# Generalisations of hyperbolicity

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## Contents




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It is an observation usually attributed to Gromov that 'a theorem that is true for all groups is either trivial or of no importance'. Our goal in these notes will be to explore classes of groups that contain a large number of relevant examples, and about which we can prove interesting statements. Those groups are geometric in nature and are inspired by Gromov's theory of hyperbolic groups.

# Talk 1 – Overview of hyperbolic groups

*Speaker:* Alexis Marchand. *Main reference:* [GdlH90].

## 1.1 Prehistory

The theory of hyperbolic groups can be argued to date back from Dehn's decision problems in group theory. Given a group presentation $\langle S \mid R \rangle$, Dehn set two questions which have become fundamental:

(i) *Word problem:* find an algorithm that decides, given a word $w$ over $S$, whether $w$ represents the trivial element in $\langle S \mid R \rangle$.

(ii) *Conjugacy problem:* find an algorithm that decides, given two words $w_1$ and $w_2$ over $S$, whether $w_1$ and $w_2$ represent conjugate elements in $\langle S \mid R \rangle$.

In 1912, Dehn proposed a solution to the word problem for fundamental groups of closed surfaces (*surface groups* for short). In modern language, he proved that surface groups have linear isoperimetric function. Later, Dehn's algorithm was abstracted and led to small cancellation theory. A *small cancellation group* is, roughly, a group $G$ with a presentation $\langle S \mid R \rangle$ in which there is little overlap between the relators. However, small cancellation groups exclude many groups of significant geometric interest. For example, they have cohomological dimension at most 2, so fundamental groups of higher rank hyperbolic manifolds cannot be small cancellation groups.

Instead, in 1987, Gromov [Gro87] proposed a new method with a much stronger geometric flavour, and that would have a long-lasting impact on the field.

## 1.2 Definition and examples

The general idea is, given a finitely generated group $G$, to consider the geometric properties of the metric spaces $X$ equipped with a *geometric action* of $G$ (i.e. a proper cocompact action by isometries). By the Švarc-Milnor, all those spaces are quasi-isometric, and they are quasi-isometric to Cayley graphs of $G$ (with respect to finite generating sets).

**Definition 1.1.** Let $(X, d)$ be a geodesic metric space. The following assertions are equivalent:

(i) *(Rips-hyperbolicity)* There is a constant $\delta \geq 0$ such that, given three points $x, y, z \in X$ and three geodesic segments $[x, y], [y, z], [z, x]$ between them, we have

$$[x, z] \subseteq \mathcal{N}_\delta\left([x, y]\right) \cup \mathcal{N}_\delta\left([y, z]\right),$$

where $\mathcal{N}_\delta(A) = \{x \in X \mid d(x, A) < \delta\}$ for $A \subseteq X$.

(ii) *(Gromov-hyperbolicity)* There is a constant $\delta \geq 0$ such that the following inequality holds for all $x, y, z, \omega \in X$:

$$(x \mid y)_\omega \geq \min\left\{(x \mid z)_\omega, (y \mid z)_\omega\right\} - \delta,$$

where $(\cdot \mid \cdot)_\omega$ is a 'metric inner product' defined by

$$(a \mid b)_\omega = \frac{d(a, \omega) + d(b, \omega) - d(a, b)}{2},$$

for $a, b \in X$.

If these assertions are satisfied, we say that $X$ is *hyperbolic*.

**Example 1.2.** (i) Trees are hyperbolic (and one can take $\delta = 0$ in both definitions).

(ii) The hyperbolic plane $\mathbb{H}^2$ is hyperbolic (where $\delta$ is the radius of a disk of area $2\pi$ for Rips-hyperbolicity).

(iii) The hyperbolic $n$-space $\mathbb{H}^n$ is hyperbolic (triangles in $\mathbb{H}^n$ are contained in copies of $\mathbb{H}^2$, so this follows from hyperbolicity of $\mathbb{H}^2$).

One reason why hyperbolicity is a very nice notion is the following result:

**Theorem 1.3.** *Hyperbolicity is a quasi-isometry invariant of geodesic metric spaces.*

This allows one to give the following definition:

**Definition 1.4.** A finitely generated group $G$ is *hyperbolic* if some (or any) of the geodesic spaces on which it acts geometrically is hyperbolic.

**Example 1.5.** The following groups are hyperbolic:

(i) Free groups (acting geometrically on trees),

(ii) Fundamental groups of closed hyperbolic surfaces (acting geometrically on $\mathbb{H}^2$),

(iii) Fundamental groups of closed hyperbolic manifolds (acting geometrically on $\mathbb{H}^n$),

(iv) $\mathcal{C}'\left(\frac{1}{6}\right)$ or $\mathcal{C}(7)$ small cancellation groups,

(v) Random groups,

(vi) Finite groups.

## 1.3 Main properties

**The Rips complex and applications.** Hyperbolic groups have very rich structural properties. Many of them can be deduced from the following fundamental theorem:

**Theorem 1.6** (Rips)**.** *If $G$ is a hyperbolic group, then there is a proper cocompact simplicial action of $G$ on a locally finite, finite-dimensional simplicial complex $X$.*

**Corollary 1.7.** *If $G$ is a hyperbolic group, then*

(i) *$G$ is finitely presented (and in fact of type* **F** *if it is torsion-free),*

(ii) *$G$ has a finite number of conjugacy classes of torsion elements,*

(iii) *$H^k(G;\mathbb{Q}) = 0$ for $k$ large enough.*

**Algorithmic properties.** Hyperbolic groups have solvable word and conjugacy problems.

**Boundary.** In the same way that one can equip the hyperbolic plane with its boundary at infinity, there is a notion of boundary for hyperbolic spaces and groups that has great relevance in the study of the dynamics of elements of the group.

**Definition 1.8.** Let $X$ be a proper hyperbolic space.

- A *geodesic ray* in $X$ is an isometric embedding $\rho : \mathbb{R}_{\geq 0} \to X$.
- We say that two geodesic rays $\rho_1, \rho_2$ in $X$ are *equivalent*, and we write $\rho_1 \sim \rho_2$, if

$$\sup_t d(\rho_1(t), \rho_2(t)) < \infty.$$

- The *(visual) boundary* of $X$ is the set

$$\partial X = \{\text{geodesic rays } \rho : \mathbb{R}_{\geq 0} \to X \text{ with } \rho(0) = x_0\} / \sim$$

  for a fixed basepoint $x_0$.

One can define a topology on the set $\bar{X} = X \cup \partial X$ that makes it a compactification of $X$. Remarkably, the boundary is a quasi-isometry invariant:

**Proposition 1.9.** *If two hyperbolic spaces $X_1$ and $X_2$ are quasi-isometric, then $\partial X_1$ and $\partial X_2$ are homeomorphic.*

**Definition 1.10.** Given a hyperbolic group $G$, its *boundary* $\partial G$ is defined to be the boundary of any space $X$ on which $G$ acts geometrically.

**Example 1.11.** (i) The boundary of a free group is a Cantor set.

(ii) The boundary of a closed hyperbolic surface group is a circle.

(iii) The boundary of a closed hyperbolic 3-manifold group is a 2-sphere.

(iv) The boundary of the fundamental group of a compact hyperbolic 3-manifold with nonempty boundary is a Sierpinski carpet.

Hence, given a hyperbolic group $G$, there is an action $G \curvearrowright \partial G$ by homeomorphisms, and we can use it to read dynamical properties of elements of $G$:

**Definition 1.12.** If $G \curvearrowright X$ is a geometric action of a hyperbolic group, and if $H \leq G$ is a subgroup, the *limit set* of $H$ is

$$\Lambda(H) = \overline{H \cdot x_0} \cap \partial X$$

for a fixed choice of basepoint $x_0 \in X$.

**Proposition 1.13.** *Let $G$ be a hyperbolic group acting geometrically on a space $X$, and let $g \in G$. Then the following are equivalent:*

(i) *$g$ has no fixed point in $\partial X$.*

(ii) *There is $x_0 \in X$ such that the orbit $\langle g \rangle \cdot x_0$ is bounded.*

(iii) *For each $x_0 \in X$, the orbit $\langle g \rangle \cdot x_0$ is bounded.*

(iv) $\Lambda(\langle g \rangle) = \varnothing$.

*We then say that $g$ is* elliptic.

**Proposition 1.14.** *Let $G$ be a hyperbolic group acting geometrically on a space $X$, and let $g \in G$. Then the following are equivalent:*

(i) *$g$ has at least one fixed point in $\partial X$.*

(ii) *There is $x_0 \in X$ such that the orbit $\langle g \rangle \cdot x_0$ is a quasi-geodesic.*

(iii) *For each $x_0 \in X$, the orbit $\langle g \rangle \cdot x_0$ is a quasi-geodesic.*

(iv) $|\Lambda(\langle g \rangle)| = 2$.

*We then say that $g$ is* loxodromic.

**Proposition 1.15.** *Let $G$ be a hyperbolic group acting geometrically on a space $X$. Given $H \leq G$, the following are equivalent:*

(i) $|\Lambda(H)| < \infty$.

(ii) $|\Lambda(H)| \in \{0, 2\}$.

(iii) *$H$ is virtually cyclic.*

*We then say that $H$ is* elementary.

**SQ-universality.** The following is another nice property that hyperbolic groups enjoy, and that will be relevant when we discuss acylindrical hyperbolicity:

**Definition 1.16.** A group $G$ is *SQ-universal* if every countable group embeds into some quotient of $G$.

**Theorem 1.17.** *Non-elementary hyperbolic groups are SQ-universal.*

SQ-universality means that hyperbolic groups have a lot of quotients in a very strong sense.

**Tits alternative.** Being hyperbolic also imposes strong restrictions on subgroups of hyperbolic groups:

**Theorem 1.18.** *If $G$ is hyperbolic and $G_1 \leq G$ is a subgroup, then either*

(i) *$G_1$ is virtually cyclic, or*

(ii) *$G_1$ contains a free subgroup of rank 2.*

It follows for instance from Theorem 1.18 that containing a copy of $\mathbb{Z}^2$ or of a Baumslag-Solitar group is an obstruction to being hyperbolic.

**Hopf property.** We conclude this hyperbolic sightseeing tour with the following:

**Definition 1.19.** A group $G$ has the *Hopf property* if every surjective morphism $G \twoheadrightarrow G$ is an isomorphism.

**Theorem 1.20.** *Torsion-free hyperbolic groups have the Hopf property.*

Note that all finitely generated, residually finite groups have the Hopf property, and it is a famous open problem to know whether every hyperbolic group is residually finite.

## 1.4 Shortcomings

There are still many groups of strong geometric significance that are not hyperbolic, even though their geometry is somehow hyperbolic-like. Here are just a few examples.

**Finite-volume hyperbolic manifolds.** If $M$ is a finite-volume cusped hyperbolic $n$-manifold (with $n \geq 3$), then $M$ may not be hyperbolic, because the cusps might contain subgroups isomorphic to $\mathbb{Z}^2$.

**Mapping class groups and $\mathrm{Out}\,(F_n)$.** Given a closed surface $S$, define its *mapping class group* by

$$\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S),$$

where $\mathrm{Homeo}(S)$ is the group of homeomorphisms of $S$, and $\mathrm{Homeo}_0(S)$ is the connected component of the identity. Then there is a simplicial complex $\mathcal{C}(S)$ — called the *curve complex* of $S$ — on which $\mathrm{MCG}(S)$ acts cocompactly — but not properly. Moreover,

**Theorem 1.21** (Masur-Minsky [MM99])**.** *$\mathcal{C}(S)$ is hyperbolic.*

However, $\mathrm{MCG}(S)$ is not hyperbolic: for instance, two Dehn twists in disjoint simple closed curves generate a copy of $\mathbb{Z}^2$.

The situation with $\mathrm{Out}\,(F_n)$ is very similar.

**Free products.** Free products of groups act on trees via Bass-Serre theory, but they may not be hyperbolic if the vertex groups are not nice.

Our goal will be to understand larger classes of groups with hyperbolic-like features that will allow us to study the above examples (and more).

# Talk 2 – Relatively hyperbolic groups: definitions inspired by hyperbolic geometry

*Speaker:* Alexis Marchand. *Main reference:* [Hru10].

**Remark 2.1.** In §2 and §3, all groups are assumed to be countable.

## 2.1 The geometry of horoballs

Before defining relatively hyperbolic groups, we need to understand horoballs in $\delta$-hyperbolic spaces. First recall that, in the Poincaré upper-half-space model of $\mathbb{H}^n$, there is a point $\infty \in \partial\mathbb{H}^n$, and a *horosphere* centered at $\infty$ is the set

$$\mathcal{H} = \left\{(x_1, \ldots, x_n) \in \mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \mid x_n = k\right\}$$

for some fixed $k > 0$. This can be used to define horospheres centered at any point of $\partial\mathbb{H}^n$ by transitivity of the action $\operatorname{Isom}(\mathbb{H}^n) \curvearrowright \partial\mathbb{H}^n$. Horospheres play an important part in hyperbolic geometry because they are stabilised by parabolic groups of isometries, and because their geometry is Euclidean.

Let $X$ be a $\delta$-hyperbolic geodesic metric spaces. After possibly changing the value of $\delta$, we may assume that ideal triangles in $\bar{X} = X \cup \partial X$ are also $\delta$-slim in the sense of Definition 1.1.

**Definition 2.2.** A *$\delta$-centre* of a geodesic triangle $\Delta$ in $X$ with vertices $x_1, x_2, x_3$ is a point $w \in X$ such that, for all $i \neq j \in \{1, 2, 3\}$,

$$d(w, [x_i, x_j]) \leq \delta.$$

Note that, by $\delta$-slimness, each side of $\Delta$ must contain a $\delta$-centre (using the continuity of the function $w \mapsto d(w, [x_i, x_j])$).

**Definition 2.3.** Given $\xi \in \partial X$, a *horofunction* about $\xi$ is a map $h : X \to \mathbb{R}$ such that there is a constant $D_0 \geq 0$, such that, for all $x, y \in X$, for each geodesic triangle $\Delta$ with vertices $x, y, \xi$, and for each $w$ centre of $\Delta$, the following inequality holds:

$$|(h(x) - h(y)) - (d(y, w) - d(x, w))| \leq D_0.$$

**Example 2.4.** (i) Let $X$ be a tree and $\xi \in \partial X$. Then one can define a horofunction about $\xi$ as follows. Pick a point $x_0 \in X$ and choose a value for $h(x_0)$. Now, given any point $x \in X$, there is a unique ideal triangle — which is in fact a tripod — with vertices $x_0, x, \xi$; let $w$ be the centre (or 0-centre) of that tripod. Then $h(x)$ is defined by the equality

$$h(x) - h(x_0) = d(x_0, w) - d(x, w).$$

See Figure 1a. The function $h : X \to \mathbb{R}$ thus defined is a horofunction with error $D_0 = 0$.

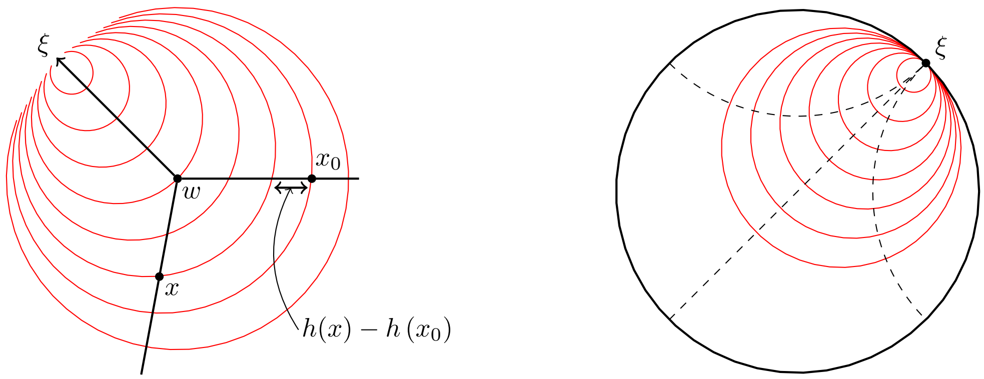
(ii) Let $X = \mathbb{H}^2$ and $\xi \in \partial\mathbb{H}^2$. One can define the *Buseman cocycle* $b_\xi : X^2 \to \mathbb{R}$ by

$$b_\xi(x, y) = \lim_{t \to \infty} \left(d(x, \gamma(t)) - d(y, \gamma(t))\right),$$

where $\gamma : \mathbb{R}_{\geq 0} \to X$ is a geodesic ray converging to $\xi$. One can then define a horofunction $h : X \to \mathbb{R}$ about $\xi$ by fixing the value of $h(x_0)$ for some $x_0 \in X$ and imposing that, for all $x \in X$,

$$h(x) - h(x_0) = b_\xi(x_0, x).$$

The level sets are horospheres in the usual sense. See Figure 1b.




(a) In a tree.

(b) In $\mathbb{H}^2$ (Poincaré disk model).

Figure 1: Level sets of horofunctions.

**Remark 2.5.** Horofunctions about a given point $\xi \in \partial X$ are far from being unique: one can always add a constant.

**Definition 2.6.** A *horoball* centered at $\xi$ is a closed subset $\mathcal{H} \subseteq X$ for which there is a horofunction $h$ about $\xi$ and a constant $D_1 \geq 0$ such that

- $\forall x \in \mathcal{H},\ h(x) \geq -D_1$,
- $\forall x \in X \smallsetminus \mathcal{H},\ h(x) \leq D_1$.

Hence, a horoball is 'almost' the set $\{x \in X \mid h(x) \geq 0\}$ for some horofunction $h$.

## 2.2 Relative hyperbolicity via cusp uniform actions

The motivation behind our first definition of relative hyperbolicity comes from the *thick-thin decomposition* in hyperbolic geometry.

Let $M$ be a complete hyperbolic $n$-manifold.

**Definition 2.7.** Given $x \in M$, the *injectivity radius* of $M$ at $x$ is defined by

$$\begin{aligned} \operatorname{inj}_x(M) &= \sup\{r > 0 \mid \exp_x : B(0,r) \subseteq T_xM \longrightarrow M \text{ is injective}\} \\ &= \frac{1}{2} \inf_{\gamma \in \pi_1 M \smallsetminus 1} d(\tilde{x}, \gamma\tilde{x}), \end{aligned}$$

for any choice of lift $\tilde{x}$ of $x$ to the universal cover $\tilde{M} = \mathbb{H}^n$.

Note that $\mathrm{inj}_x(M) > 0$ for all $x$.

**Definition 2.8.** Given $\varepsilon > 0$,

- The *$\varepsilon$-thin part* of $M$ is $M_{(0,\varepsilon]} = \{x \in M \mid 2\,\mathrm{inj}_x(M) \leq \varepsilon\}$,
- The *$\varepsilon$-thick part* of $M$ is $M_{[\varepsilon,\infty)} = \{x \in M \mid 2\,\mathrm{inj}_x(M) \geq \varepsilon\}$,

**Theorem 2.9** (Thick-thin decomposition)**.** *For each $n \geq 1$, there is an $\varepsilon > 0$ (the* Kazhdan-Margulis constant*) such that, for each complete hyperbolic $n$-manifold $M$, the $\varepsilon$-thin part of $M$ consists of a disjoint union of*

- Truncated cusps*, i.e. quotients $\mathcal{H}/\Gamma$, where $\mathcal{H}$ is a horoball centered at $\xi \in \partial\mathbb{H}^n$, and $\Gamma$ is a discrete subgroup of* $\mathrm{Stab}(\xi) \leq \mathrm{Isom}\,(\mathbb{H}^n)$*, and*
- Tubes*, i.e. neighbourhoods of simple closed geodesics.*

*See Figure 2 for an illustration when $n = 2$.*

(a) A truncated cusp.    (b) A tube.

Figure 2: Thin parts of hyperbolic surfaces.

The thick-thin decomposition says that a finite-volume complete hyperbolic manifold can be split into a thick part, which is compact and has a lower bound on the injectivity radius, and a thin part, which consists of cusps and tubes only.

Now the idea is that, given a finite-volume complete hyperbolic manifold $M$ with cusps, the action $\pi_1 M \curvearrowright \mathbb{H}^n$ is not cocompact, but it will be after removing the cusps. Hence, we want to consider that $\pi_1 M$ is 'hyperbolic relative to the cusps'. Since truncated cusps lift to horoballs in the universal cover, this leads to the following definition of relative hyperbolicity.

**Definition 2.10.** Given a properly discontinuous action of a group $G$ on a proper $\delta$-hyperbolic space, a *parabolic subgroup* is a subgroup $H \leq G$ such that $\Lambda(H)$ is a singleton $\{\xi\}$. We then say that $\xi$ is the corresponding *parabolic point*.

**Definition 2.11** ($RH_3$)**.** Let $G$ be a group and $\mathbb{P}$ a collection of subgroups of $G$. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if there is a properly discontinuous action of $G$ on a proper $\delta$-hyperbolic space $X$ for which

- $\mathbb{P}$ is a set of representatives of conjugacy classes of maximal parabolic subgroups, and
- There is a $G$-equivariant collection of disjoint horoballs centered at the parabolic points of the subgroups in $\mathbb{P}$, whose union $U$ is open in $X$, and such that the quotient $G\backslash(X \smallsetminus U)$ is compact.

We then say that the action $G \curvearrowright X$ is *cusp uniform* and that $X \smallsetminus U$ is a *truncated space* for the action.

## 2.3 Relative hyperbolicity via convergence group actions

Our next definition of relative hyperbolicity is motivated by the dynamics of the action of discrete groups of hyperbolic isometries on the boundary $\partial\mathbb{H}^n$ of hyperbolic $n$-space.

**Definition 2.12** (Beardon-Maskit)**.** A *convergence group action* is an action of a group $M$ on a compact metrisable space $M$ such that:

- If $M$ is empty, then $G$ is finite.
- If $M$ is a singleton, then $G$ can be any countable group.
- If $M$ is a pair, then $G$ is virtually cyclic.
- In all other cases, the action of $G$ on the space of distinct unordered triples in $M$ is properly discontinuous.

**Theorem 2.13** (Tukia)**.** *Every properly discontinuous action of a group $G$ on a $\delta$-hyperbolic space $X$ induces a convergence group action $G \curvearrowright \partial X$.*

Let $G \curvearrowright M$ be a convergence group action.

**Definition 2.14.**

- An element $g \in G$ is *loxodromic* if it has infinite order and fixes exactly two points in $M$.
- A subgroup $P \leq G$ is *parabolic* if it is infinite and has no loxodromic element. This implies that $P$ has a unique fixed point in $M$, which we call the *parabolic point* of $P$.

Note that, if $p \in M$ is a parabolic point, then its stabiliser $\mathrm{Stab}(p)$ is maximal parabolic.

**Definition 2.15.**

- A parabolic point $p \in M$ is *bounded* if its stabiliser $\mathrm{Stab}(p)$ acts cocompactly on $M \setminus p$.
- A point $\xi \in M$ is a *conical limit point* if there is a sequence $(g_i)_{i \geq 1}$ of elements of $G$, and points $\zeta_0, \zeta_1 \in M$ such that $g_i(\xi) \to \zeta_0$ and, for all $\eta \in M \setminus \xi$, $g_i(\eta) \to \zeta_1$.

**Example 2.16.**

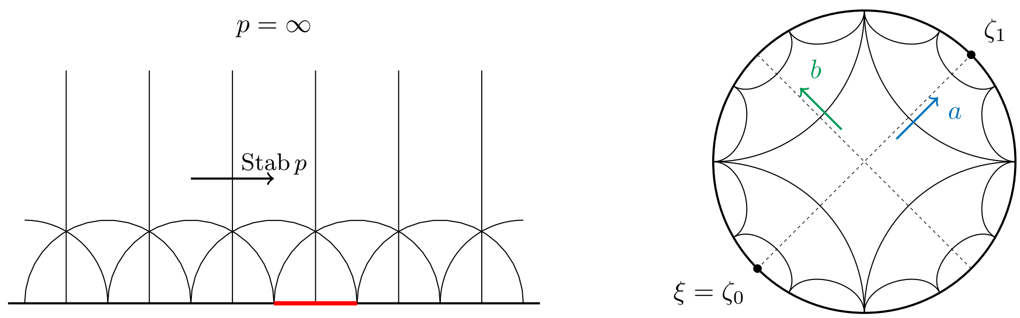
(i) The action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}^2$ by Möbius transforms induces a convergence group action on $M = \partial\mathbb{H}^2$. In the upper-half-plane model, let $p = \infty$. Then the stabiliser of $p$ is given by

$$\mathrm{Stab}(p) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Hence, the action $\mathrm{Stab}(p) \curvearrowright M \setminus p$ is just $\mathbb{Z} \curvearrowright \mathbb{R}$, which is cocompact (with fundamental domain the red segment in Figure 3a).

(ii) Let $\Sigma_{1,1}$ be the once-punctured torus. Endowing $\Sigma_{1,1}$ with a hyperbolic structure yields an action of $\pi_1\Sigma_{1,1} = F_2$ on $\mathbb{H}^2$, inducing a convergence group action on $M = \partial\mathbb{H}^2$. Let $a, b$ be the usual free generating set of $\pi_1\Sigma_{1,1}$, and let $\xi = \zeta_0$ and $\zeta_1$ be the opposite endpoints of a geodesic axis of $a$, as shown in Figure 3b. Then $\xi$ is a conical limit point, and one can pick $g_i = a^i$.

**Definition 2.17.** A convergence group action $G \curvearrowright M$ is *geometrically finite* if each point of $M$ is either a conical limit point or a bounded parabolic point.




(a) $PSL_2\mathbb{Z} \curvearrowright \partial\mathbb{H}^2$ (upper-half-plane model).

(b) $\pi_1\Sigma_{1,1} \curvearrowright \partial\mathbb{H}^2$ (Poincaré disk model).

Figure 3: Two examples of convergence group actions.

We are now ready to give our second definition of relative hyperbolicity:

**Definition 2.18** ($RH_1$)**.** Let $G$ be a group and $\mathbb{P}$ a collection of subgroups of $G$. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if there is a geometrically finite convergence group action of $G$ such that $\mathbb{P}$ is a set of representatives of conjugacy classes of maximal parabolic subgroups.

In fact, in this definition, we can always assume that the convergence group action comes from an action on a $\delta$-hyperbolic space as in Tukia's Theorem:

**Definition 2.19** ($RH_2$)**.** Let $G$ be a group and $\mathbb{P}$ a collection of subgroups of $G$. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if there is a properly discontinuous action of $G$ on a proper $\delta$-hyperbolic space $X$ such that the induced convergence group action on $\partial X$ is geometrically finite, and $\mathbb{P}$ is a set of representatives of conjugacy classes of maximal parabolic subgroups.

**Example 2.20** (continuing 2.16)**.** (i) Let $G = PSL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2$, and let

$$\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in PSL_2(\mathbb{Z}).$$

Then the pair $(G, \langle \tau \rangle)$ is relatively hyperbolic.

(ii) Let $G = \pi_1\Sigma_{1,1} = F(a,b)$, and let $g = aba^{-1}b^{-1} \in G$. Then the pair $(G, \langle g \rangle)$ is relatively hyperbolic.

Note that in both cases, $G$ is a hyperbolic group, so the pair $(G, \varnothing)$ is also relatively hyperbolic! This illustrates the fact that there can be several different relatively hyperbolic structures on a given group $G$.

# Talk 3 – Relatively hyperbolic groups via actions on graphs

*Speaker:* Will Cohen. *Main reference:* [Hru10].

## 3.1 Fine hyperbolic graphs

The next definition of relative hyperbolicity is due to Bowditch and relies on the following graph-theoretic property:

**Definition 3.1.** A graph $\Gamma$ is *fine* if for each edge $e$ of $\Gamma$ and for each integer $n \in \mathbb{N}$, $e$ is contained in only finitely many loops of length $n$.

**Example 3.2.** (i) Trees are fine (because they have no loop), as are locally finite graphs (because they have only finitely many paths of a given length starting at a fixed vertex).

(ii) A simple example of a non-fine graph can be constructed by taking two vertices and joining them by infinitely many edges.

**Definition 3.3** ($RH_4$)**.** Let $G$ be a group and $\mathbb{P}$ a collection of subgroups of $G$. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if there is an action of $G$ on a fine hyperbolic graph $\Gamma$, with finitely many orbits of edges and finite edge stabilisers, and such that $\mathbb{P}$ is a set of representatives of conjugacy classes of infinite vertex stabilisers.

**Example 3.4.** (i) If $G$ is a group, then $(G, \{G\})$ is a relatively hyperbolic pair because the action $G \curvearrowright \{\mathrm{pt}\}$ satisfies ($RH_4$).

(ii) If $G$ is a hyperbolic group, then $(G, \varnothing)$ is a relatively hyperbolic pair because the action of $G$ on a Cayley graph satisfies ($RH_4$).

(iii) Consider the free product $G = \mathbb{Z} * H$ for a fixed infinite group $H$. Denote by $t$ a generator of $\mathbb{Z}$. Let $\Gamma$ be the Bass-Serre tree of $G$, seen as a trivial HNN extension of $H$: the graph $\Gamma$ has one vertex for each (left) coset of $H$, with an edge between $aH$ and $atH$ for each $a \in G$; the action $G \curvearrowright \Gamma$ is given by $g \cdot (aH) = (ga)\,H$. This action satisfies ($RH_4$), showing that $(G, \{H\})$ is relatively hyperbolic.

**Remark 3.5.** The fineness condition might seem unnatural. To understand why we need it, take an infinite group $H$, and consider the group $G = \mathbb{Z} \times H$ acting on the graph $\Gamma$ of Figure 4. The graph $\Gamma$ is hyperbolic (it is a quasi-tree), the action is simply transitive on edges, and vertex stabilisers are all equal to $H$. However, we *do not want* the pair $(G, \{H\})$ to be relatively hyperbolic, and it is not according to our definition because $\Gamma$ is not a fine graph.

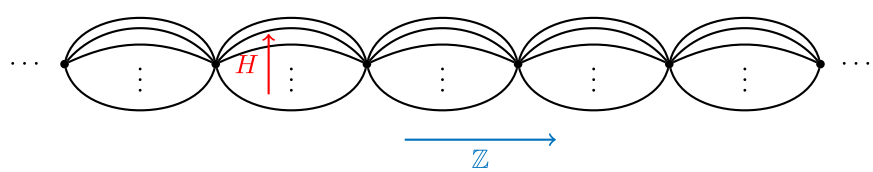



Figure 4: An action of $\mathbb{Z} \times H$ on a non-fine hyperbolic graph.

## 3.2 The coned-off Cayley graph and bounded coset penetration

We consider a pair $(G, \mathbb{P})$ consisting of a group $G$ and a collection $\mathbb{P}$ of subgroups of $G$.

**Definition 3.6.** A subset $S \subseteq G$ is a *relative generating set* for $(G, \mathbb{P})$ if $G = \left\langle S \cup \bigcup_{P \in \mathbb{P}} P \right\rangle$.

**Definition 3.7.** Let $S \subseteq G$ be a subset. The *coned-off Cayley graph* $\hat{\Gamma}\,(G, \mathbb{P}, S)$ is constructed as follows. Start with the (usual) Cayley graph $\Gamma(G, S)$, then for each coset $gP$ of subgroups $P \in \mathbb{P}$, add a vertex $v_{gP}$, and connect $v_{gP}$ to every element of $gP$ by an edge of length $\frac{1}{2}$.

**Remark 3.8.** $\hat{\Gamma}(G, \mathbb{P}, S)$ is connected if and only if $S$ is a relative generating set for $(G, \mathbb{P})$.

**Example 3.9.** Consider $G = \mathbb{Z}^2$ with a basis $(x, y)$, let $\mathbb{P} = \{\langle y \rangle\}$ and $S = \{x\}$. The coned-off Cayley graph $\hat{\Gamma}(G, \mathbb{P}, S)$ is represented in Figure 5, with the Cayley graph in black and the added vertices and edges in red.

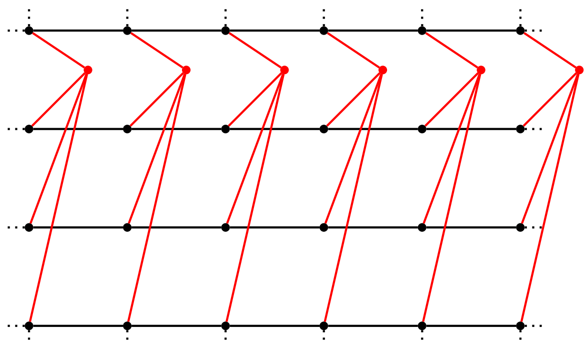


Figure 5: The coned-off Cayley graph of $\left(\mathbb{Z}^2, \{\langle y \rangle\}, \{x\}\right)$.

We need a technical condition on coned-off Cayley graphs:

**Definition 3.10.** Let $\gamma$ be a path in $\hat{\Gamma}(G, \mathbb{P}, S)$.

- We say that $\gamma$ *enters* the coset $gP$ if $\gamma$ passes through the vertex $v_{gP}$. In this case, the vertex preceding $v_{gP}$ is called the *entering vertex* and the vertex succeeding $v_{gP}$ is called the *exiting vertex*.
- We say that $\gamma$ is *without backtracking* if it enters each coset at most once.

**Definition 3.11.** The triple $(G, \mathbb{P}, S)$ is said to satisfy *bounded coset penetration* if for each $\lambda \geq 1$, there is some $a > 0$ such that for any pair $(\gamma_1, \gamma_2)$ of $(\lambda, 0)$-quasi-geodesics without backtracking in $\hat{\Gamma}(G, \mathbb{P}, S)$ with the same initial point and endpoints within distance 1 of each other, we have:

(i) If $\gamma_1$ penetrates a coset $gP$ and $\gamma_2$ does not, then the entering vertex and the exiting vertex in $\gamma_1$ are at $S$-distance at most $a$ from each other.

(ii) If both $\gamma_1$ and $\gamma_2$ penetrate a coset $gP$, then their entering vertices (resp. exiting vertices) are at $S$-distance at most $a$ from each other.

Loosely, bounded coset penetration can be understood as the right condition to ensure that when each coset is collapsed to a point, the resulting graph is fine.

This leads to a definition of relative hyperbolicity in terms of the coned-off Cayley graph:

**Definition 3.12** ($RH_5$)**.** Assume $G$ is finitely generated relative to $\mathbb{P}$, and each $P \in \mathbb{P}$ is infinite. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if for some (hence every) finite relative generating set $S$, the coned-off Cayley graph $\hat{\Gamma}(G, \mathbb{P}, S)$ is hyperbolic and satisfies bounded coset penetration.

**Example 3.13.** (i) Consider the free product $G = \mathbb{Z} * H$ for an infinite group $H$. If $\mathbb{Z} = \langle t \rangle$, then $\{t\}$ is a relative generating set for $(G, \{H\})$. The coned-off Cayley graph looks like the Bass-Serre tree of the trivial HNN extension of $H$ (see Example 3.4.(iii)), but where each vertex is replaced by a copy of $H$, and with an extra vertex representing the corresponding coset. This is hyperbolic and satisfies bounded coset penetration because quasi-geodesics with the same endpoints must penetrate the same cosets with the same entering and exiting vertices.

(ii) Let $(x, y)$ be a basis of $G = \mathbb{Z}^2$, let $\mathbb{P} = \{\langle y \rangle\}$ and $S = \{x\}$ as in Example 3.9. Figure 6 shows an example of two geodesics in the coned-off Cayley graph which do not satisfy bounded coset penetration; indeed, the blue geodesic penetrates a coset with entering and exiting vertices at an infinite $S$-distance from each other, while the green geodesic does not penetrate this coset.

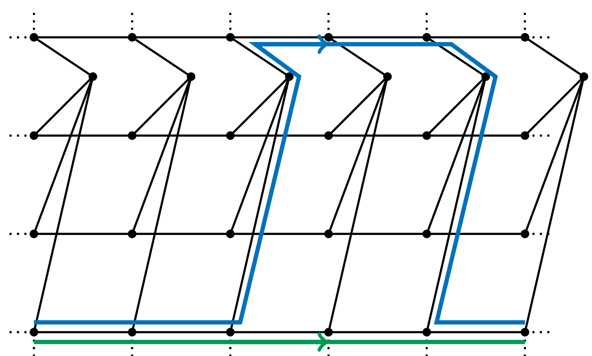


Figure 6: Two geodesics in $\hat{\Gamma}\left(\mathbb{Z}^2, \{\langle y \rangle\}, \{x\}\right)$ violating bounded coset penetration.

### 3.3 Relative Dehn function

Recall that a group is hyperbolic if and only if it admits a finite presentation with linear Dehn function. We now describe a relative analogue of this phenomenon.

**Definition 3.14.** Let $G$ be a group and let $\mathbb{P}$ be a collection of subgroups of $G$. If $S \subseteq G$ is a relative generating set, there is a natural epimorphism

$$K = F(S) * \mathop{\ast}_{P \in \mathbb{P}} P \overset{p}{\twoheadrightarrow} G.$$

Given a set $R \subseteq K$ of relators, we say that $G$ has a *relative presentation* $\langle \mathbb{P}, S \mid R \rangle$ if $\operatorname{Ker} p$ is the normal subgroup generated by $R$.

The relative presentation $\langle \mathbb{P}, S \mid R \rangle$ is called *finite* if both $S$ and $R$ are finite.

**Definition 3.15.** A function $f : \mathbb{N} \to \mathbb{N}$ is a *relative isoperimetric function* for a relative presentation $\langle \mathbb{P}, S \mid R \rangle$ if for each $m \in \mathbb{N}$ and for each word $w$ over $S \amalg \coprod_{P \in \mathbb{P}} P$ of length at most $m$, there are elements $g_1, \ldots, g_k \in K$ and relators $r_1, \ldots, r_k \in R$ such that $k \leq f(m)$ and

$$w \overset{K}{=} \prod_{i=1}^{k} g_i^{-1} r_i g_i.$$

The *relative Dehn function* of $\langle \mathbb{P}, S \mid R \rangle$ is the smallest possible relative isoperimetric function if it exists.

This allows us to define relative hyperbolicity via linear relative Dehn functions.

**Definition 3.16** ($RH_6$)**.** Let $\mathbb{P}$ be a finite collection of subgroups of $G$. We say that the pair $(G, \mathbb{P})$ is *relatively hyperbolic* if $(G, \mathbb{P})$ has a finite relative presentation for which the relative Dehn function is well-defined and linear.

# Talk 4 – CAT(0) cube complexes, contact graphs, and quasi-arboreality

*Speaker:* Ana Isakovic. *Main reference:* [Hag14].

The goal of this talk is to show how to construct a 'hierarchy' of hyperbolic spaces for CAT(0) cube complexes, and in particular for right-angled Artin groups.

## 4.1 Weak hyperbolicity

We start by introducing the notion of weak hyperbolicity. We give two equivalent definitions, which are parallel to definitions $RH_4$ and $RH_5$ of relative hyperbolicity (see Definitions 3.3 and 3.12).

**Definition 4.1** ($WH_1$, Farb [Far94])**.** A group $G$ is *weakly hyperbolic* with respect to a finite collection $\mathbb{P}$ of conjugacy-invariant subgroups if there is a relative generating set $S$ such that the coned-off Cayley graph $\hat{\Gamma}\left(G, \mathbb{P}, S\right)$ is hyperbolic.

Note that, contrary to relative hyperbolicity, we do not impose that the coned-off Cayley graph satisfy bounded coset penetration.

**Example 4.2.** $\mathbb{Z}^2 = \langle x, y \rangle$ is weakly hyperbolic with respect to $\{\langle y \rangle\}$ because the coned-off Cayley graph $\hat{\Gamma}\left(\mathbb{Z}^2, \{\langle y \rangle\}, \{x\}\right)$ is hyperbolic (see Figure 5).

This example shows in particular that a group that is weakly hyperbolic with respect to hyperbolic subgroups need not be hyperbolic.

**Definition 4.3** ($WH_2$, Bowditch [Bow12])**.** A group $G$ is *weakly hyperbolic* with respect to a collection $\mathbb{P}$ of subgroups if $G$ acts on a connected hyperbolic graph $\Gamma$ with finitely many edge orbits, and such that each $P \in \mathbb{P}$ fixes a vertex $v$ of $\Gamma$, and each vertex stabiliser contains an element of $\mathbb{P}$ as a finite index subgroup.

If in addition $\Gamma$ is quasi-isometric to a tree, we say that $G$ is *quasi-arboreal* with respect to $\mathbb{P}$.

Note that we do not require the graph $\Gamma$ to be fine, nor the edge stabilisers to be finite.

## 4.2 Cube complexes and contact graphs

**Definition 4.4.**

- A *cube complex* $X$ is a Euclidean complex (i.e. a metric cell complex obtained by gluing Euclidean polytopes) in which all $n$-cells are isometric to $n$-cubes $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$.
- A *midcube* (or *hypercube*) of an $n$-cube is an $(n-1)$-cube which is obtained by restricting exactly one coordinate to zero.

- If $X$ is CAT(0), a *hyperplane* $W$ in $X$ is a maximal connected collection of midcubes of $X$ such that, for every cube $C$ in $X$, the intersection $W \cap C$ is either empty or one of the midcubes of the collection.

- The *carrier* $N(W)$ of a hyperplane $W$ is the union of all cubes which $W$ intersects.

**Proposition 4.5.** *If $X$ is a CAT(0) cube complex, then every hyperplane $W$ is*

(i) *Two-sided,*

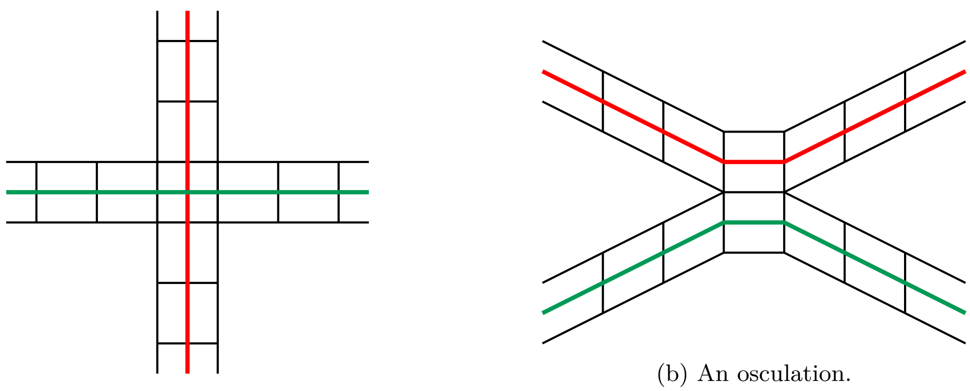
(ii) *Separated,*

(iii) *A CAT(0) cube complex.*

*Moreover, every midcube is in exactly one hyperplane.*

**Definition 4.6.** Two hyperplanes $V, W$ in $X$ are said to *contact* if $N(V) \cap N(W) \neq \varnothing$.

There are two types of contacts: crossings and osculations. See Figure 7.



(a) A crossing.

(b) An osculation.

Figure 7: Contacts in CAT(0) cube complexes.

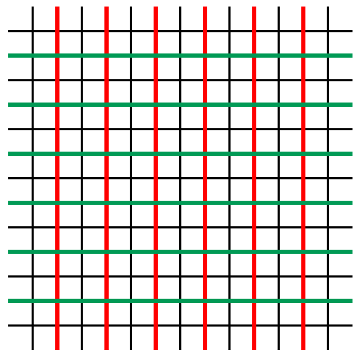
**Definition 4.7.** The *contact graph* of a CAT(0) cube complex $X$ is the graph $\Gamma$ with vertices corresponding to hyperplanes of $X$, and with edges given by contact. See Figure 8 for an example.

**Theorem 4.8** (Hagen [Hag14])**.** *The contact graph of a CAT(0) cube complex is quasi-isometric to a tree.*

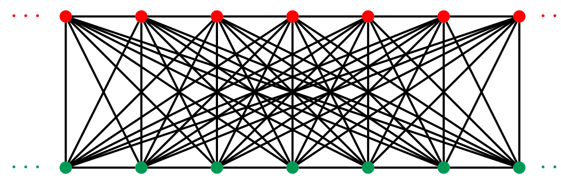
We do not prove Theorem 4.8 here, but we will just say that it relies on the following:

**Lemma 4.9** (Manning's Bottleneck Criterion)**.** *Let $Q$ be a geodesic metric space. Assume that there is a constant $\Delta \geq 0$ such that, for all $x, y \in Q$, there is a path $[x, y]$ between $x$ and $y$ that is contained in the $\Delta$-neighbourhood of every path from $x$ to $y$. Then $Q$ is quasi-isometric to a tree, and $\Delta$ is called a* bottleneck constant *for $Q$.*

As a consequence of Theorem 4.8, we obtain:



(a) The CAT(0) cube complex $X$.



(b) The contact graph of $X$.

Figure 8: A CAT(0) cube complex and its contact graph.

**Corollary 4.10.** *If a group $G$ acts properly and cocompactly on a CAT(0) cube complex, then it is quasi-arboreal with respect to hyperplane stabilisers.*

Corollary 4.10 hints at what a hierarchy is going to be: our group $G$ is quasi-arboreal with respect to hyperplane stabilisers; but hyperplanes are also CAT(0) cube complexes, so we can iterate. We will now see a more specific example of this in the case of right-angled Artin groups.

## 4.3 Right-angled Artin groups

**Definition 4.11.** Let $N$ be a simplicial graph.

- The *right-angled Artin group* defined by $N$ is

$$A_N = \langle V(N) \mid [v,w] = 1 \text{ whenever } \{v,w\} \in E(N) \rangle ,$$

where $V(N)$ and $E(N)$ denote the vertex and edge set respectively of $N$.

- The *Salvetti complex* $\mathcal{S}_N$ of $A_N$ is the cube complex obtained by starting with a 0-cell, adding one 1-cell for each vertex $v \in V(N)$, and one $k$-cube for each $k$-complete subgraph of $N$.

**Example 4.12.** (i) If $N = \bullet\ \bullet$, then $A_N$ is the free group of rank 2, $\mathcal{S}_N$ is the wedge of two circles, and the universal cover $\tilde{S}_N$ is the regular 4-valent tree.

(ii) If $N = \bullet\!\!-\!\!\bullet$, then $A_N$ is $\mathbb{Z}^2$, $\mathcal{S}_N$ is the (2-dimensional) torus, and the universal cover $\tilde{S}_N$ is $\mathbb{R}^2$ with the same cubulation as in Figure 8a.

**Proposition 4.13.** *The Salvetti complex $\mathcal{S}_N$ is nonpositively curved, so its universal cover $\tilde{S}_N$ is CAT(0).*

An important observation is that, in $\tilde{S}_N$, hyperplane stabilisers are again right-angled Artin groups. Again, this hints at what the hierarchy may be. In fact, we have the following (so far imprecise) theorem:

**Theorem 4.14** (Behrstock-Hagen-Sisto [BHS17])**.** *Let $X$ be a CAT(0) cube complex with a* factor system *$\mathcal{F}$, i.e. a collection of 'nice' convex subcomplexes. Then $X$ is hierarchically hyperbolic with respect to the set of* factored contact graphs *$\widehat{CW}$ for $W \in \mathcal{F}$, where $\widehat{CW}$ denotes the contact graph of $W$ with coned-off smaller factors.*

## Talk 5 – Mapping class groups and curve complexes

*Speaker:* Jason Behrstock. *Main references:* [MM99], [MM00].

### 5.1 Mapping class groups

We denote by $S = S_{g,p}$ an oriented surface of genus $g$ with $p$ punctures.

**Definition 5.1.** The *mapping class group* of $S$ is the group $\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$.

**Example 5.2.** Let $S = S_{0,4}$ be the 4-punctured sphere. Figure 9 shows two elements $a$ and $b$ of $\mathrm{MCG}(S)$. Those elements are called *half-twists*.

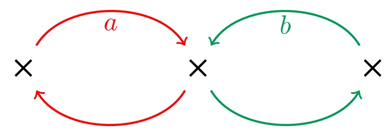



Figure 9: Two half-twists on the 4-punctured sphere (with one puncture at infinity).

We can understand the mapping class $ab$ by considering a simple closed curve $\gamma$ on $S$ and its iterates under $ab$, as in Figure 10.

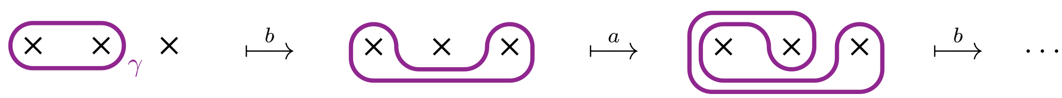



Figure 10: Applying successive half-twists to a simple closed curve on the 4-punctured sphere.

All the iterates $(ab)^k\gamma$ will be carried by the 'train track' represented in Figure 11.

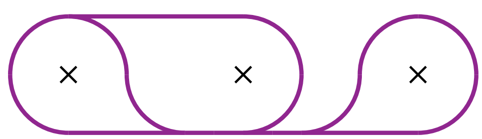


Figure 11: The invariant train track of the mapping class $ab$.

### 5.2 Curve complexes

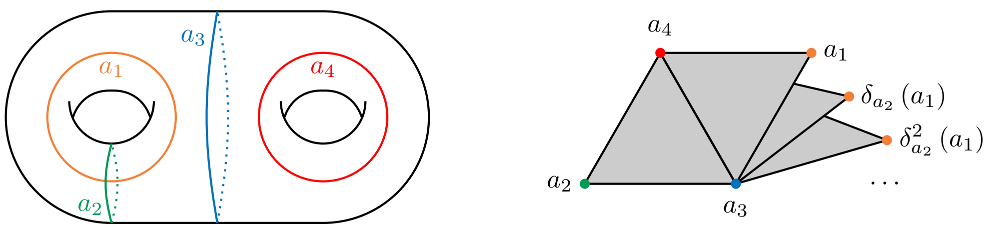
The mapping class group $\mathrm{MCG}(S)$ acts on an important geometric object:

**Definition 5.3.** The *curve complex* of $S$ is the simplicial complex $\mathcal{C}(S)$ where

- Vertices are homotopy classes of essential, non-peripheral simple closed curves on $S$,
- Edges correspond to simple closed curves which can be realised disjointly on $S$,

- $n$-simplices correspond to collections of $(n+1)$ disjoint simple closed curves.

See Figure 12.




(a) $S_{2,0}$. (b) $\mathcal{C}(S_{2,0})$.

Figure 12: A part of the curve complex of $S_{2,0}$ ($\delta_{a_2}$ denotes the Dehn twist in $a_2$).

We first give some basic properties of the curve complex:

**Proposition 5.4.** *Let $S = S_{g,p}$.*

(i) $\dim \mathcal{C}(S) = \xi(S) - 1$, *where* $\xi(S) = 3g + p - 3$ *is the* complexity *of $S$.*

(ii) *$\mathcal{C}(S)$ is locally infinite (see Figure 12).*

(iii) *$\mathcal{C}(S)$ is connected, except for the exceptional cases where $S = S_{1,0}, S_{1,1}, S_{0,4}$.*

**Remark 5.5.** In the exceptional cases $S = S_{1,0}, S_{1,1}, S_{0,4}$, the curve complex $\mathcal{C}(S)$ as we defined it has no edge. This can be fixed by replacing 'can be realised disjointly' by 'can be realised with minimal intersection among all pairs of curves on $S$' in Definition 5.3 (in $S_{1,0}$ or $S_{1,1}$, this means 'with one intersection point'; in $S_{0,4}$, this means 'with two intersection points'; in all other cases, this means 'with no intersection point').

**Example 5.6.** Let $S = S_{1,0}$ be the torus. Edges of $\mathcal{C}(S)$ correspond to pairs of simple closed curves with a single intersection. Observe that, by seeing the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, simple closed curves on $S$ correspond to elements of $\mathbb{Q} \cup \{\infty\}$, where each rational $q \in \mathbb{Q} \cup \{\infty\}$ is associated to the curve given by the image in $\mathbb{R}^2/\mathbb{Z}^2$ of the line of slope $q$ in $\mathbb{R}^2$. It turns out that $\mathcal{C}(S)$ is the *Farey graph*. See Figure 13. In fact, $\mathcal{C}(S_{1,0}) = \mathcal{C}(S_{1,1}) = \mathcal{C}(S_{0,4})$.

The following are deeper results on the curve complex:

**Proposition 5.7.** *Let $S = S_{g,p}$.*

(i) *$\mathcal{C}(S)$ has infinite diameter.*

(ii) (Masur-Minsky [MM99]) *$\mathcal{C}(S)$ is hyperbolic.*

## 5.3 Hierarchy paths

There is another way to understand $\mathrm{MCG}(S)$: one can pick a pants decomposition of $S$ and think of a mapping class as the image of that pants decomposition.

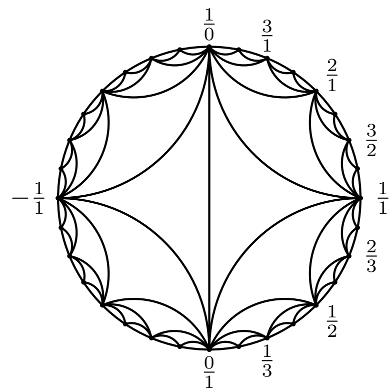



Figure 13: The curve graph of the torus.

**Example 5.8.** Take $S = S_{0,5}$. A pants decomposition of $S$ is given by the choice of two simple closed curves $a_1, a_2$. Applying a mapping class gives two new simple closed curves $b_1, b_2$. Now consider a geodesic $\gamma$ in $\mathcal{C}(S)$ from $a_1$ to $b_1$. The first point $x_1$ of $\gamma$ is a simple closed curve in $S \setminus a_1 = S_{0,3} \amalg S_{0,4}$. Note that $\mathcal{C}(S_{0,3})$ is a point, but there is a copy of $\mathcal{C}(S_{0,4})$ containing both $x_1$ and $a_2$. We can pick a geodesic in that copy of $\mathcal{C}(S_{0,4})$ from $a_2$ to $x_1$. Then we look at the next point $x_2$ on $\gamma$: it lies in a copy of $\mathcal{C}(S_{0,4})$ which also contains $a_1$. Repeating this process, we get the picture of Figure 14. The red path is called a *hierarchy path*. It really is a path in the pants

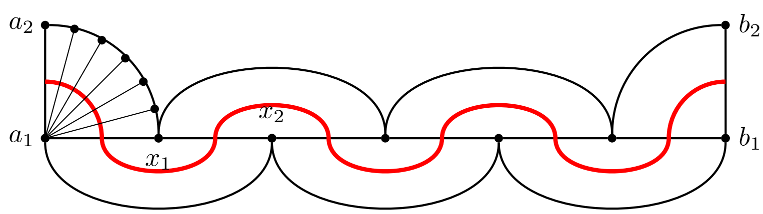



Figure 14: A hierarchy path in $\mathcal{C}(S_{0,5})$.

graph of $S$, which can be identified with MCG($S$).

The construction of a hierarchy path in Example 5.8 implicitly uses the fact that, given a subsurface $W \subseteq S$, there is a Lipschitz map

$$\pi_W : \mathrm{MCG}(S) \to \mathcal{C}(W).$$

This map is given by viewing an element of MCG($S$) as a pants decomposition of $S$, restricting it to $W$, and applying surgery.

The following coarse equality now tells us that hierarchy paths are quasi-geodesics in MCG($S$).

**Theorem 5.9** (Masur-Minsky [MM00]). *There is a constant $T_0 > 0$ such that for all $T \geq T_0$, there* *are constants $K, C$, such that for all $a, b \in \mathrm{MCG}(S)$,*

$$d_{\mathrm{MCG}(S)}(a,b) \asymp_{K,C} \sum_{\substack{Y \subseteq S \\ essential}} \left[\left[d_{\mathcal{C}(Y)}\left(\pi_Y(a), \pi_Y(b)\right)\right]\right]_T .$$

In the above theorem, $\asymp_{K,C}$ denotes a $(K,C)$-coarse inequality, and we write

$$[[x]]_T = \begin{cases} x & \text{if } x > T \\ 0 & \text{otherwise} \end{cases} .$$

One might then ask, given two essential subsurfaces $Y, Z \subseteq S$, what can be said about the image of the map

$$\mathrm{MCG}(S) \xrightarrow{\pi_Y \times \pi_Z} \mathcal{C}(Y) \times \mathcal{C}(Z).$$

If $Y$ and $Z$ can be realised disjointly, then the above map is coarsely onto.

Otherwise, the image of $\mathrm{MCG}(S)$ contains the wedge of $\mathcal{C}(Y)$ and $\mathcal{C}(Z)$ along the point

$$\left(\pi_Y\left(\partial Z\right), \pi_Z\left(\partial Y\right)\right) \in \mathcal{C}(Y) \times \mathcal{C}(Z).$$

Actually, this is coarsely an equality. In this case, we say that $Y$ and $Z$ are *overlapping* or *transverse*: this happens exactly when $Y$ and $Z$ intersect and are non-nested.

# Talk 6 – Hierarchically hyperbolic spaces

*Speaker:* Jason Behrstock. *Main reference:* [BHS17].

## 6.1 Contracting geodesics

Consider a geodesic $\gamma$ in a hyperbolic space. Then $\gamma$ satisfies the following:

**Definition 6.1.** A quasi-geodesic $\gamma$ in a metric space $X$ is said to be *contracting* if there is a map $\pi_\gamma : X \to \gamma$, and constants $A, D$ such that

(i) For all $x \in X$, $\operatorname{diam} \pi_\gamma\left(B_{A \cdot d(x,\gamma)}(x)\right) < D$,

(ii) $\pi_\gamma$ is coarsely idempotent,

(iii) $\pi_\gamma$ is coarsely Lipschitz.

See Figure 15.

The contracting property has the following applications.

**Proposition 6.2.** *Let $\gamma$ be a contracting quasi-geodesic. Then:*

(i) *$\gamma$ is* Morse*, i.e. quasi-geodesically stable. This means that any quasi-geodesic with endpoints on $\gamma$ lies within a bounded distance of $\gamma$.*

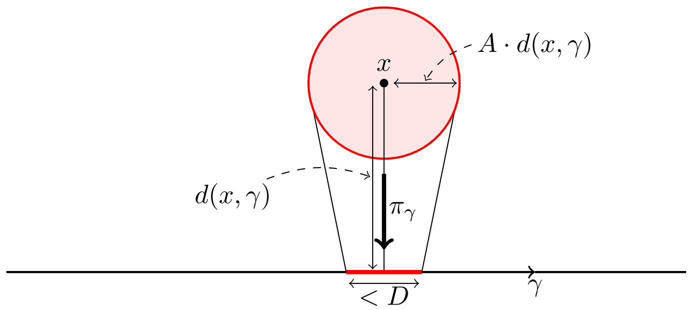



Figure 15: The contracting property for quasi-geodesics.

(ii) *$\gamma$ has at least quadratic divergence. This means that, given a ball $B$ centred on $\gamma$ of radius $r$, the shortest path in $X \smallsetminus B$ between two opposite points on $\gamma \smallsetminus B$ has length of order at least $r^2$.*

Our goal for this talk is to describe spaces where we can characterise the contracting quasi-geodesics.

**Remark 6.3.** There is also a notion of 'strongly contracting' quasi-geodesics, in which one demands that *all* balls (without any restriction on the radius) are projected to $\gamma$ with diameter less than $D$. This property is very strong but, contrary to the contracting property, it is not a quasi-isometry invariant.

## 6.2 Hierarchically hyperbolic spaces

A *hierarchically hyperbolic space* (or *HHS*) is a quasi-geodesic metric space $X$ (e.g. $X = \mathrm{MCG}(S)$), together with the following data:

- An indexing set $\mathfrak{S}$ (playing the role of the set of essential subsurfaces $Y \subseteq S$),
- A collection $\{\mathcal{C}(Y)\}_{Y \in \mathfrak{S}}$ of uniformly hyperbolic spaces (playing the role of the curve complexes),
- A partial order $\sqsubseteq$ on $\mathfrak{S}$ (playing the role of inclusion of subsurfaces), with a largest element $S$, and such that the lengths of chains are bounded,
- A symmetric, non-reflexive relation $\perp$ on $\mathfrak{S}$, called *orthogonality* (playing the role of disjointness), and such that there is a bound on the size of pairwise orthogonal subsets of $\mathfrak{S}$,
- A *transversality* relation: $Y$ and $Z$ are transverse if they are neither nested nor orthogonal,
- Coarsely Lipschitz maps $\pi_{\mathcal{C}(Y)} : X \to 2^{\mathcal{C}(Y)}$ (for each $Y \in \mathfrak{S}$) that sends points to bounded diameter sets in $\mathcal{C}(Y)$ (playing the role of subsurface projections).

In addition, those objects must satisfy some natural geometric assumptions.

The following result generalises Theorem 5.9 and says that, in a HHS, one can compute distances by computing a collection of distances in some hyperbolic spaces.

**Theorem 6.4** (Distance formula, Behrstock-Hagen-Sisto [BHS17])**.** *Let $(X, \mathfrak{S})$ be a HHS. There is a constant $T_0 > 0$ such that for all $T \geq T_0$, there are constants $K, C$, such that for all $a, b \in X$,*

$$d_X(a,b) \asymp_{K,C} \sum_{Y \in \mathfrak{S}} \left[\left[d_{\mathcal{C}(Y)}\left(\pi_Y(a), \pi_Y(b)\right)\right]\right]_T.$$

In the above theorem, $\asymp_{K,C}$ denotes a $(K, C)$-coarse inequality, and we write

$$[[x]]_T = \begin{cases} x & \text{if } x > T \\ 0 & \text{otherwise} \end{cases}.$$

The following gives a characterisation of hyperbolic spaces among HHSs. It says that, in a hyperbolic space, all products must look like strips.

**Theorem 6.5** (Behrstock-Hagen-Sisto [BHS17])**.** *Let $(X, \mathfrak{S})$ be a HHS. Then $X$ is hyperbolic if and only if there is some $q > 0$ such that, whenever $U \perp Y$,*

$$\min\{\operatorname{diam}\mathcal{C}(U), \operatorname{diam}\mathcal{C}(Y)\} < q.$$

There is a similar criterion for relative hyperbolicity, due to Russell.

**Remark 6.6.** There is also a notion of relative hierarchical hyperbolicity, where we require that $\mathcal{C}(Y)$ be hyperbolic only if $Y$ is not a minimal element for $\sqsubseteq$.

## 6.3 Examples of hierarchically hyperbolic spaces

The main motivating examples of HHSs are mapping class groups and right-angled Artin groups, as hinted at in Talks 4 and 5, but there are many others.

**Example 6.7.** The following are hierarchically hyperbolic spaces:

(i) Hyperbolic spaces, where $\mathfrak{S}$ has only one element.

(ii) Mapping class groups, with the structure described in §6.2.

(iii) Right-angled Artin groups $A_\Gamma$, where $\mathfrak{S} = \{gA_\Lambda \mid g \in A_\Gamma,\ \Lambda \subseteq \Gamma\}$ /parallelism, $\mathcal{C}(gA_\Lambda)$ is the factored contact graph of the Salvetti complex associated to $A_\Lambda$, $\sqsubseteq$ is the inclusion up to parallelism, $g_1A_{\Lambda_1} \perp g_2A_{\Lambda_2}$ if and only if $A_{\Lambda_1}$ and $A_{\Lambda_2}$ span a direct product, and $\pi_{\mathcal{C}(gA_\Lambda)}(x)$ is the set of hyperplanes containing $x$ (this set is a simplex in $\mathcal{C}(gA_\Lambda)$). See Talk 4.

(iv) Most non-positively curved cube complexes (including all special ones) with a geometric action. Conjecturally all non-positively curved cube complexes?

(v) Fundamental groups of 3-manifolds without any Nil or Sol component.

(vi) Leary-Minasyan groups.

(vii) Separating curve graphs.

In addition, there are combination theorems to produce new HHSs out of old ones.

There are also some obstructions to being a hierarchically hyperbolic group or space:

**Proposition 6.8.** *Every hierarchically hyperbolic group has quadratic Dehn function.*

### 6.4 Characterisation of contracting geodesics

As promised, we have a complete criterion for detecting contracting quasi-geodesics in HHSs:

**Theorem 6.9** (Abbott-Behrstock-Durham [ABD21])**.** *Let $X$ be a hierarchically hyperbolic group (i.e. a group that is a HHS, where all the structure is $G$-equivariant), or a hierarchically hyperbolic space with some minor additional assumptions. Then for all $D > 0$, there is $D' > 0$, such that for each quasi-geodesic $\gamma$ in $X$, the following are equivalent:*

(i) *$\gamma$ is $D$-contracting,*

(ii) *For each $U \in \mathfrak{S} \smallsetminus \{S\}$, diam $\left(\pi_{\mathcal{C}(U)}(\gamma)\right) < D'$.*

**Corollary 6.10.** *Every hierarchically hyperbolic group $X$ is universally acylindrically hyperbolic, in the sense that every element of $X$ that acts loxodromically for some acylindrical action of $X$ acts acylindrically on $X$ itself.*

## Talk 7 – Acylindrically hyperbolic groups: definitions

*Speaker:* Julian Wykowski. *Main reference:* [Osi16].

The goal of this talk is to explain all the terms in the following:

**Theorem 7.1** (Osin [Osi16])**.** *Let $G$ be a group. Then the following are equivalent:*

($AH_1$) *There is a (not necessarily finite) generating set $S$ for $G$ such that the Cayley graph $\Gamma(G,S)$ is hyperbolic, $|\partial\Gamma(G,S)| > 2$, and the action $G \curvearrowright \Gamma(G,S)$ is acylindrical.*

($AH_2$) *$G$ acts acylindrically and non-elementarily on a hyperbolic metric space.*

($AH_3$) *$G$ is not virtually cyclic and has an action on a hyperbolic space with at least one loxodromic element acting weakly properly discontinuously.*

($AH_4$) *$G$ contains a hyperbolically embedded infinite proper subgroup.*

### 7.1 Acylindrical actions

**Definition 7.2.** An action of a group $G$ on a geodesic metric space $X$ is *acylindrical* if for all $\varepsilon > 0$, there are $R > 0$ and $N \geq 1$ such that, for all $x, y \in X$ with $d(x,y) \geq R$, we have

$$|\{g \in G \mid d(x,gx) < \varepsilon \text{ and } d(y,gy) < \varepsilon\}| \leq N.$$

See Figure 16.

Note that any group $G$ acts acylindrically on a point. But acylindricity is not very interesting for small actions: that is why we assume in ($AH_2$) that the action is non-elementary (recall that an action of a group $G$ on a hyperbolic space $X$ is *elementary* if $|\Lambda(G)| \leq 2$, see also Proposition 1.15 in the case of a geometric action).

**Proposition 7.3.** *Any non-elementary geometric action on a hyperbolic space is in fact acylindrical.*

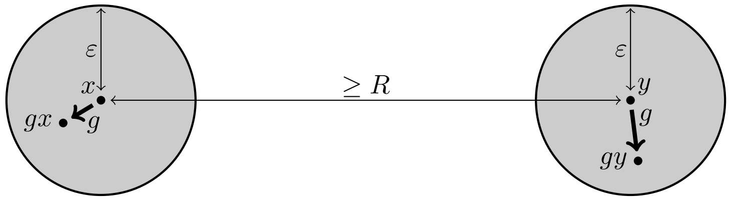



Figure 16: An action is acylindrical if there are few elements $g \in G$ acting as on the picture.

## 7.2 Weak proper discontinuity

**Definition 7.4.** Let $G$ be a group acting on a hyperbolic space $X$. A loxodromic element $g \in G$ is said to satisfy *weak proper discontinuity* (or *WPD*) if for all $\varepsilon > 0$ and for all $x \in X$, there is $M \geq 1$ such that the set

$$\left\{h \in G \mid d(x, hx) < \varepsilon \text{ and } d\left(g^M, hg^M x\right) < \varepsilon\right\}$$

is finite. See Figure 17.

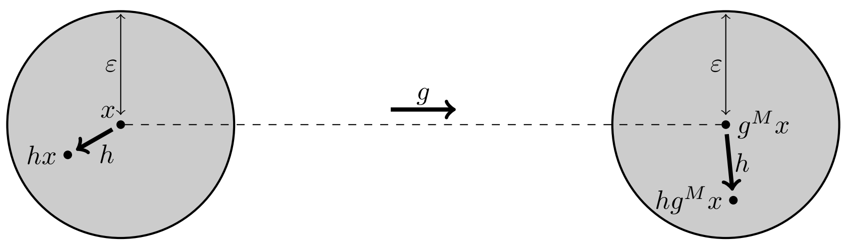



Figure 17: The element $g$ satisfies WPD if the set of elements $h$ as on the picture is finite.

Note that, if $G$ acts geometrically on $X$, then every loxodromic element satisfies WPD by properness of the action.

We have now defined all the words in the first three statements of Theorem 7.1. It is an exercise to check that $(AH_1) \Rightarrow (AH_2) \Rightarrow (AH_3)$.

## 7.3 Hyperbolically embedded subgroups

Let $G$ be a group and let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a collection of subgroups of $G$.

Given a subset $X \subseteq G$ such that $G = \langle X \cup \bigcup_\lambda H_\lambda \rangle$, we denote by

$$\Gamma\left(G, X \amalg \coprod_{\lambda \in \Lambda} H_\lambda\right)$$

the Cayley graph of $G$ with edges in $X \amalg \coprod_\lambda H_\lambda$ (note that, if the $H_\lambda$s intersect, we add edges multiple times for elements in the intersection).

For each $\lambda \in \Lambda$, there is an embedding $\Gamma(H_\lambda, H_\lambda) \hookrightarrow \Gamma\left(G, X \amalg \coprod_{\lambda\in\Lambda} H_\lambda\right)$, and the *relative metric*

$$\hat{d}_\lambda : H_\lambda \times H_\lambda \to [0, \infty]$$

on $H_\lambda$ is defined as follows: given $x, y \in H_\lambda$, $\hat{d}_\lambda(x, y)$ is the infimum of the lengths of the paths from $x$ to $y$ in $\Gamma\left(G, X \amalg \coprod_{\lambda\in\Lambda} H_\lambda\right)$ that do not contain any edge in $\Gamma(H_\lambda, H_\lambda)$.

**Definition 7.5.** The collection $\{H_\lambda\}_{\lambda\in\Lambda}$ is said to be *hyperbolically embedded* in $(G, X)$, which we denote $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$, if

(i) $\Gamma\left(G, X \amalg \coprod_{\lambda\in\Lambda} H_\lambda\right)$ is hyperbolic, and

(ii) For each $\lambda \in \Lambda$, the metric space $\left(H_\lambda, \hat{d}_\lambda\right)$ is proper (i.e. any ball of finite radius is finite).

**Example 7.6.** (i) For any group $G$, we have $G \hookrightarrow_h (G, \varnothing)$ because $\hat{d} = \infty$.

(ii) Given $H \leq G$, we have $H \hookrightarrow_h (G, G)$ whenever $H$ is finite.

(iii) Consider $G = H \times \langle t \rangle$ and $X = \{t\}$. Then $\left(H, \hat{d}\right)$ has diameter $\leq 3$.

It follows that $H \hookrightarrow_h (H \times \langle t \rangle, \{t\})$ if and only if $H$ is finite.

(iv) Consider $G = H * \langle t \rangle$ and $X = \{t\}$. Then it is always true that $H \hookrightarrow_h (H * \langle t \rangle, \{t\})$.

Observe that Definition 3.12 can be reinterpreted as saying that the group $G$ is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ if and only if there is a relative generating set $X$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$.

### 7.4 Examples of acylindrically hyperbolic groups

**Definition 7.7.** We say that a group $G$ is *acylindrically hyperbolic* if any (hence every) of the statements $(AH_1) - (AH_4)$ of Theorem 7.1 hold.

We now give a few examples:

**Example 7.8.** (i) Non-elementary hyperbolic groups are acylindrically hyperbolic.

(ii) Non-virtually cyclic groups that are hyperbolic relative to a finite collection of proper peripheral subgroups are acylindrically hyperbolic.

(iii) $\mathrm{MCG}(S_{g,p})$ is acylindrically hyperbolic unless $g = 0$ and $p \leq 3$ (because the action on the curve complex is acylindrical).

(iv) $\mathrm{Out}(F_n)$ is acylindrically hyperbolic for $n \geq 2$.

(v) (Hamenstädt) Any group acting properly on a proper hyperbolic space of bounded growth is either acylindrically hyperbolic or virtually nilpotent.

## Talk 8 – Acylindrically hyperbolic groups: applications and bounded cohomology

*Speaker:* Julian Wykowski. *Main reference:* [Osi16].

## 8.1 Classification of acylindrical actions

The following theorem classifies acylindrical actions:

**Theorem 8.1** (Osin [Osi16])**.** *Let $G$ be a group with an acylindrical action on a metric space $X$. Then exactly one of the following holds:*

(i) *The action has bounded orbits.*

(ii) *$G$ is virtually cyclic and has a loxodromic elements.*

(iii) *The action has infinitely many loxodromic elements that are* independent *(i.e. their limit points are pairwise disjoint).*

Theorem 8.1 says that an acylindrical action cannot be parabolic or *quasi-parabolic* (i.e. where the limit set is infinite but all loxodromic elements share a limit point).

We now give two examples of non-acylindrical action where the theorem fails.

**Example 8.2.** Let $G$ be a group with a finite generating set $S$. We construct a graph $\mathcal{H}(G,S)$, with vertex set $G \times \mathbb{N}_{\geq 0}$, and with two types of edges:

- For all $g \in G$ and $k \in \mathbb{N}_{\geq 0}$, there is an edge $(g,k) \leftrightarrow (g,k+1)$,
- For all $g,h \in G$ and $k \in \mathbb{N}_{\geq 0}$, if $0 < d_S(g,h) \leq 2^k$, then we add an edge $(g,k) \leftrightarrow (h,k)$.

The graph $\mathcal{H}(G,S)$ can be thought of as putting together Cayley graphs $\Gamma\left(G, S^{k+1}\right)$ for all $k \in \mathbb{N}_{\geq 0}$.

It turns out that $\mathcal{H}(G,S)$ is always hyperbolic. Thus we have a proper action of $G$ on a hyperbolic space, which is not acylindrical unless $G$ is finite.

This action does not satisfy the conclusion of Theorem 8.1 since the action $G \curvearrowright \mathcal{H}(G,S)$ is parabolic. Note in particular that, if $G$ is a finitely generated infinite torsion group, then all $g \in G$ have bounded orbits, but $G$ has unbounded orbits.

**Example 8.3.** Consider $BS(1,2) = \langle a, t \mid t^{-1}at = a^2 \rangle$. There is an embedding $BS(1,2) \hookrightarrow SL_2(\mathbb{R})$ via

$$a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

This gives an action $BS(1,2) \curvearrowright \mathbb{H}^2$, which is free and non-elementary.

However, any two loxodromic elements share a limit point, so the action is quasi-parabolic.

## 8.2 More examples, and some applications

Continuing Example 7.8, here are some more acylindrically hyperbolic groups:

**Example 8.4.** (vi) (Minasyan) Many graphs of groups are acylindrically hyperbolic. It follows in particular that

- One-relator groups with at least three generators are acylindrically hyperbolic,
- For any field $k$, $\mathrm{Aut}\,(k[x,y])$ is acylindrically hyperbolic.

(vii) (Wilton-Zalesski) If $M$ is a compact 3-manifold, then $\pi_1 M$ is either virtually polycyclic, or acylindrically hyperbolic, or there is a short exact sequence

$$1 \to \mathbb{Z} \to \pi_1 M \to Q \to 1,$$

with $Q$ acylindrically hyperbolic.

We give one selected application of acylindrical hyperbolicity. Recall the following from Definition 1.16:

**Definition 8.5.** A group $G$ is *SQ-universal* if every countable group embeds into some quotient of $G$.

**Theorem 8.6** (Dahmani-Guirardel-Osin)**.** *Let $G$ be an acylindrically hyperbolic group. Then:*

(i) *$G$ is SQ-universal.*

(ii) *The elementary theory of $G$ is not superstable.*

## 8.3 Bounded cohomology

We'll give one application of acylindrical hyperbolicity to bounded cohomology. We start by recalling some definitions.

Let $X$ be a topological space and let $V$ be a Banach space over $\mathbb{R}$.

- $S_n(X)$ is the set of singular simplices on $X$, i.e. continuous maps $\Delta^n \to X$.
- $C_n(X)$ is the free $\mathbb{Z}$-module with basis $S_n(X)$.
- $C^n(X, V)$ is the $\mathbb{Z}$-module of homomorphisms $C_n(X) \to V$, which correspond to maps $S_n(X) \to V$.
- $C_b^n(X, V)$ is the set of bounded maps $S_n(X) \to V$.

Recall that $C^\bullet(X, V)$ is equipped with a differential $d^{n+1} : C^n(X, V) \to C^{n+1}(X, V)$. Observe that $C_b^n(X, V) \subseteq C^n(X, V)$, and $C^\bullet(X, V)$ is preserved by the differential. Hence, we get a cochain complex $(C_b^\bullet(X, V), d)$.

**Definition 8.7.** The *bounded cohomology* of $X$ with coefficients in $V$ is defined by

$$H_b^n(X, V) = H^n\left(C_b^\bullet(X, V)\right).$$

Note that the inclusion $C_b^\bullet(X, V) \hookrightarrow C^\bullet(X, V)$ commutes with the differential by construction, so it descends to a map on cohomology groups.

**Definition 8.8.** The *comparison map* is the homomorphism

$$\psi^n : H_b^n(X, V) \to H^n(X, V)$$

induced by the inclusion $C_b^\bullet(X, V) \hookrightarrow C^\bullet(X, V)$.

There is a seminorm $\|\cdot\|_\infty$ on $H_b^n(X, V)$, defined by

$$\|[\varphi]\|_\infty = \inf_{f \in [\varphi]} \sup_{\sigma \in S_n(X)} \|f(\sigma)\|_V.$$

Bounded cohomology can be defined for groups as well. Suppose $G$ is a group acting on the Banach space $V$ by isometries.

- If the action $G \curvearrowright V$ is trivial, one can set $H_b^n(G,V) = H_b^n\left(K(G,1),V\right)$.
- In general, one can define $H_b^n(G,V)$ as the $n$-th cohomology group of the cochain complex $C_b^\bullet(G,V)$, where $C_b^n(G,V)$ is the set of bounded functions $G^n \to V$.

It turns out that hyperbolicity can be completely characterized in terms of bounded cohomology:

**Theorem 8.9** (Mineyev)**.** *A finitely presented group $G$ is hyperbolic if and only if the comparison map*

$$H_b^2(G,V) \to H^2(G,V)$$

*is surjective for any Banach $G$-module $V$ that is* bounded *(i.e. elements of $G$ act on $V$ with uniformly bounded norms).*

Acylindrical hyperbolicity also has consequences in bounded cohomology:

**Theorem 8.10** (Bestvina-Fujiwara, Hamenstädt)**.** *If $G$ is acylindrically hyperbolic and $V = \mathbb{R}$ or $V = \ell^p(G)$ $(1 \le p < \infty)$, then* $\operatorname{Ker}\left(H_b^2(G,V) \to H^2(G,V)\right)$ *is infinite-dimensional.*

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