Stable commutator length, surfaces, and rationality

Alexis Marchand

Supervisor: Henry Wilton

UNIVERSITY OF CAMBRIDGE

What is scl?

Algebraic definition. Given $w \in G$, set

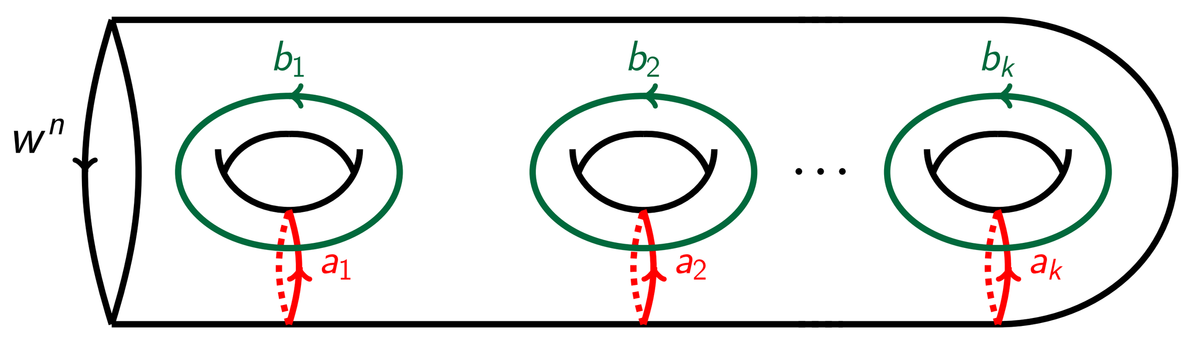
$$\mathrm{cl}_G(w) := \inf\{k \geq 1 \mid \exists a_1, b_1, \dots, a_k, b_k \in G,\ w = [a_1, b_1] \cdots [a_k, b_k]\} \in \mathbb{N}_{\geq 0} \cup \{\infty\}.$$

The stable commutator length of w is

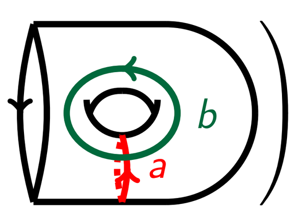
$$\mathrm{scl}_G(w) := \lim_{n \to \infty} \frac{\mathrm{cl}_G(w^n)}{n}.$$

Topological interpretation. Assume that $G = \pi_1 X$. Then

$$\mathrm{scl}_G(w) = \inf\left\{ \frac{-\chi^-(\Sigma)}{2n(\Sigma)} \,\middle|\, \substack{f:(\Sigma,\partial\Sigma)\to(X,w) \\ \Sigma \text{ or. comp. surface} \\ [f_{|\partial\Sigma}]=[w^{n(\Sigma)}]} \right\}.$$



Example. Given $a, b \in G$,

$\mathrm{scl}_G([a,b]) \leq -\frac{1}{2}\chi^-\Big($  $\Big) = \frac{1}{2}.$

In fact, if $G = F(a,b)$, then $\mathrm{cl}_G([a,b]^n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ (!)

Dual interpretation. A quasimorphism is a map $\phi : G \to \mathbb{R}$ such that

$$D(\phi) := \sup_{a,b \in G} |\phi(ab) - \phi(a) - \phi(b)| < \infty.$$

We denote by $Q(G)$ the set of quasimorphisms on G, satisfying in addition $\phi(w^n) = n \cdot \phi(w)$ for all $w \in G$ and $n \in \mathbb{Z}$.

(Bavard '91) $\mathrm{scl}_G(w) = \sup\left\{\frac{1}{2}\phi(w)/D(\phi) \mid \phi \in Q(G),\ D(\phi) \neq 0\right\}$.

Scl as a measure of curvature

Definition. A group G is said to have a spectral gap for scl if

$$\exists \varepsilon > 0,\ \forall w \in G,\ \mathrm{scl}_G(w) \in \{0\} \cup [\varepsilon, \infty].$$

Theorem. The following groups have spectral gaps:

- **(Duncan–Howie '91)** Free groups (with $\varepsilon = \frac{1}{2}$),
- **(Calegari–Fujiwara '10)** Hyperbolic groups,
- **(Bestvina–Bromberg–Fujiwara '16)** Mapping class groups,
- **(Heuer '19)** Right-angled Artin groups (with $\varepsilon = \frac{1}{2}$),
- **(Chen–Heuer)** 3-manifold groups.

Most of the above results were proved by constructing quasimorphisms.

- **(M. [3])** A new topological proof of Heuer's (sharp) spectral gap for RAAGs by constructing non-positively curved angle structures on surfaces bounding some power of w.

General philosophy. Non-positively curved groups have large values of scl.

Opposite situation. If G is amenable, then $\mathrm{scl}_G(w) = 0$ for all $w \in [G,G]$.

Computing scl

Theorem. Scl is computable and has rational values in

- **(Calegari '09)** Free groups,
- **(Chen '20)** Graphs of groups with infinite cyclic vertex and edge groups.

Question. What about closed surface groups?

Website: dpmms.cam.ac.uk/~aptm3

Email: aptm3@cam.ac.uk

Isometric embeddings

Idea. Construct embeddings $f : F_r \hookrightarrow \pi_1 S_g$ from free groups to closed surface groups that are scl-isometric in the sense that

$$\forall w \in [F_r, F_r],\ \mathrm{scl}_{\pi_1 S_g}(f(w)) = \mathrm{scl}_{F_r}(w).$$

Theorem 1 (M. [2])

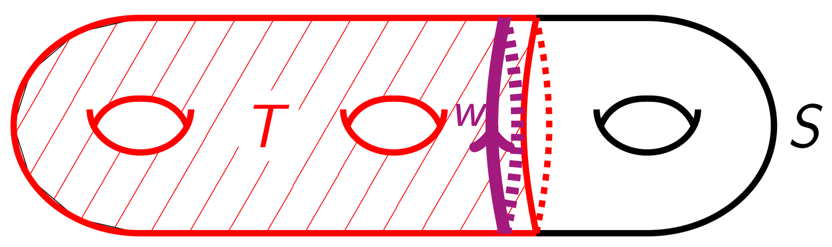
Let S be an oriented compact surface with $\partial S \neq \varnothing$ and let $T \subseteq S$ be a π_1-injective subsurface. Then

$$\pi_1 T \hookrightarrow \pi_1 S$$

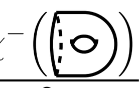
is an scl-isometric embedding.

The relative ℓ^1-seminorm

Example. If S is closed, Theorem 1 does not hold.



$$\mathrm{scl}_{\pi_1 T}(w) = \frac{-\chi^-(T)}{2} = \frac{3}{2}.$$

$\mathrm{scl}_{\pi_1 S}(w) \leq \frac{-\chi^-(\text{}}{2}$  $\frac{)}{} = \frac{1}{2}.$

But there is a way to generalise...

Definition. The relative ℓ^1-seminorm on $H_2(X, w)$ is defined by

$$\|\alpha\|_1 := \inf\left\{ \frac{-2\chi^-(\Sigma)}{n(\Sigma)} \,\middle|\, \substack{f:(\Sigma,\partial\Sigma)\to(X,w) \\ \Sigma \text{ or. comp. surface} \\ f_*[\Sigma]=n(\Sigma)\alpha} \right\}.$$

Observation. $4\,\mathrm{scl}_{\pi_1 X}(w) = \inf\left\{\|\alpha\|_1 \mid \alpha \in H_2(X,w),\ \partial\alpha = [w]\right\}$.

Theorem 2 (M. [2])

Let S be an oriented compact surface, $T \subseteq S$ a π_1-injective subsurface. Given $w \in \pi_1 T$,

$$H_2(T, w) \hookrightarrow H_2(S, w)$$

is an ℓ^1-isometric embedding.

Finding classes with rational ℓ^1-seminorm

Theorem (Calegari '09). Let S be a compact hyperbolic surface with $\partial S \neq \emptyset$. If a power of $w \in \pi_1 S$ is bounded by an immersed surface $f : (\Sigma, \partial\Sigma) \to (S, w)$, then

$$\mathrm{scl}_{\pi_1 S}(w) = \frac{-\chi^-(\Sigma)}{2n(\Sigma)} = \frac{1}{2}\mathrm{rot}_S(w).$$

Theorem 3 (M. [1])

Let S be a compact hyperbolic surface, $w \in \pi_1 S$. If a multiple of $\alpha \in H_2(S, w)$ is represented by an immersed surface Σ, then

$$\|\alpha\|_1 = \frac{-2\chi^-(\Sigma)}{n(\Sigma)} = -2\left\langle \mathrm{eu}_b^{\mathbb{R}}(S), \alpha \right\rangle.$$

Acknowledgements

I would like to thank Macarena Arenas and Julian Wykowski for their feedback on a previous version of this poster, as well as my supervisor Henry Wilton for his constant support.

References

[1] A. Marchand. Bavard duality for the relative Gromov seminorm. Preprint available on arXiv:2305.02149, 2023.

[2] A. Marchand. Isometric embeddings of surfaces for scl. Preprint available on arXiv:2302.04133. To appear in *Groups Geom. Dyn.*, 2023.

[3] A. Marchand. From letter-quasimorphisms to angle structures and spectral gaps. In preparation, 2024.