The Loop & Sphere Theorems

Reading group on 3-manifolds

Alexis Marchand

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Suppose that we have a loop $\gamma : S^1 \to M$ on a 3-manifold M and assume that γ is null-homotopic. This means that there is a map $f : D^2 \to M$ with $f_{|S^1} = \gamma$, where D^2 is the 2-dimensional disc, with $\partial D^2 = S^1$. The general problem that we are concerned with is, assuming that γ is regular enough, to upgrade f to a regular map. More specifically, we would like to answer the following:

Question 1. If $\gamma : S^1 \hookrightarrow M$ is a null-homotopic *embedded* loop, is there an embedding $f : D^2 \hookrightarrow M$ such that $f_{|S^1} = \gamma$?

Remark 2. If M is a surface, then the Jordan-Schönflies Theorem gives an affirmative answer to Question 1.

Dehn's Lemma, the Loop and Sphere Theorems

For 3-manifolds, an answer to Question 1 is given by Dehn's Lemma:

Lemma 3 (Dehn's Lemma [Deh10])**.** *Let $f : D^2 \to M$ be a map that restricts to an embedding on some neighbourhood of $\partial D^2 = S^1$. Then $f_{|S^1} : S^1 \to M$ extends to an embedding $D^2 \hookrightarrow M$.*

Dehn's original proof had a gap, but this was fixed by Papakyriakopoulos with the Loop and Sphere Theorems. We give a stronger version of the Loop's Theorem, due to Stallings:

Theorem 4 (Loop Theorem [Pap57, Sta60])**.** *Let M be a 3-manifold and let B be a connected component of ∂M. Let N be a normal subgroup of $\pi_1 B$. If $\mathrm{Ker}(\pi_1 B \to \pi_1 M) \not\subseteq N$, then there is an embedding*

$$g : \left(D^2, S^1\right) \hookrightarrow (M, B)$$

such that $\left[g_{|S^1}\right] \notin N$.

We first explain why the Loop Theorem implies Dehn's Lemma:

Proof (Loop ⇒ Dehn). Let $f : D^2 \to M$ be a map that restricts to an embedding on some neighbourhood of $\partial D^2 = S^1$. Consider R a regular neighbourhood of $f(S^1)$ in M, and let

$$M_1 = \overline{M \setminus R}.$$

Hence M_1 is a 3-manifold, with boundary homeomorphic to the 2-torus T^2. Apply the Loop Theorem to M_1, with $N = 1$, $B = \partial M_1 \cong T^2$. Note that $\pi_1 B \cong \mathbb{Z}^2$, with basis $\{a, b\}$, where a and b are represented by simple closed curves on B, with $a = \left[f_{|S^1}\right]$. In particular, the existence of the map $f : D^2 \to M$ shows that a maps to 1 under $\pi_1 B \to \pi_1 M_1$. Hence, the Loop Theorem gives an embedding

$$g : \left(D^2, S^1\right) \hookrightarrow (M, B)$$

such that $\left[g_{|S^1}\right] \neq 1$. Set $c = \left[g_{|S^1}\right] \in \pi_1 B$, and write $c = ka + \ell b$, with $k, \ell \in \mathbb{Z}$. Since a and c map to 1 in $\pi_1 M_1$ and b does not, we must have $\ell = 0$. Moreover, c admits a simple closed curve representative in B, from which it follows that $k = \pm 1$. After possibly changing orientations, we may assume that $k = 1$, proving that $\left[g_{|S^1}\right] = \left[f_{|S^1}\right]$. After performing a homotopy in B, we obtain the result. □

Papakyriakopoulos also proved the following:

Theorem 5 (Sphere Theorem)**.** *Let M be an orientable 3-manifold, and let $N \lneq \pi_2 M$ be a proper $\pi_1 M$-invariant subgroup. Then there is an embedding $g : S^2 \hookrightarrow M$ such that $[g] \notin N$.*

Remark 6. The action of the fundamental group on higher homotopy groups is illustrated in Figure 1, interpreting $\pi_n(M, x)$ as the set of homotopy classes of maps $(D^n, \partial D^n) \to (M, x)$. In particular, if $n = 1$, then we recover the action

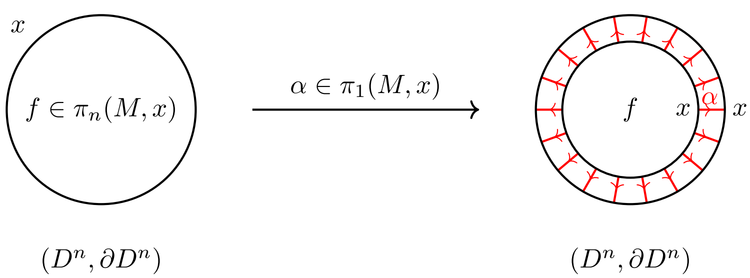


Figure 1: The action $\pi_1(M, x) \curvearrowright \pi_n(M, x)$.

of $\pi_1(M, x)$ by conjugation on itself.

Applying the Sphere Theorem with $N = 1$ yields a corollary worth stating separately:

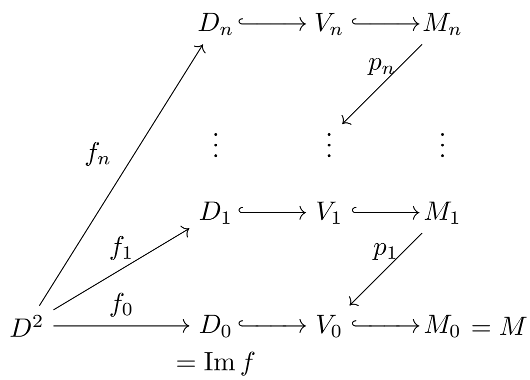
Corollary 7. *Let M be an orientable 3-manifold. If $\pi_2 M \neq 1$, then there is an embedding $S^2 \hookrightarrow M$ that is not null-homotopic.*

Our goal for this talk is to give the main ideas of the proof of the Loop Theorem (in the case $N = 1$). We will follow [Hat07] and [Sta60].

The tower argument

Proof of Theorem 4 (for $N = 1$). Start with a map $f : \left(D^2, S^1\right) \to (M, \partial M)$ with $\left[f_{|S^1}\right] \neq 1$ in $\pi_1 B$. The goal is to upgrade f to an embedding.

The strategy is to construct a *tower*: starting with $f_0 = f : D^2 \to M$, pick a regular neighbourhood V_0 of $D_0 = \operatorname{Im} f_0$ in $M_0 = M$. Then take a (connected) 2-sheeted cover $p_1 : M_1 \to V_0$. Since D^2 is simply-connected, f_0 lifts to a map $f_1 : D^2 \to M_1$. Now pick a regular neighbourhood V_1 of $D_1 = \operatorname{Im} f_1$ in M_1, and repeat the construction. One obtains the following commutative diagram:



(1)

Each V_i is a regular neighbourhood of $D_i = \operatorname{Im} f_i$, and each $p_{i+1} : M_{i+1} \to V_i$ is a (connected) 2-sheeted cover.

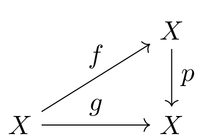
Claim. The tower construction eventually terminates.

Proof. The idea is to measure the *singularity* of the maps f_i, and show that this becomes smaller at each step.

Given a (set-theoretic) map $f : X \to Y$, the *singularity* of f is the set

$$\mathcal{S}(f) = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}.$$

Observe that, given a commutative diagram



we have $\mathcal{S}(f) \subseteq \mathcal{S}(g)$, with equality if and only if the restriction of p to $\operatorname{Im} f$ is injective.

Therefore, we have a chain of proper inclusions

$$\cdots \subsetneq \mathcal{S}(f_{i+1}) \subsetneq \mathcal{S}(f_i) \subsetneq \cdots \subsetneq \mathcal{S}(f_1) \subsetneq \mathcal{S}(f_0).$$

For now, this is no more than a chain of closed subsets of $D^2 \times D^2$, so it could decrease indefinitely. To make the inductive argument work, we need to make the situation discrete. To do this, we *triangulate* the manifold M and the disc D^2 in such a way that the map $f : D^2 \to M$ is simplicial, and we perform the construction of the tower in such a way that all resulting spaces and maps are simplicial.

Therefore, $(\mathcal{S}(f_i))_{i \geq 0}$ is a strictly descending chain of subcomplexes of $D^2 \times D^2$, so it is eventually constant and the construction must terminate. □

Now we have a finite tower as in (1). The next step is to see that, at the top of tower, the topology has become simpler and we can show there that $f_n : D^2 \to M_n$ is an embedding.

Lemma 8. *Let M be a compact 3-manifold that has no connected 2-sheeted cover. Then every component of ∂M is a 2-sphere.*

Proof. Note that connected 2-sheeted covers of M correspond to maps $\pi_1 M \to \mathbb{Z}/2$, which correspond to maps $H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$. Hence, the assumption that M has no connected 2-sheeted cover means that

$$H_1(M; \mathbb{Z}/2) = 0.$$

By Poincaré-Lefschetz Duality and the Universal Coefficient Theorem, it follows that $H_2(M, \partial M; \mathbb{Z}/2) \cong H^1(M; \mathbb{Z}/2) = 0$. Therefore, the long exact sequence of $(M, \partial M)$ is

$$\cdots \to \underbrace{H_2(M, \partial M; \mathbb{Z}/2)}_{=0} \to H_1(\partial M; \mathbb{Z}/2) \to \underbrace{H_1(M; \mathbb{Z}/2)}_{=0} \to \cdots.$$

It follows that $H_1(\partial M; \mathbb{Z}/2) = 0$. Since the only compact connected surface Σ with $H_1(\Sigma; \mathbb{Z}/2) = 0$ is the 2-sphere, it follows that every component of ∂M is a 2-sphere. □

Hence, every component of ∂M_n is a 2-sphere. Now consider the preimage of ∂M_0 inside M_n. This is a planar surface F, so its fundamental group is generated by some loops in ∂F. One of these loops must be nontrivial in $\pi_1 F$ (because of the assumption that $\left[f_{|S^1}\right] \neq 1$ in $\pi_1 B$), and this loop bounds an embedded disc in ∂V_n. This yields an embedding

$$g^{(n)} : \left(D^2, S^1\right) \hookrightarrow (M_n, F)$$

with $\left[g^{(n)}_{|S^1}\right] \neq 1$ in $\pi_1 F$.

We have an embedding at the top of the tower; the final step is to descend back to M_0. At each step of the tower, we produce at most double singularities, and they can be eliminating by some surgery operations. See [Hat07] or [Sta60] for more details. In the end, we obtain an embedding

$$g^{(0)} : \left(D^2, S^1\right) \hookrightarrow (M, B)$$

with $\left[g^{(0)}_{|S^1}\right] \neq 1$ in $\pi_1 B$. □

Application of the Loop Theorem

We conclude with a simple application of the Loop Theorem to the classification of 3-manifolds. We are interested in classifying all prime 3-manifolds, and the following classifies those that have infinite cyclic fundamental group:

Proposition 9. *Let M be an orientable compact connected 3-manifold. Assume that M is prime and $\pi_1 M \cong \mathbb{Z}$. Then $M \cong S^1 \times S^2$ (if $\partial M = \emptyset$) or $M \cong S^1 \times D^2$ (if $\partial M \neq \emptyset$).*

Proof. Case 1: $\partial M \neq \emptyset$. Note that the only prime 3-manifold with a spherical boundary component is B^3 (which has $\pi_1 B^3 = 1$), so ∂M contains no 2-sphere. Moreover, $\dim H_1(M; \mathbb{Q}) = 1$, so Poincaré-Lefschetz Duality and the Universal Coefficient Theorem give $\dim H_2(M, \partial M; \mathbb{Q}) = \dim H^1(M; \mathbb{Q}) = 1$. Now consider the following commutative diagram, with vertical arrows given by Poincaré-Lefschetz Duality:

$$\begin{array}{ccccccc} \cdots \to H_2(M, \partial M; \mathbb{Q}) & \xrightarrow{\partial} & H_1(\partial M; \mathbb{Q}) & \xrightarrow{i_*} & H_1(M; \mathbb{Q}) & \longrightarrow \cdots \\ \text{PD} \downarrow \wr\vert & & \text{PD} \downarrow \wr\vert & & \text{PD} \downarrow \wr\vert & \\ \cdots \longrightarrow H^1(M; \mathbb{Q}) & \xrightarrow{i^*} & H^1(\partial M; \mathbb{Q}) & \xrightarrow{\delta} & H^2(M, \partial M; \mathbb{Q}) & \to \cdots \end{array}$$

Observe that

$$\begin{aligned} \operatorname{rk} \partial &= \dim \operatorname{Im} \partial = \dim \operatorname{Ker} i_* = \dim \operatorname{Coker} i^* = \dim \operatorname{Coker} \partial \\ &= \dim H_1(\partial M; \mathbb{Q}) - \operatorname{rk} \partial. \end{aligned}$$

Therefore

$$\dim H_1(\partial M; \mathbb{Q}) = 2 \operatorname{rk} \partial \leq 2 \dim H_2(M, \partial M; \mathbb{Q}) = 2.$$

It follows that ∂M is a 2-torus. In particular, the map $\pi_1(\partial M) \to \pi_1 M$ is not injective, so the Loop Theorem gives an embedding

$$g : (D^2, S^1) \hookrightarrow (M, \partial M)$$

with $[g_{|S^1}] \neq 1$ in $\pi_1(\partial M)$. Cutting M along $g(D^2)$ yields a splitting

$$M = N \sharp (S^1 \times D^2).$$

But M is prime, so $N \cong S^3$.

Case 2: $\partial M = \emptyset$. Then we need the

Fact. Every class in $H_2(M)$ is represented by an embedded oriented closed surface $\Sigma \to M$, with every component of Σ mapping to M π_1-injectively.

By Poincaré Duality and the Universal Coefficient Theorem, we have

$$H_2(M) \cong H^1(M) \cong H_1(M) \cong \mathbb{Z}.$$

Hence pick a class in $H_2(M) \setminus 0$, and represent it by an embedded oriented closed surface $\Sigma \to M$ where each component is π_1-injective. But $\pi_1 M \cong \mathbb{Z}$, so each component of Σ is a 2-sphere. Again, this gives a splitting

$$M = N\sharp\left(S^1 \times S^2\right),$$

and M is prime, so $N \cong S^3$. $\square$

References

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