \in [c_2(T), c_1(T)]$, then $|t - T| = d\left(c_1(t), c_1(T)\right) \leq d\left(c_1(t), y\right) + d\left(y, c_1(T)\right) \leq \delta + d$ (for the first equality, we need to assume that $T_1 = T$, which we may do without loss of generality), so

$$\begin{aligned} d\left(c_1(t), c_2(t)\right) &\leq d\left(c_1(t), y\right) + d\left(y, c_2(T)\right) + d\left(c_2(T), c_2(t)\right) \\ &\leq \delta + d + |t - T| = 2\left(\delta + d\right). \end{aligned}$$

- If $y = c_2\left(t'\right) \in c_2\left([0,T]\right)$, then

$$\left|t - t'\right| = \left|d\left(c_1(0), c_1(t)\right) - d\left(c_1(0), c_2\left(t'\right)\right)\right| \leq d\left(c_1(t), c_2\left(t'\right)\right) \leq \delta,$$

  so
$$d\left(c_1(t), c_2(t)\right) \leq d\left(c_1(t), c_2\left(t'\right)\right) + d\left(c_2\left(t'\right), c_2(t)\right) \leq \delta + \left|t - t'\right| \leq 2\delta. \qquad \square$$

We next need the following result about quasi-convexity in hyperbolic groups. We follow the proofs of [Wil09].

Lemma 1.17. *Let Γ be a hyperbolic group with finite generating set S.*

(i) *Let H_1, H_2 be two subgroups of Γ that are quasi-convex as subsets of $\mathrm{Cay}(\Gamma, S)$. Then $H_1 \cap H_2$ is also quasi-convex.*

(ii) *If $g \in \Gamma$, then the centraliser $C(g)$ is quasi-convex in $\mathrm{Cay}(\Gamma, S)$.*

Proof. Let $\delta \geq 0$ such that $\mathrm{Cay}\left(\Gamma, S\right)$ is Rips-δ-hyperbolic.

(i) Let $k \geq 0$ be a quasi-convexity constant for both H_1 and H_2. It suffices to show that, for all $h \in H_1 \cap H_2$, vertices of $\mathrm{Cay}(\Gamma, S)$ on a geodesic segment $[e, h]$ remain at a bounded distance from $H_1 \cap H_2$. Write $h = s_1 \cdots s_\ell$ with $s_i \in S$ and $\ell = |h|_S$. Fix $1 \leq i \leq \ell$ and write $w_i = s_1 \cdots s_i$; we want to bound $d_S\left(w_i, H_1 \cap H_2\right)$. By quasi-convexity, there exist $h_1^{(0)} \in H_1$ and $h_2^{(0)} \in H_2$ such that

$$d_S\left(w_i, h_1^{(0)}\right), d_S\left(w_i, h_2^{(0)}\right) \leq k.$$

Let $\mu \in H_1 \cap H_2$ minimising the distance to w_i; write $\mu = w_i \sigma_1 \cdots \sigma_D$ with $\sigma_i \in S$ and $D = d_S\left(w_i, H_1 \cap H_2\right)$. For each $0 \leq j \leq D$, let $\mu_j = w_i \sigma_1 \cdots \sigma_j$. Since the geodesic triangle with vertices $w_i, \mu, h_1^{(0)}$ is δ-thin, there exist $p \in \left[w_i, h_1^{(0)}\right] \cup \left[h_1^{(0)}, \mu\right]$ such that $d_S\left(\mu_j, p\right) \leq \delta$. If $p \in \left[h_1^{(0)}, \mu\right]$, then by quasi-convexity of H_1, there exists

$h_1^{(j)} \in H_1$ such that $d_S\left(p, h_1^{(j)}\right) \leq k$; if $p \in \left[w_i, h_1^{(0)}\right]$, we may just take $h_1^{(j)} = h_1^{(0)}$ and still have $d_S\left(p, h_1^{(j)}\right) \leq k$. Hence $d_S\left(\mu_j, h_1^{(j)}\right) \leq k + \delta$.

We may perform the same construction in H_2 and thus obtain $h_1^{(0)}, \ldots, h_1^{(D)} \in H_1$ and $h_2^{(0)}, \ldots, h_2^{(D)} \in H_2$ with $d_S\left(\mu_j, h_1^{(j)}\right), d_S\left(\mu_j, h_2^{(j)}\right) \leq k + \delta$. Therefore, there exist $u_0, \ldots, u_D, v_0, \ldots, v_D \in \Gamma$ with $|u_j|_S, |v_j|_S \leq k + \delta$ such that

$$\mu_j = h_1^{(j)} u_j = h_2^{(j)} v_j.$$

Let B be the closed ball with centre e and radius $k + \delta$ in (Γ, d_S). Note that $u_0, \ldots, u_D, v_0, \ldots, v_D \in B$. Therefore if $D > |B|^2$, then there must exist $0 \leq j_1 < j_2 \leq D$ such that $u_{j_1} = u_{j_2}$ and $v_{j_1} = v_{j_2}$.

Consider $\nu = \mu_{j_1} \mu_{j_2}^{-1} \mu$; we claim that $\nu \in H_1 \cap H_2$ and $d_S(w_i, \nu) < d_S(w_i, \mu)$, contradicting the choice of μ. To prove this claim, note that

$$\nu = h_1^{(j_1)} u_{j_1} u_{j_2}^{-1} \left(h_1^{(j_2)}\right)^{-1} \mu = h_1^{(j_1)} \left(h_1^{(j_2)}\right)^{-1} \mu \in H_1,$$

and similarly $\nu \in H_2$. Moreover,

$$\begin{aligned} d_S(w_i, \nu) = d_S\left(\mu_0, \mu_{j_1} \mu_{j_2}^{-1} \mu_D\right) &\leq d_S(\mu_0, \mu_{j_1}) + d_S\left(\mu_{j_1}, \mu_{j_1} \mu_{j_2}^{-1} \mu_D\right) \\ &= d_S(\mu_0, \mu_{j_1}) + d_S(\mu_{j_2}, \mu_D) \\ &\leq j_1 + (D - j_2) < D = d_S(w_i, \mu), \end{aligned}$$

which proves the claim and yields the desired contradiction.

Therefore, $H_1 \cap H_2$ is $|B|^2$-quasi-convex.

(ii) We first claim that there is a nondecreasing function $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (depending only on Γ) such that, if two elements $u, v \in \Gamma$ are conjugate, then there exists $\rho \in \Gamma$ with

$$v = \rho u \rho^{-1} \qquad \text{and} \qquad |\rho|_S \leq M(|u|_S, |v|_S).$$

Indeed, since u, v are conjugate, we may write $v = \hat{\rho} u \hat{\rho}^{-1}$ for some $\hat{\rho} \in \Gamma$. Write $\hat{\rho} = s_1 \cdots s_\ell$ with $s_i \in S$ and $\ell = |\hat{\rho}|_S$ and set $\hat{\rho}_t = s_1 \cdots s_t$. Consider also a geodesic segment c from e to $\hat{\rho} u = v \hat{\rho}$. Lemma 1.16 implies that

$$\begin{aligned} d_S(\hat{\rho}_t, c(t)) &\leq 2(\delta + d_S(\hat{\rho}, \hat{\rho} u)) = 2(\delta + |u|_S), \\ d_S(v \hat{\rho}_t, c(|\hat{\rho} u|_S - \ell + t)) &\leq 2(\delta + d_S(e, v)) = 2(\delta + |v|_S). \end{aligned}$$

Therefore,

$$\begin{aligned} d_S(\hat{\rho}_t, v \hat{\rho}_t) &\leq 4\delta + 2(|u|_S + |v|_S) + d_S(c(t), c(|\hat{\rho} u|_S - \ell + t)) \\ &= 4\delta + 2(|u|_S + |v|_S) + |\hat{\rho} u|_S - \ell \\ &\leq 4\delta + 2(|u|_S + |v|_S) + |u|_S = R(|u|_S, |v|_S), \end{aligned}$$

with $R : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing. Hence, $\left|\hat{\rho}_t^{-1} v \hat{\rho}_t\right|_S \leq R(|u|_S, |v|_S)$ for all $t \leq |\hat{\rho}|_S$. Let $M(|u|_S, |v|_S)$ be the size of the (closed) ball centred at e with radius $R(|u|_S, |v|_S)$ in (Γ, S). The above implies that, if $|\hat{\rho}|_S > M(|u|_S, |v|_S)$, then there must exist $0 \leq t_1 < t_2 \leq \ell$ such that $\hat{\rho}_{t_1}^{-1} v \hat{\rho}_{t_1} = \hat{\rho}_{t_2}^{-1} v \hat{\rho}_{t_2}$, or in other words

$$v = \left(\hat{\rho}_{t_1} \hat{\rho}_{t_2}^{-1} \hat{\rho}\right) u \left(\hat{\rho}_{t_1} \hat{\rho}_{t_2}^{-1} \hat{\rho}\right)^{-1} = (s_1 \cdots s_{t_1} s_{t_2+1} \cdots s_\ell)\, u\, (s_1 \cdots s_{t_1} s_{t_2+1} \cdots s_\ell)^{-1}.$$

We can therefore make $\hat{\rho}$ shorter by replacing it by $(s_1 \cdots s_{t_1} s_{t_2+1} \cdots s_\ell)$. Iterating this process proves the claim.

Now let $h \in C(g)$. We want to prove that every element $h_0 \in \Gamma \cap [e, h]$ is at a bounded distance from $C(g)$. The same argument as in the proof of the claim yields $\left|h_0^{-1} g h_0\right|_S \leq R_1(g)$, where $R_1(g)$ is a constant depending only on g. Since g and $h_0^{-1} g h_0$ are conjugate, the claim also implies the existence of $\rho \in \Gamma$ with $h_0^{-1} g h_0 = \rho^{-1} g \rho$ and

$$|\rho|_S \leq M\left(|g|_S, \left|h_0^{-1} g h_0\right|_S\right) \leq M\left(|g|_S, R_1(g)\right) = R_2(g).$$

The equality $h_0^{-1} g h_0 = \rho^{-1} g \rho$ means that $h_0 \rho^{-1} \in C(g)$, so that

$$d_S\left(h_0, C(g)\right) \leq d_S\left(h_0, h_0 \rho^{-1}\right) = |\rho|_S \leq R_2(g). \qquad \square$$

We now obtain the fact that infinite cyclic subgroups of hyperbolic groups are quasi-isometrically embedded.

Proposition 1.18. *Let Γ be a hyperbolic group with finite generating set S and let $g \in \Gamma$ have infinite order.*

(i) *The inclusion $\left(\langle g \rangle, d_{\{g\}}\right) \hookrightarrow (\Gamma, d_S)$ is a quasi-isometric embedding.*

(ii) *The map $c : \mathbb{R} \to \mathrm{Cay}(\Gamma, S)$ given by $t \mapsto g^{\lfloor t \rfloor}$ is a quasi-geodesic.*

(iii) *There exists a constant K such that in $\mathrm{Cay}(\Gamma, S)$, for all $m \geq 1$,*

$$\mathscr{D}_H\left([e, g^m], \left\{g^i,\ 0 \leq i \leq m\right\}\right) \leq K,$$

where $[e, g^m]$ is a geodesic segment between e and g^m, and $\mathscr{D}_H$ is the Hausdorff distance.

Proof. (i) By Lemma 1.17, $C(g)$ is quasi-convex in $\mathrm{Cay}(\Gamma, S)$. In particular, $C(g)$ has a finite generating set T (by Lemma 1.15). Now

$$Z\left(C(g)\right) = C(g) \cap \bigcap_{t \in T} C(t).$$

It follows from Lemma 1.17 that $Z\left(C(g)\right)$ is quasi-convex in $\mathrm{Cay}(\Gamma, S)$. Therefore, $Z\left(C(g)\right)$ is quasi-isometrically embedded in Γ. In particular, $Z\left(C(g)\right)$ is finitely generated, abelian and hyperbolic. The Classification Theorem for finitely generated abelian groups implies that $Z\left(C(g)\right)$ is virtually cyclic (because hyperbolicity prevents it from being quasi-isometric to $\mathbb{Z}^d$ for $d \geq 2$), so $\langle g \rangle$ has finite index and therefore is quasi-isometrically embedded in $Z\left(C(g)\right)$, and thus in Γ.

(ii) This is a reformulation of (i).

(iii) This follows from (ii) and the stability of geodesics (Theorem 1.8). $\square$

The following argument of [Sho91] now leads us to the desired result on centralisers in hyperbolic groups.

Theorem 1.19. *Let Γ be a hyperbolic group and let $g \in \Gamma$ have infinite order. Then $\langle g \rangle$ has finite index in its centraliser $C(g)$.*

Proof. Fix a finite generating set S for Γ, and let $\delta \geq 0$ such that $\mathrm{Cay}(\Gamma, S)$ is Rips-δ-hyperbolic. Let K be the constant given by Proposition 1.18.(iii).

We are going to prove that, for all $h \in C(g)$, the coset $h\langle g\rangle$ intersects the closed ball $\overline{B}(e, 2K+2\delta)$ in $\mathrm{Cay}(\Gamma, S)$. The result will follow because this ball has a finite number of vertices.

Let $h \in C(g)$. Since the inclusion $\langle g\rangle \hookrightarrow \Gamma$ is quasi-isometric (by Proposition 1.18), we may choose $m \geq 1$ such that $|g^m|_S > 4|h|_S + 4\delta$. Now consider two geodesic triangles with vertices e, g^m, hg^m and e, h, hg^m respectively. Let u be the midpoint of $[e, g^m]$. Since the triangle with vertices e, g^m, hg^m is δ-thin, there exists $v \in [e, hg^m] \cup [hg^m, g^m]$ such that $d(u, v) \leq \delta$. If $v \in [g^m, hg^m]$, then

$$\begin{aligned}\frac{1}{2}|g^m|_S &= d(g^m, u) \leq d(g^m, v) + d(v, u)\\ &\leq d(g^m, hg^m) + d(u, v) = d(g^m, g^m h) + d(u, v)\\ &\leq |h|_S + \delta < \frac{1}{4}|g^m|_S,\end{aligned}$$

a contradiction. Hence, $v \in [e, hg^m]$ and $d(e, v) > \delta + 2|h|_S$. Again, the triangle with vertices e, h, hg^m is δ-thin, so there exists $w \in [e, h] \cup [h, hg^m]$ such that $d(v, w) \leq \delta$. If $w \in [e, h]$, we deduce a contradiction as before. It follows that $w \in [h, hg^m]$.

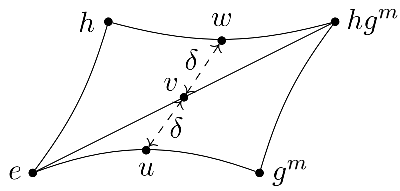

Figure 6: In hyperbolic spaces, rectangles are also thin

Hence, we have $u \in [e, g^m]$ and $w \in [h, hg^m]$ such that $d(u, w) \leq 2\delta$. But Proposition 1.18 implies the existence of integers $0 \leq i, j \leq m$ such that $d(u, g^i), d(h^{-1}w, g^j) \leq K$. Therefore

$$\left|hg^{i-j}\right|_S = d\left(g^i, hg^j\right) \leq d\left(g^i, u\right) + d(u, w) + d\left(w, hg^j\right) \leq 2K + 2\delta,$$

i.e. $hg^{i-j} \in h\langle g\rangle \cap \overline{B}(e, 2K+2\delta)$ as wanted. □

Corollary 1.20. *Any torsion-free abelian subgroup of a hyperbolic group is cyclic.*

Proof. Let Γ be a hyperbolic group and $A \leq \Gamma$ be torsion-free and abelian; we may assume that A is non-trivial. We pick $\gamma \in A \setminus \{e\}$. Then $A \subseteq C(\gamma)$, so Theorem 1.19 implies that

$$[A : \langle\gamma\rangle] \leq [C(\gamma) : \langle\gamma\rangle] < \infty.$$

In particular, A is finitely generated, abelian, and torsion-free, so $A \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$. But A is virtually cyclic, so $r \leq 1$, and A is cyclic. □

We end this chapter by giving a much stronger result for later use.

Theorem 1.21 (Tits alternative)**.** *Let Γ_1 be a subgroup of a hyperbolic group. Then one of the following assertions holds:*

(i) Γ_1 *is virtually cyclic.*

(ii) Γ_1 *contains a free subgroup of rank* 2*.*

Proof. See [GdlH90, 8.37]. $\square$

Chapter 2

Hausdorff-Gromov convergence

The main aim of this essay will be to prove that, if a group Γ is hyperbolic and has enough automorphisms, then not only does Γ act on a hyperbolic space (namely its Cayley graph), but we can actually strengthen this to obtain an action (though with weaker properties) on a 0-hyperbolic space, or in other words a real tree. This real tree will be constructed as a limit of Cayley graphs under a process called **Hausdorff-Gromov convergence**. We now develop this tool, with the aim of obtaining a compactness criterion, i.e. a way to get a convergent subsequence from a 'bounded' sequence of spaces.

2.1 Hausdorff-Gromov distance

The study of hyperbolic metric spaces has already led us to encounter the **Hausdorff distance**: given a metric space X and two subsets $A, B \subseteq X$, their Hausdorff distance is defined by

$$\mathscr{D}_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

This may not define a metric on the power set of X; for instance $\mathscr{D}_H\left(A, \overline{A}\right) = 0$ for all $A \subseteq X$. However, if we consider the set $\mathcal{C}(X)$ of closed subsets of X, then $\mathscr{D}_H$ is a metric on $\mathcal{C}(X)$.

The Hausdorff distance has the following compactness criterion, which we do not prove here because the proof is a slightly simplified version of the compactness criterion that we will prove for the Hausdorff-Gromov distance.

Proposition 2.1. *Let X be a compact metric space. Then the space $\mathcal{C}(X)$ of closed subsets of X is compact when equipped with the Hausdorff distance $\mathscr{D}_H$.*

Proof. See [BS94, 1.2]. □

The Hausdorff distance is useful to compare spaces living in a common ambient space X. However, we would like a notion of distance allowing us to compare abstract spaces which may not have any relation with one another; that is why we introduce the Hausdorff-Gromov distance.

Let A, B be two metric spaces. For $\varepsilon \geq 0$, an **ε-approximation** between A and B is a relation $R \subseteq A \times B$ such that

(i) The projection maps $R \to A$ and $R \to B$ are surjective,

(ii) For all $(a,b), (a',b') \in R$, we have $|d_A(a,a') - d_B(b,b')| \leq \varepsilon$.

We also write aRb for $(a,b) \in R$. If there is an ε-approximation between A and B, then we write $A \sim_\varepsilon B$. An ε-approximation can be understood as a relation $R \subseteq A \times B$ which looks approximately like the 'diagonal', even though there is no actual diagonal if $A \neq B$.

The **Hausdorff-Gromov distance** between A and B is defined by

$$\mathscr{D}_{HG}(A,B) = \inf \{\varepsilon \geq 0,\ A \sim_\varepsilon B\}.$$

We say that a sequence $(A_n)_{n\geq 1}$ of metric spaces converges to A in the Hausdorff-Gromov topology, and we write $A_n \xrightarrow[n\to\infty]{HG} A$, if $\mathscr{D}_{HG}(A_n, A) \xrightarrow[n\to\infty]{} 0$.

The following example shows that the Hausdorff-Gromov distance forgets irrelevant embedding information and only takes into account the isometry type of metric spaces.

Example. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers with $a_n \leq b_n$ for all n.*

(i) *If $a_n \xrightarrow[n\to\infty]{} a$ and $b_n \xrightarrow[n\to\infty]{} b$, then $\mathscr{D}_H([a_n,b_n],[a,b]) \xrightarrow[n\to\infty]{} 0$ in $\mathbb{R}$.*

(ii) *If $|b_n - a_n| \xrightarrow[n\to\infty]{} \ell$, then $\mathscr{D}_{HG}([a_n,b_n],[0,\ell]) \xrightarrow[n\to\infty]{} 0$.*

We now give a separation property for the Hausdorff-Gromov topology. Its proof (following an argument of [BS94]) hints at the techniques that will be used to construct limiting spaces in the Hausdorff-Gromov Topology.

Proposition 2.2. *Let A and B be two compact metric spaces. Then A and B are isometric if and only if $\mathscr{D}_{HG}(A,B) = 0$.*

Proof. ($\Rightarrow$) If there is an isometry $j : A \xrightarrow{\cong} B$, define $R = \{(a, j(a)),\ a \in A\} \subseteq A \times B$. Then R is a 0-approximation between A and B, so $\mathscr{D}_{HG}(A,B) = 0$.

($\Leftarrow$) Assume that $\mathscr{D}_{HG}(A,B) = 0$. Let $(a_m)_{m\geq 1}$ be a dense sequence in A. For $n \geq 1$, there exists a $\frac{1}{n}$-approximation $R_n \subseteq A \times B$ between A and B. By definition, for $m, n \geq 1$, there exists $b_{m,n} \in B$ such that $a_m R_n b_{m,n}$. Since B is compact, there is a subsequence $\left(b_{1,\phi_1(n)}\right)_{n\geq 1}$ that converges to $b_{1,\infty}$ as $n \to \infty$. Likewise, we may construct $\phi_1, \phi_2, \ldots$ such that

$$b_{m,\phi_1\cdots\phi_m(n)} \xrightarrow[n\to\infty]{} b_{m,\infty}.$$

Hence, for $m \leq m'$,

$$\begin{aligned}
&\left|d_B\left(b_{m,\infty}, b_{m',\infty}\right) - d_A\left(a_m, a_{m'}\right)\right| \\
&\qquad = \lim_{n\to\infty} \left|d_B\left(b_{m,\phi_1\cdots\phi_{m'}(n)}, b_{m',\phi_1\cdots\phi_{m'}(n)}\right) - d_A\left(a_m, a_{m'}\right)\right| \\
&\qquad \leq \limsup_{n\to\infty} \frac{1}{\phi_1\cdots\phi_{m'}(n)} = 0.
\end{aligned}$$

Therefore, the unique extension $j : A \to B$ of $a_m \mapsto b_{m,\infty}$ is an isometric embedding. By symmetry, there is also an isometric embedding $i : B \to A$. Hence, $j \circ i : B \to B$ is an isometric embedding. Since B is compact, $j \circ i$ is onto, which implies that j is onto (and therefore an isometry). $\square$

We close this section with a result, stated without proof in [BS94], that will not be strictly necessary, but that sheds light on the concept of Hausdorff-Gromov convergence and its relation with the Hausdorff distance.

Proposition 2.3. *Let A and B be two compact metric spaces. We define*

$$\mathscr{D}_h(A,B) = \inf_{A,B \hookrightarrow X} \mathscr{D}_H(A,B),$$

where the infimum is taken over all compact metric spaces X containing isometric copies of A and B, and $\mathscr{D}_H$ is the Hausdorff distance in X. Then

$$\mathscr{D}_{HG}(A,B) = 2\mathscr{D}_h(A,B).$$

Proof. ($\leq$) Let $\varepsilon > \mathscr{D}_h(A,B)$; we will show that $2\varepsilon \geq \mathscr{D}_{HG}(A,B)$. Since $\varepsilon > \mathscr{D}_h(A,B)$, there exists a compact metric space X containing isometric copies of A and B such that $\mathscr{D}_H(A,B) < \varepsilon$ in $\mathcal{C}(X)$. Now consider

$$R = \{(a,b) \in A \times B,\ d_X(a,b) < \varepsilon\}.$$

Since $\mathscr{D}_H(A,B) < \varepsilon$, the projections $R \to A$ and $R \to B$ are onto; moreover, for $(a,b),(a',b') \in R$, we have

$$|d_A(a,a') - d_B(b,b')| = |d_X(a,a') - d_X(b,b')| \leq d_X(a,b) + d_X(a',b') < 2\varepsilon.$$

Hence, R is a 2ε-approximation between A and B, so $2\varepsilon \geq \mathscr{D}_{HG}(A,B)$.

($\geq$) Let $\varepsilon > \mathscr{D}_{HG}(A,B)$; we will show that $\frac{\varepsilon}{2} \geq \mathscr{D}_h(A,B)$. Since $\varepsilon > \mathscr{D}_{HG}(A,B)$, there exists an ε-approximation $R \subseteq A \times B$ between A and B. We construct a metric d_X on the metric space $X = A \amalg B$ by setting

$$d_X(a,b) = \min_{(\alpha,\beta)\in\overline{R}} \left(d_A(a,\alpha) + \frac{\varepsilon}{2} + d_B(\beta,b) \right),$$

for all $(a,b) \in A \times B$ (with $d_{X|A\times A} = d_A$ and $d_{X|B\times B} = d_B$). To show that d_X defines a metric on X, the only non-obvious fact is the triangle inequality: if, say, $a_1, a_2 \in A$ and $b \in B$, there exist $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \overline{R}$ such that $d_X(a_i,b) = d_A(a_i,\alpha_i) + \frac{\varepsilon}{2} + d_B(\beta_i,b)$. Now

$$\begin{aligned}
d_X(a_1,a_2) = d_A(a_1,a_2) &\leq d_A(a_1,\alpha_1) + d_A(\alpha_1,\alpha_2) + d_A(\alpha_2,a_2) \\
&\leq d_B(\beta_1,\beta_2) + |d_A(\alpha_1,\alpha_2) - d_B(\beta_1,\beta_2)| + \sum_{i\in\{1,2\}} d_A(a_i,\alpha_i) \\
&\leq |d_A(\alpha_1,\alpha_2) - d_B(\beta_1,\beta_2)| + \sum_{i\in\{1,2\}} (d_A(a_i,\alpha_i) + d_B(\beta_i,b)) \\
&\leq \varepsilon + \sum_{i\in\{1,2\}} (d_A(a_i,\alpha_i) + d_B(\beta_i,b)) = d(a_1,b) + d(b,a_2).
\end{aligned}$$

Therefore, (X,d) is a metric space and the inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$ are isometric embeddings; and X is compact because $X = A \amalg B$. Thus,

$$\mathscr{D}_h(A,B) \leq \mathscr{D}_H(A,B) \leq \frac{\varepsilon}{2}. \qquad \square$$

2.2 Compactness criterion

We now prove our main result about the Hausdorff-Gromov topology, namely a criterion allowing us to extract convergent subsequences in certain cases. We follow the method of proof outlined in [BS94].

We say that a family $(A_i)_{i\in I}$ of compact metric spaces is **uniformly compact** if

(i) There exists $M \geq 0$ such that $\operatorname{diam} A_i \leq M$ for all $i \in I$,

(ii) For all $\varepsilon > 0$, there exists $N(\varepsilon) \geq 0$ such that, for all $i \in I$, A_i can be covered by $N(\varepsilon)$ open balls of radius ε.

Theorem 2.4. *Let $(A_p)_{p\geq 1}$ be a uniformly compact sequence of compact metric spaces. Then $(A_p)_{p\geq 1}$ has a subsequence which converges in the Hausdorff-Gromov topology.*

Proof. For $\varepsilon > 0$, by uniform compactness, there is $N(\varepsilon) \geq 0$ such that, for all $p \geq 1$, the space A_p can be covered by $N(\varepsilon)$ open balls of radius ε. Hence, there are points $(a_{n,p,j})_{1\leq j\leq N(1/n)}$ such that

$$A_p = \bigcup_{j=1}^{N(1/n)} B\left(a_{n,p,j}, \frac{1}{n}\right).$$

Moreover, we can choose $a_{n+1,p,j} = a_{n,p,j}$ for $j \leq N\left(\frac{1}{n}\right)$ (we can do this for example by increasing the number $N\left(\frac{1}{n+1}\right)$ of balls). We write $a_{p,j} = a_{n,p,j}$ for $j \leq N\left(\frac{1}{n}\right)$.

The sequence $\left(d_{A_p}\left(a_{p,1}, a_{p,2}\right)\right)_{p\geq 1}$ is bounded because the $(A_p)_{p\geq 1}$ have bounded diameter; we may therefore extract a convergent subsequence:

$$d_{A_{\psi_2(p)}}\left(a_{\psi_2(p),1}, a_{\psi_2(p),2}\right) \xrightarrow[p\to\infty]{} \Delta_{1,2}.$$

Likewise, the sequence $\left(d_{A_{\psi_2(p)}}\left(a_{\psi_2(p),1}, a_{\psi_2(p),3}\right), d_{A_{\psi_2(p)}}\left(a_{\psi_2(p),2}, a_{\psi_2(p),3}\right)\right)_{p\geq 1}$ lives in a bounded subset of $\mathbb{R}^2$, so it has a convergent subsequence. Iterating, we construct $\psi_2, \psi_3, \ldots$ such that, for all $1 \leq i \leq j$, we have

$$d_{A_{\psi_2\cdots\psi_j(p)}}\left(a_{\psi_2\cdots\psi_j(p),i}, a_{\psi_2\cdots\psi_j(p),j}\right) \xrightarrow[p\to\infty]{} \Delta_{i,j}.$$

Now take abstract symbols $a_{\infty,j}$ for $j \geq 1$ and consider the set $A_\infty = \{a_{\infty,j},\ j \geq 1\}/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation defined by $a_{\infty,i}\mathcal{R}a_{\infty,j}$ if $\Delta_{i,j} = 0$. Equip A_∞ with a metric structure defined by

$$d_{A_\infty}\left(a_{\infty,i}, a_{\infty,j}\right) = \Delta_{i,j}.$$

This defines a metric on A_∞. Let $\overline{A}_\infty$ be the completion of A_∞ for this metric.

Note that, if $n \geq 1$ is fixed, then for all $j \geq 1$ and for all $p \geq 1$, there exists $j' \leq N\left(\frac{1}{n}\right)$ such that

$$d_{A_p}\left(a_{p,j}, a_{p,j'}\right) \leq \frac{1}{n}.$$

After passing to the limit, the above result remains true when p is replaced by ∞.

Write $\phi(p) = \psi_2 \cdots \psi_p(p)$. We now claim that

$$A_{\phi(p)} \xrightarrow[p\to\infty]{HG} \overline{A}_\infty.$$

Let $n \geq 1$, let $N = N\left(\frac{1}{n}\right)$. Then there exists $p_0 \geq N$ such that, for all $p \geq p_0$ and for all $1 \leq i, j \leq N$, we have

$$\left| d_{A_{\psi_2 \cdots \psi_N(p)}}\left(a_{\psi_2 \cdots \psi_N(p),i}, a_{\psi_2 \cdots \psi_N(p),j}\right) - d_{\overline{A}_\infty}\left(a_{\infty,i}, a_{\infty,j}\right)\right| \leq \frac{1}{n}. \tag{$*$}$$

For $p \geq p_0$, we set

$$R = \left\{(x,y) \in A_{\phi(p)} \times \overline{A}_\infty,\ \exists j \leq N,\ d_{A_{\phi(p)}}\left(x, a_{\phi(p),j}\right) \leq \frac{2}{n} \text{ and } d_{\overline{A}_\infty}\left(a_{\infty,j}, y\right) \leq \frac{2}{n}\right\}.$$

We claim that R is a $\frac{9}{n}$-approximation; hence $\mathscr{D}_{HG}\left(A_{\phi(p)}, \overline{A}_\infty\right) \leq \frac{9}{n}$.

- The projection $R \to A_{\phi(p)}$ is onto: if $x \in A_{\phi(p)}$, then there exists $j \leq N\left(\frac{1}{n}\right)$ such that $d_{A_{\phi_p(p)}}\left(x, a_{\phi_p(p),j}\right) \leq \frac{1}{n}$; hence $xRa_{\infty,j}$.
- The projection $R \to \overline{A}_\infty$ is onto: if $y \in \overline{A}_\infty$, then since A_∞ is dense in $\overline{A}_\infty$, there exists $j \geq 1$ such that $d_{\overline{A}_\infty}\left(y, a_{\infty,j}\right) \leq \frac{1}{n}$. Now there exists $j' \leq N\left(\frac{1}{n}\right)$ such that $d_{\overline{A}_\infty}\left(a_{\infty,j}, a_{\infty,j'}\right) \leq \frac{1}{n}$, so that $d_{\overline{A}_\infty}\left(y, a_{\infty,j'}\right) \leq \frac{2}{n}$ and therefore $a_{\phi(p),j}Ry$.
- Finally, let $(x,y), (w,z) \in R$. By definition, there exist $i, j \leq N\left(\frac{1}{n}\right)$ such that

  $$d_{A_{\phi(p)}}\left(x, a_{\phi(p),i}\right), d_{A_{\phi(p)}}\left(w, a_{\phi(p),j}\right), d_{\overline{A}_\infty}\left(a_{\infty,i}, y\right), d_{\overline{A}_\infty}\left(a_{\infty,j}, z\right) \leq \frac{2}{n}.$$

  Therefore, by $(*)$,

  $$\begin{aligned}\left| d_{A_{\phi(p)}}(x,w) - d_{\overline{A}_\infty}(y,z)\right| \leq\ & d_{A_{\phi(p)}}\left(w, a_{\phi(p),j}\right) + d_{A_{\phi(p)}}\left(x, a_{\phi(p),i}\right) \\ & + \left| d_{A_{\phi(p)}}\left(a_{\phi(p),i}, a_{\phi(p),j}\right) - d_{\overline{A}_\infty}\left(a_{\infty,i}, a_{\infty,j}\right)\right| \\ & + d_{\overline{A}_\infty}\left(a_{\infty,i}, y\right) + d_{\overline{A}_\infty}\left(a_{\infty,j}, z\right) \\ \leq\ & \frac{9}{n}.\end{aligned}$$

  □

2.3 Convergence of hyperbolic spaces

Recalling that we are interested in making hyperbolic spaces shrink to real trees by Hausdorff-Gromov convergence, we need one more ingredient: we want to understand what happens when a sequence of hyperbolic spaces converges in the Hausdorff-Gromov topology, with hyperbolicity constant shrinking to zero. The following proposition will do that for us. It is also taken from [BS94], but we make the proof of the second statement slightly simpler by using Gromov's definition of hyperbolicity.

Proposition 2.5. *Let $(A_p)_{p \geq 1}$ and A be compact metric spaces such that*

$$A_p \xrightarrow[p \to \infty]{HG} A.$$

(i) *If A_p is geodesic for all p, then A is geodesic.*

(ii) *If we assume in addition that each A_p is Gromov-δ_p-hyperbolic (resp. Rips-δ_p-hyperbolic) with $\delta_p \xrightarrow[p \to \infty]{} 0$, then A is a real tree.*

Proof. (i) For $p \geq 1$, let $R_p \subseteq A_p \times A$ be an ε_p-approximation between A_p and A, with $\varepsilon_p \xrightarrow[p\to\infty]{} 0$. Let $x, y \in A$; we want to construct a geodesic segment from x to y. For $p \geq 1$, let $x_p, y_p \in A_p$ such that $x_p R_p x$ and $y_p R_p y$. Since A_p is geodesic, there exists a geodesic segment $\gamma_p : I_p \longrightarrow A_p$ from x_p to y_p, with $I_p = \left[0, d_{A_p}(x_p, y_p)\right]$. We have

$$I_p \xrightarrow[p\to\infty]{HG} I = [0, d_A(x, y)]$$

since $d_{A_p}(x_p, y_p) \xrightarrow[p\to\infty]{} d_A(x, y)$. Let $L_p = \{a \in A,\ \exists a_p \in \gamma_p(I_p),\ aR_p a_p\} \subseteq A$ and let $K_p = \overline{L_p} \subseteq A$. Then

$$\mathscr{D}_{HG}(K_p, I_p) = \mathscr{D}_{HG}(K_p, \gamma_p(I_p)) \leq \varepsilon_p \xrightarrow[p\to\infty]{} 0.$$

Now, using the compactness of $\mathcal{C}(A)$ (Proposition 2.1), we may assume that $(K_p)_{p\geq 1}$ converges to $K \subseteq A$. Hence

$$\mathscr{D}_{HG}(K, I) \leq \mathscr{D}_{HG}(K, K_p) + \mathscr{D}_{HG}(K_p, I_p) + \mathscr{D}_{HG}(I_p, I) \xrightarrow[p\to\infty]{} 0,$$

so K is isometric to I by Proposition 2.2. But note that $x, y \in \bigcap_{p\geq 1} L_p \subseteq K$, so the isometry $I = [0, d_A(x, y)] \to K \subseteq A$ gives a geodesic from x to y.

(ii) We assume first that each A_p is Gromov-δ_p-hyperbolic. Let $\omega, x, y, z \in A$. For $p \geq 1$, let $\omega_p, x_p, y_p, z_p \in A_p$ such that $\omega_p R_p \omega$, etc. Then $d_{A_p}(\omega_p, x_p) \xrightarrow[p\to\infty]{} d_A(\omega, x)$, etc. It follows that

$$(x_p \mid y_p)_{\omega_p} \xrightarrow[p\to\infty]{} (x \mid y)_\omega\,,$$

and this remains true when one replace (x, y) by (x, z) or (y, z); therefore

$$\begin{aligned}(x \mid y)_\omega &- \min\left\{(x \mid z)_\omega, (y \mid z)_\omega\right\} \\ &= \lim_{p\to\infty}\left((x_p \mid y_p)_{\omega_p} - \min\left\{(x_p \mid z_p)_\omega, (y_p \mid z_p)_{\omega_p}\right\}\right) \geq \liminf_{p\to\infty}(-\delta_p) = 0.\end{aligned}$$

This means that A is Gromov-0-hyperbolic, so it is a real tree.

If we assume instead that each A_p is Rips-δ_p-hyperbolic, then A_p is Gromov-$3\delta_p$-hyperbolic by Proposition 1.3, so the result follows from the above. □

Chapter 3

Paulin's Theorem and rigidity

With the tool of Hausdorff-Gromov convergence in hand, we are ready to show, following [BS94], that given a hyperbolic group with large automorphism group, we can extract a 'nice' action on a real tree. This is this chapter's first aim. We then proceed to show how the Rips machine can be applied to this context to obtain Paulin's characterisation of rigidity.

3.1 Construction of a limiting action on a real tree

Given a group Γ, we denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ and by $\mathrm{Inn}(\Gamma)$ the group of **inner automorphisms**, or automorphisms given by $\gamma \mapsto \vartheta^{-1}\gamma\vartheta$ for some $\vartheta \in \Gamma$. It is easy to check that $\mathrm{Inn}(\Gamma)$ is a normal subgroup of $\mathrm{Aut}(\Gamma)$, and we define the group of **outer automorphisms** of Γ by

$$\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\,\mathrm{Inn}(\Gamma).$$

We say that Γ is **rigid** if $\mathrm{Out}(\Gamma)$ is finite.

The first idea, expressed by the following lemma, is that, if a group is not rigid, then we may construct an infinite sequence of shrinking Cayley graphs with distorted distances.

Lemma 3.1. *Let Γ be a non-rigid group with finite generating set S; let $X = \mathrm{Cay}\,(\Gamma, S)$. Then there is a sequence $(\phi_n)_{n\geq 0}$ of automorphisms of Γ, a sequence $(x_n)_{n\geq 0}$ of vertices or midpoints of edges in the Cayley graph X and a generator $s_0 \in S$ such that:*

(i) *The $(\phi_n)_{n\geq 0}$ represent distinct outer automorphisms, none of them being inner.*

(ii) *For all $n \geq 0$, we have*

$$\inf_{x\in X} \max_{s\in S} d_X\left(x, \phi_n(s)x\right) = d_X\left(x_n, \phi_n\left(s_0\right) x_n\right).$$

The above quantity is denoted by λ_n.

(iii) *The sequence $(\lambda_n)_{n\geq 0}$ is strictly increasing and diverges to $+\infty$.*

In this context, we define X_n to be the metric Γ-space X with basepoint x_n, with metric $d_n = \frac{d_X}{\lambda_n}$ and where $\gamma \in \Gamma$ acts on X_n via $x \mapsto \phi_n(\gamma)x$.

Proof. Since $\mathrm{Out}(\Gamma)$ is infinite, we may choose a sequence $(\phi_n)_{n\geq 0}$ of automorphisms of Γ satisfying (i). For $n \geq 0$, we consider $f_n : X \to [0, +\infty)$ defined by

$$f_n(x) = \max_{s\in S} d_X\left(x, \phi_n(s)x\right).$$

We note that f_n takes non-negative integer values at vertices and endpoints of edges, and is linear on half-edges. Therefore, f_n has a minimum at some point $x_n \in X$, which is a vertex or the midpoint of an edge. We now have

$$\lambda_n = \max_{s \in S} d_X\left(x_n, \phi_n(s) x_n\right).$$

Hence, for each $n \geq 0$, there is a generator $s_n \in S$ such that

$$\lambda_n = d_X\left(x_n, \phi_n\left(s_n\right) x_n\right).$$

Since S is finite, we may assume by passing to a subsequence that $\left(s_n\right)_{n \geq 0}$ is constant, say $s_n = s_0 \in S$ for all n. This proves (ii).

We now claim that the sequence $\left(\lambda_n\right)_{n \geq 0}$ is unbounded. If this is true, then we may pass to a subsequence that is strictly increasing and diverges to $+\infty$. Hence, up to the extraction of a subsequence, (iii) will be satisfied.

To prove the claim, assume for contradiction that there is a uniform bound $M \geq 0$ for $\left(\lambda_n\right)_{n \geq 0}$. For $n \geq 0$, let $y_n \in \Gamma \subseteq X$ be a vertex of the Cayley graph minimising the distance to x_n (so that $d_X\left(x_n, y_n\right) \leq \frac{1}{2}$). For $s \in S$, we have

$$\begin{aligned} \left|y_n^{-1} \phi_n(s) y_n\right|_S &= d_X\left(y_n, \phi_n(s) y_n\right) \\ &\leq d_X\left(y_n, x_n\right) + d_X\left(x_n, \phi_n(s) x_n\right) + d_X\left(\phi_n(s) x_n, \phi_n(s) y_n\right) \\ &\leq 1 + \lambda_n \leq M + 1. \end{aligned}$$

But the ball of centre e and radius $M+1$ in Γ is finite, and so is S, so there must exist $m \neq n$ such that

$$y_m^{-1} \phi_m(s) y_m = y_n^{-1} \phi_n(s) y_n$$

for all $s \in S$, and therefore $y_m^{-1} \phi_m y_m = y_n^{-1} \phi_n y_n$, contradicting the fact that ϕ_m and ϕ_n represent distinct outer automorphisms. $\square$

We now combine the above lemma with the compactness criterion for the Hausdorff-Gromov topology to obtain an action on a real tree.

Proposition 3.2. *Let Γ be a non-rigid hyperbolic group with finite generating set S; let $X = \operatorname{Cay}(\Gamma, S)$. Then the sequence $\left(X_n\right)_{n \geq 0}$ of Lemma 3.1 has a subsequence which converges in the Hausdorff-Gromov topology to a real tree X_∞ that can be equipped with an action of Γ by isometries.*

Proof. Let $\delta \geq 0$ such that X is Rips-δ-hyperbolic.

Step 1. We choose an exhaustion

$$\{e\} = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_m \subseteq \cdots$$

of Γ by finite subsets. For $n \geq 0$, we define inductively a sequence $\left(Q_{m,n}\right)_{m \geq 0}$ of subsets of X_n by $Q_{0,n} = \{x_n\}$, and $Q_{m,n}$ is obtained from $Q_{m-1,n}$ by adding a choice of geodesic segments from x_n to each element of $\{\phi_n(\gamma) x_n,\ \gamma \in P_m \setminus P_{m-1}\}$. We denote by $W_{m,n}$ the closed δ-neighbourhood of $Q_{m,n}$ in X_n (for the metric d_X) and $d_{m,n}$ the path metric induced by d_X on $W_{m,n}$.

We claim that, for all $x, y \in W_{m,n}$, the following inequalities hold:

$$d_X(x, y) \leq d_{m,n}(x, y) \leq d_X(x, y) + 4\delta. \tag{$\star$}$$

The first inequality is simply a consequence of the general fact that the induced metric on a connected subspace is no greater than the induced path metric.

For the second one, we first note that, by hyperbolicity, any geodesic segment in X_n joining two points of $Q_{m,n}$ must lie within $W_{m,n}$. Now let $x, y \in W_{m,n}$. Choose points $x', y' \in Q_{m,n}$ minimising the distance to x, y respectively. Choose geodesic segments $[x, x'], [x', y'], [y', y]$. Each of those is contained in $W_{m,n}$, and therefore so is their concatenation. But the latter is a path from x to y in $W_{m,n}$ of length at most $d_X(x', y') + 2\delta \leq d_X(x, y) + 4\delta$, from which the second inequality follows.

Step 2. The inequalities $(\star)$ imply that the inclusion $(W_{m,n}, d_{m,n}) \hookrightarrow (X, d_X)$ is a $(1, 4\delta)$-quasi-isometric embedding. It follows from Corollary 1.10 that $(W_{m,n}, d_{m,n})$ is Rips-η-hyperbolic, where η depends on δ only.

Moreover, we claim that for fixed $m \geq 0$, the spaces $\left(W_{m,n}, \frac{d_{m,n}}{\lambda_n}\right)_{n\geq 0}$ are uniformly compact.

To see this, let

$$\mu_m = \max_{\gamma \in P_m} |\gamma|_S .$$

Note that, for $\gamma \in P_m$, if $\gamma = s_1 \cdots s_\ell$, where $s_1, \ldots, s_\ell \in S$ and $\ell = |\gamma|_S$, then we have

$$d_X(x_n, \phi_n(\gamma)x_n) \leq \sum_{i=0}^{\ell-1} d_X(\phi_n(s_1 \cdots s_i)x_n, \phi_n(s_1 \cdots s_{i+1})x_n) \leq \ell\lambda_n \leq \mu_m\lambda_n.$$

In other words, the geodesic segments used to construct $Q_{m,n}$ have length at most $\mu_m\lambda_n$. This implies that, for any $\varepsilon > 0$, we can cover $Q_{m,n}$ by $\frac{1}{\varepsilon}|P_m|\mu_m$ segments of length $\lambda_n\varepsilon$ (for the metric d_X). Hence, $W_{m,n}$ can be covered by $\frac{1}{\varepsilon}|P_m|\mu_m$ balls of radius $\lambda_n\varepsilon + \delta$ for d_X. Since $\lambda_n \xrightarrow[n\to\infty]{} \infty$, we have $\lambda_n\varepsilon > 2\delta$ for n large enough. But by $(\star)$, the metrics $\frac{d_X}{\lambda_n}$ and $\frac{d_{m,n}}{\lambda_n}$ only differ by an additive constant $\frac{4\delta}{\lambda_n} < 2\varepsilon$, which shows that $\left(W_{m,n}, \frac{d_{m,n}}{\lambda_n}\right)$ can be covered by $\frac{1}{\varepsilon}|P_m|\mu_m$ balls of radius 4ε. Since the spaces $\left(W_{m,n}, \frac{d_{m,n}}{\lambda_n}\right)_{n\geq 0}$ are geodesic, it follows that they have diameter bounded by $4|P_m|\mu_m$, which completes the proof of uniform compactness.

Step 3. For all $m \geq 0$, the spaces $\left(W_{m,n}, \frac{d_{m,n}}{\lambda_n}\right)_{n\geq 0}$ are geodesic, $\frac{\eta}{\lambda_n}$-hyperbolic and uniformly compact. By Theorem 2.4, we may assume by passing to a subsequence and using a diagonal argument that $\left(W_{m,n}, \frac{d_{m,n}}{\lambda_n}\right)_{n\geq 0}$ converges in the Hausdorff-Gromov topology to $(W_{m,\infty}, d_m)$. By Proposition 2.5, $(W_{m,\infty}, d_m)$ is a real tree. The construction of the limit $W_{m,\infty}$ in the proof of Theorem 2.4 shows that we may assume that there are points $\gamma x_\infty \in W_{m,\infty}$ for all $\gamma \in P_m$ such that

$$\frac{1}{\lambda_n} d_{m,n}(\phi_n(\gamma)x_n, \phi_n(\gamma')x_n) \xrightarrow[n\to\infty]{} d_m(\gamma x_\infty, \gamma' x_\infty).$$

Note that by $(\star)$, $d_{m,n}(\phi_n(\gamma)x_n, \phi_n(\gamma')x_n)$ is 4δ-close to $d_X(\phi_n(\gamma)x_n, \phi_n(\gamma')x_n)$, which is independent of m, so $d_m(\gamma x_\infty, \gamma' x_\infty)$ does not depend on m. The inequalities $(\star)$ also imply that $\mathscr{D}_{HG}(W_{m,n}, Q_{m,n}) \leq \frac{4\delta+\delta}{\lambda_n}$, from which it follows that

$$\left(Q_{m,n}, \frac{d_X}{\lambda_n}\right) \xrightarrow[n\to\infty]{HG} (W_{m,\infty}, d_m).$$

Since $Q_{m,n}$ is the convex hull of $\{\phi_n(\gamma)x_n,\ \gamma \in P_m\}$, we deduce that $W_{m,\infty}$ is the convex hull of $\{\gamma x_\infty,\ \gamma \in P_m\}$. We can therefore use the fact that $d_m(\gamma x_\infty, \gamma' x_\infty)$ does not depend on m to isometrically embed $W_{m,\infty} \hookrightarrow W_{m+1,\infty}$ for all m. Now define

$$X_\infty = \bigcup_{m \geq 0} W_{m,\infty}.$$

The metric space X_∞ is a real tree (as a nondecreasing union of real trees). Moreover, Γ acts by isometries on the set $\{\gamma x_\infty,\ \gamma \in \Gamma\}$, and since the convex hull of this set is X_∞, this extends to an isometric action of Γ on X_∞. □

Having constructed an action on a real tree, we wish to investigate the group-theoretical properties of this action. We reach this essay's main theorem.

Theorem 3.3. *Let Γ be a non-rigid hyperbolic group. Then Γ acts by isometries on a real tree X_∞ with the following properties:*

(i) *There is no point of X_∞ fixed by all elements of Γ.*

(ii) *The stabiliser of every non-trivial segment in X_∞ is virtually cyclic.*

Proof. We prove that the action of Γ on the real tree X_∞ of Proposition 3.2 satisfies the desired properties.

(i) Assume for contradiction that there is a point $\omega_\infty \in X_\infty$ such that $\gamma\omega_\infty = \omega_\infty$ for all $\gamma \in \Gamma$. Since X_∞ was constructed as the convex hull of $\{\gamma x_\infty,\ \gamma \in \Gamma\}$, there exist $\gamma, \gamma' \in \Gamma$ and $m \geq 0$ such that $\omega_\infty \in [\gamma x_\infty, \gamma' x_\infty] \subseteq W_{m,\infty}$. Up to the extraction of a subsequence, the geodesic segments $[\phi_n(\gamma)x_n, \phi_n(\gamma')x_n] \subseteq W_{m,n}$ converge for the Hausdorff-Gromov topology to $[\gamma x_\infty, \gamma' x_\infty]$ as $n \to \infty$, and we may choose points $\omega_n \in [\phi_n(\gamma)x_n, \phi_n(\gamma')x_n]$ that converge to ω_∞. We may assume by choosing m, m' large enough that $S \subseteq P_m$ and $P_m P_m \subseteq P_{m'}$. Now, for every generator $s \in S$, the geodesic segments $s[\phi_n(\gamma)x_n, \phi_n(\gamma')x_n]$ in $W_{m',n}$ converge to $s[\gamma x_\infty, \gamma' x_\infty]$ in $W_{m',\infty}$. Since $\phi_n(s)\omega_n$ converges to $s\omega_\infty = \omega_\infty$ for all $s \in S$, it follows that $d_{X_n}(\phi_n(s)\omega_n, \omega_n) < \frac{1}{4}$ for n large enough, so $d_X(\phi_n(s)\omega_n, \omega_n) < \frac{\lambda_n}{4}$. This contradicts the definition of λ_n.

(ii) Consider a subgroup H of Γ stabilising a segment η with endpoints u, v in X_∞. Then $H_0 = \{\gamma \in H,\ \gamma u = u \text{ and } \gamma v = v\}$ fixes η pointwise, and it has index at most 2 in H. Since H_0 is virtually cyclic if and only if H is, we may replace H by H_0 and assume that H fixes η pointwise. We write $D = d_{X_\infty}(u, v)$.

We claim that the set of commutators $\{[g,h],\ g, h \in H\} \subseteq [H,H]$ is finite. The same argument will apply to any subgroup of H, and therefore H cannot contain a non-abelian free subgroup. Theorem 1.21 will imply that H is virtually cyclic.

To prove the claim, fix $\varepsilon > 0$ and let $g, h \in H$. Let P be the set of all finite products of length at most 4 in g, h, g^{-1}, h^{-1}. By construction of X_∞, there is a P-equivariant ε-approximation R_n between η and a compact subset of X_n for n sufficiently large. Choose points u_n, v_n such that $u_n R_n u$ and $v_n R_n v$. For $p \in P$, we have

$$d_{X_n}(\varphi_n(p)u_n, u_n) \leq \varepsilon + d_{X_\infty}(pu, u) = \varepsilon,$$

and similarly for v_n. Moreover, $d_{X_n}(u_n, v_n) \geq D - \varepsilon$.

The idea is to use the thinness of the rectangle with vertices $u_n, v_n, \varphi_n(p)v_n, \varphi_n(p)u_n$ to show that, for a in some non-empty subsegment θ_n of $[u_n, v_n]$, there exists $p_*a \in [u_n, v_n]$ such that
$$d_X\left(\varphi_n(p)a, p_*a\right) \leq 2\delta.$$
It follows in particular that, for all $a, b \in \theta_n$,
$$\left|d_X\left(p_*a, p_*b\right) - d_X\left(a, b\right)\right| \leq 4\delta.$$
Viewing $\theta_n \subseteq [u_n, v_n]$ as a subsegment of $\mathbb{R}$, the above inequality implies that $p_* : \theta_n \to [u_n, v_n]$ is a quasi-geodesic. Now the stability of geodesics (Theorem 1.8) implies that there is a constant $\kappa \geq 0$ and an isometry τ_p of the geodesic line extending $[u_n, v_n]$ such that
$$\mathscr{D}_H\left(p_*\theta_n, \tau_p\theta_n\right) \leq \kappa.$$
We may assume that τ_p is a translation (because we can compose it with a reflection without changing its image). Therefore, after possibly restricting θ_n to make sure that there are points of θ_n for which we can apply four successive maps of the form p_*, the composite $g_*h_*g_*^{-1}h_*^{-1}$ is at a bounded distance from $\tau_g\tau_h\tau_g^{-1}\tau_h^{-1} = \mathrm{id}$. In addition, it is at a bounded distance from $\left(ghg^{-1}h^{-1}\right)_*$, which is itself at a bounded distance from $\varphi_n\left(ghg^{-1}h^{-1}\right)$. It follows that $\varphi_n\left(ghg^{-1}h^{-1}\right)$ only moves some points of θ_n by a bounded distance, so the set $\{\varphi_n([g,h]),\ g, h \in H\}$ is finite because the action of Γ on the Cayley graph X is free, and hence the set $\{[g,h],\ g, h \in H\}$ of commutators is also finite. We refer the reader to [BS94] for more details. $\square$

3.2 Paulin's Theorem

Given a non-rigid hyperbolic group Γ, Theorem 3.3 provides us with an action of Γ on a real tree X_∞ with specific properties. We now make use of theoretical tools developped by Rips which analyse the algebraic structure of groups acting on real trees. More precisely, we shall use without proof the following result.

Theorem 3.4 (Rips)**.** *Let Γ be a hyperbolic group acting on a real tree with no global fixed point and virtually cyclic segment stabilisers. Then Γ splits as an amalgamated free product or HNN extension over a virtually cyclic subgroup.*

Proof. See [BF95, 1.1]. $\square$

We now restrict our attention to the case of torsion-free groups for simplicity. We first prove that, in this context, we can replace virtual cyclicity by cyclicity. Towards this aim, we will need the following theorem.

Theorem 3.5 (Schur)**.** *Let Γ be a group whose centre $Z(\Gamma)$ has finite index. Then the derived subgroup $\Gamma' = [\Gamma, \Gamma]$ is finite.*

Proof. Assume that $[\Gamma : Z(\Gamma)] = n$ and let $\{h_1, \ldots, h_n\}$ be a set of left coset representatives for $Z(\Gamma)$ in Γ. Note that for all $z, z' \in Z(\Gamma)$, we have $[h_iz, h_jz'] = [h_i, h_j]$, from which it follows that Γ' is finitely generated by $\{[h_i, h_j],\ 1 \leq i < j \leq n\}$. But
$$[\Gamma' : \Gamma' \cap Z(\Gamma)] \leq [\Gamma : Z(\Gamma)] < \infty,$$

so $\Gamma' \cap Z(\Gamma)$ is finitely generated in addition to being abelian. Now $Z(\Gamma) \trianglelefteq \Gamma$ and $|\Gamma/Z(\Gamma)| = n$, so there is a well-defined map

$$f : g \in \Gamma \longmapsto g^n \in Z(\Gamma).$$

It turns out that this map is a group homomorphism (see [Rob82, 10.1.3] for more details). But $Z(\Gamma)$ is abelian, so $\Gamma' \leq \operatorname{Ker} f$. In particular, Γ' is torsion. Hence, $\Gamma' \cap Z(\Gamma)$ is a finitely generated torsion abelian group, so $\Gamma' \cap Z(\Gamma)$ is finite, and Γ' is also finite since $[\Gamma' : \Gamma' \cap Z(\Gamma)] < \infty$. ☐

Corollary 3.6. *Let Γ be a torsion-free virtually cyclic group. Then Γ is trivial or $\Gamma \cong \mathbb{Z}$.*

Proof. We assume that Γ is non-trivial. Let $H \leq \Gamma$ be a cyclic subgroup of finite index; note that $H \cong \mathbb{Z}$. Since Γ is virtually $\mathbb{Z}$, it has a finite generating set $\{s_1, \ldots, s_r\}$. For $1 \leq i \leq r$, consider the centraliser $C(s_i)$ of s_i in Γ. Note that $[C(s_i) : H \cap C(s_i)] \leq [\Gamma : H] < \infty$. Since $C(s_i)$ is infinite, $H \cap C(s_i)$ is infinite; in particular it is a non-trivial subgroup of $H \cong \mathbb{Z}$, so $H \cap C(s_i)$ has finite index in H and therefore in Γ. Hence

$$[\Gamma : Z(\Gamma)] = \left[\Gamma : \bigcap_{i=1}^{r} C(s_i)\right] \leq \prod_{i=1}^{r} [\Gamma : C(s_i)] < \infty.$$

Therefore, Theorem 3.5 implies that the derived group $\Gamma' \leq \Gamma$ is finite. But Γ' is torsion-free, so Γ' is trivial. This means that Γ is abelian; since it is also finitely-generated and torsion-free, $\Gamma \cong \mathbb{Z}^d$ for some d, but Γ is virtually $\mathbb{Z}$, so $d = 1$. ☐

Together with the previous construction of limiting actions on real trees, Theorem 3.4 shows that non-rigid hyperbolic groups split. We are now interested in the converse: if Γ splits over 1 or $\mathbb{Z}$, we will show that $\operatorname{Out}(\Gamma)$ is infinite by constructing an algebraic analogue of Dehn twists. We need to exclude the case where $\Gamma = \mathbb{Z}$, because $\mathbb{Z}$ is a rigid hyperbolic group, but $\mathbb{Z}$ does act freely on the real tree $\mathbb{R}$, and its only splitting is the trivial HNN extension $\mathbb{Z} = 1{*_1}$. Since $\mathbb{Z}$ is in fact the only torsion-free abelian hyperbolic group by Corollary 1.20, this amounts to assuming that Γ is non-abelian.

We obtain Paulin's Theorem, which gives a full characterisation of rigidity.

Theorem 3.7 (Paulin)**.** *Let Γ be a torsion-free non-abelian hyperbolic group. Then the following assertions are equivalent.*

(i) *Γ is non-rigid, i.e. $\operatorname{Out}(\Gamma)$ is infinite.*

(ii) *Γ acts on a real tree with no global fixed point and cyclic segment stabilisers.*

(iii) *Γ splits non-trivially as an amalgamated free product or HNN extension over 1 or $\mathbb{Z}$ (and Γ is not of the form $\mathbb{Z} *_{k\mathbb{Z}} B$ with $k > 1$).*

Proof. Note that (i) ⇒ (ii) ⇒ (iii) is a consequence of Theorems 3.3 and 3.4, together with the fact (Corollary 3.6) that any virtually cyclic subgroup of Γ is cyclic. Moreover, it can be shown that the case $\mathbb{Z} *_{k\mathbb{Z}} B$ cannot arise from the Rips machine, but this is beyond the scope of this text. Hence, it suffices to prove (iii) ⇒ (i).

By assumption, Γ is of the form $A{*_1}$, $A{*_\mathbb{Z}}$, $A{*_1}B$ or $A{*_\mathbb{Z}}B$, with A, B non-trivial. If A is abelian, then Corollary 1.20 implies that $A \cong \mathbb{Z}$; it follows that $A{*_\mathbb{Z}}B$ is of the form $\mathbb{Z}{*_{k\mathbb{Z}}}B$, which is excluded. Likewise, if A and B are both abelian, then $A *_1 B \cong \mathbb{Z} *_1 \mathbb{Z} \cong \mathbb{Z}{*_1}$. Hence, it suffices to study the following four cases.

Case 1. $\Gamma = A*_1$ with A non-trivial. Let t be the stable letter, so that $\Gamma = A * \langle t\rangle$. Since A is non-trivial and torsion-free, it contains an element a_0 of infinite order. Consider the group homomorphism $\varrho : \Gamma \to \Gamma$ defined by

$$\begin{aligned} a \in A &\longmapsto a \\ t &\longmapsto ta_0. \end{aligned}$$

The map ϱ is an automorphism of Γ with inverse defined by $a \mapsto a$ and $t \mapsto ta_0^{-1}$. Moreover, for $m \in \mathbb{Z} \smallsetminus \{0\}$, $\varrho^m \notin \mathrm{Inn}(\Gamma)$, so ϱ has infinite order in $\mathrm{Out}(\Gamma)$.

Case 2. $\Gamma = A*_{\mathbb{Z}}$ with A non-trivial. Then there are elements $u_0, v_0 \in A \smallsetminus \{e\}$ and $t \in \Gamma \smallsetminus \{e\}$, such that

$$\Gamma = A * \langle t\rangle \,/\, \left\langle\!\left\langle t^{-1}u_0tv_0^{-1} \right\rangle\!\right\rangle.$$

Consider the group homomorphism $\varrho : \Gamma \to \Gamma$ defined by

$$\begin{aligned} a \in A &\longmapsto a \\ t &\longmapsto tv_0. \end{aligned}$$

This is well-defined since $\varrho\left(t^{-1}u_0tv_0^{-1}\right) = \left(v_0^{-1}t^{-1}\right)u_0\,(tv_0)\,v_0^{-1} = 1$. The map ϱ is an automorphism of Γ with inverse defined by $a \mapsto a$ and $t \mapsto tv_0^{-1}$. Moreover, for $m \in \mathbb{Z} \smallsetminus \{0\}$, $\varrho^m \notin \mathrm{Inn}(\Gamma)$, so ϱ has infinite order in $\mathrm{Out}(\Gamma)$.

Case 3. $\Gamma = A *_1 B$ with A non-trivial and B non-abelian. For $b_0 \in B$, consider the group homomorphism $\varrho_{b_0} : \Gamma \to \Gamma$ defined by

$$\begin{aligned} a \in A &\longmapsto b_0ab_0^{-1} \\ b \in B &\longmapsto b. \end{aligned}$$

This defines a group homomorphism $\xi : B \to \mathrm{Out}(\Gamma)$ given by $b_0 \mapsto \varrho_{b_0}$, and we have $\mathrm{Ker}\,\xi = Z(B)$. But Theorem 1.19 implies that $Z(B) = 1$, for otherwise B would be virtually cyclic, and hence cyclic by Corollary 3.6. Therefore, we have an injection $B \hookrightarrow \mathrm{Out}(\Gamma)$, so $\mathrm{Out}(\Gamma)$ is infinite.

Case 4. $\Gamma = A *_{\mathbb{Z}} B$ with A, B non-abelian. Let $C = A \cap B \cong \mathbb{Z}$. For $c_0 \in C$, consider the group homomorphism $\varrho_{c_0} : \Gamma \to \Gamma$ defined by

$$\begin{aligned} a \in A &\longmapsto c_0ac_0^{-1} \\ b \in B &\longmapsto b. \end{aligned}$$

This is well-defined because $\varrho_{c_0}(c) = c_0cc_0^{-1} = c$ if $c \in A \cap B$. Hence, we define a group homomorphism $\xi : C \to \mathrm{Out}(\Gamma)$ given by $c_0 \mapsto \varrho_{c_0}$. Using the fact (as in the third case) that $Z(A) = Z(B) = 1$, we have $\mathrm{Ker}\,\xi = 1$, so there is an injection $C \hookrightarrow \mathrm{Out}(\Gamma)$ and $\mathrm{Out}(\Gamma)$ is infinite. $\square$

Having no action on a real tree could in fact be taken as a definition of rigidity, as in [RS94]; this allows one to get more general results without the assumption that groups are torsion-free. We choose however to keep this assumption, which will simplify arguments related to abelian subgroups of hyperbolic groups.

3.3 The co-Hopf property

Using the idea of Paulin's Theorem, we now proceed to show that rigid hyperbolic groups are co-Hopfian, following [RS94]. We start with the following observation, taken from [Sel95], whose point is that the construction of the limiting tree does not really need Γ to have infinite outer automorphism group, but can actually be carried out with a weaker assumption.

Lemma 3.8. *Let Γ be a torsion-free non-abelian hyperbolic group. Then the following assertions are equivalent.*

(i) Γ *is rigid, i.e.* $\mathrm{Out}(\Gamma)$ *is finite.*

(ii) *For any hyperbolic group H, the set of conjugacy classes of monomorphisms $\Gamma \hookrightarrow H$ is finite.*

Proof. (ii) $\Rightarrow$ (i) This is clear because $\mathrm{Out}(\Gamma)$ is included in the set of conjugacy classes of monomorphisms $\Gamma \hookrightarrow \Gamma$.

(i) $\Rightarrow$ (ii) Assume that there is a hyperbolic group H with infinitely many conjugacy classes of monomorphisms $\Gamma \hookrightarrow H$. Then, using the same argument as in Lemma 3.1, we may construct a sequence of distorted Cayley graphs of H equipped with a Γ-action, with hyperbolicity constant converging to zero. Following the rest of the construction of the limiting tree as before, we obtain an action of Γ on a real tree with no global fixed point and cyclic segment stabilisers. Hence, Theorem 3.7 implies that Γ is non-rigid. $\square$

Theorem 3.9 (Rips-Sela)**.** *Let Γ be a rigid torsion-free non-abelian hyperbolic group. Then Γ is co-Hopfian.*

Proof. Assume that there is a non-surjective monomorphism $\varphi : \Gamma \hookrightarrow \Gamma$. It follows that

$$\varphi^{n+1}(\Gamma) \subsetneq \varphi^{n}(\Gamma)$$

for all $n \geq 0$. Moreover Γ is non-abelian, so $\varphi^n(\Gamma)$ is non-abelian for all n because $\varphi^n(\Gamma) \cong \Gamma$. Consider the centraliser $C(\varphi^n(\Gamma))$ of $\varphi^n(\Gamma)$. If $h \in C(\varphi^n(\Gamma)) \smallsetminus \{e\}$, then $C(h) \supseteq \varphi^n(\Gamma)$, so $\varphi^n(\Gamma)$ would be cyclic by Theorem 1.19; this is a contradiction, so

$$C(\varphi^n(\Gamma)) = 1.$$

Now Lemma 3.8 tells us that there are only finitely many classes of monomorphisms $\Gamma \hookrightarrow \Gamma$; therefore, there exist $k \geq 0$, $\ell \geq 1$ and $t \in \Gamma$ such that

$$t\varphi^k(\gamma)t^{-1} = \varphi^{k+\ell}(\gamma) \tag{$*$}$$

for all $\gamma \in \Gamma$. It follows that, for all $\gamma \in \Gamma$,

$$t\varphi^{k+\ell}(\gamma)t^{-1} = t\varphi^k\left(\varphi^\ell(\gamma)\right)t^{-1} = \varphi^{k+\ell}\left(\varphi^\ell(\gamma)\right) = \varphi^\ell\left(\varphi^{k+\ell}(\gamma)\right) = \varphi^\ell(t)\varphi^{k+\ell}(\gamma)\varphi^\ell(t)^{-1}.$$

This implies that $t^{-1}\varphi^\ell(t) \in C\left(\varphi^{k+\ell}(\Gamma)\right) = 1$, so $\varphi^\ell(t) = t$. Therefore $t = \varphi^{m\ell}(t)$ for all $m \geq 0$, so

$$t \in \bigcap_{m\geq 0} \varphi^{m\ell}(\Gamma) = \bigcap_{n \geq 0} \varphi^n(\Gamma).$$

The equality $(*)$ now implies that $\varphi^k(\Gamma) \subseteq \varphi^{k+\ell}(\Gamma) \subsetneq \varphi^k(\Gamma)$, a contradiction. $\square$

Final words

This essay started with a general algebraic question on groups: what can we say about groups which are non-rigid, or in other words, which have a large number of automorphisms? Introducing the geometric idea of hyperbolicity and the analytic idea of Hausdorff-Gromov convergence led us to positive results on those groups. This can be taken as an illustration of the strength of the geometric method in group theory: thinking of groups as geometric objects and using techniques from metric geometry can be a powerful way to solve both geometric and algebraic problems.

More specifically, hyperbolicity of groups is an elementary geometric condition on the Cayley graph, yet it yields surprisingly deep theorems: we have seen that hyperbolic groups have no large abelian subgroups, Paulin's Theorem classifies non-rigid hyperbolic groups, and the Rips-Sela Theorem shows that rigid hyperbolic groups are co-Hopfian. There are many more such results, and Gromov's ideas have been leading to active research in group theory. This essay can therefore be taken as an invitation to further exploration of hyperbolic groups and the wider universe of geometric group theory.

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