Stable commutator length and spectrum problems

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What is scl?

Algebraic definition. Let $w \in G$ be an element in a group.

$$\mathrm{cl}_G(w) := \inf\{k \geq 0 \mid \exists a_i, b_i \in G,\ w = [a_1, b_1] \cdots [a_k, b_k]\} \in \mathbb{N}_{\geq 0} \cup \{\infty\}.$$

The **stable commutator length** of w is

$$\mathrm{scl}_G(w) := \liminf_{n \to \infty} \frac{\mathrm{cl}_G(w^n)}{n} \in [0, +\infty].$$

Remark. $\mathrm{cl}_G(w) < +\infty \Leftrightarrow w \mapsto 0$ in $H_1(G; \mathbb{Z})$,
$\mathrm{scl}_G(w) < +\infty \Leftrightarrow w \mapsto 0$ in $H_1(G; \mathbb{Q})$.

Topological interpretation. Let $G = \pi_1 X$, and $w = [\gamma : S^1 \to X]$. Then

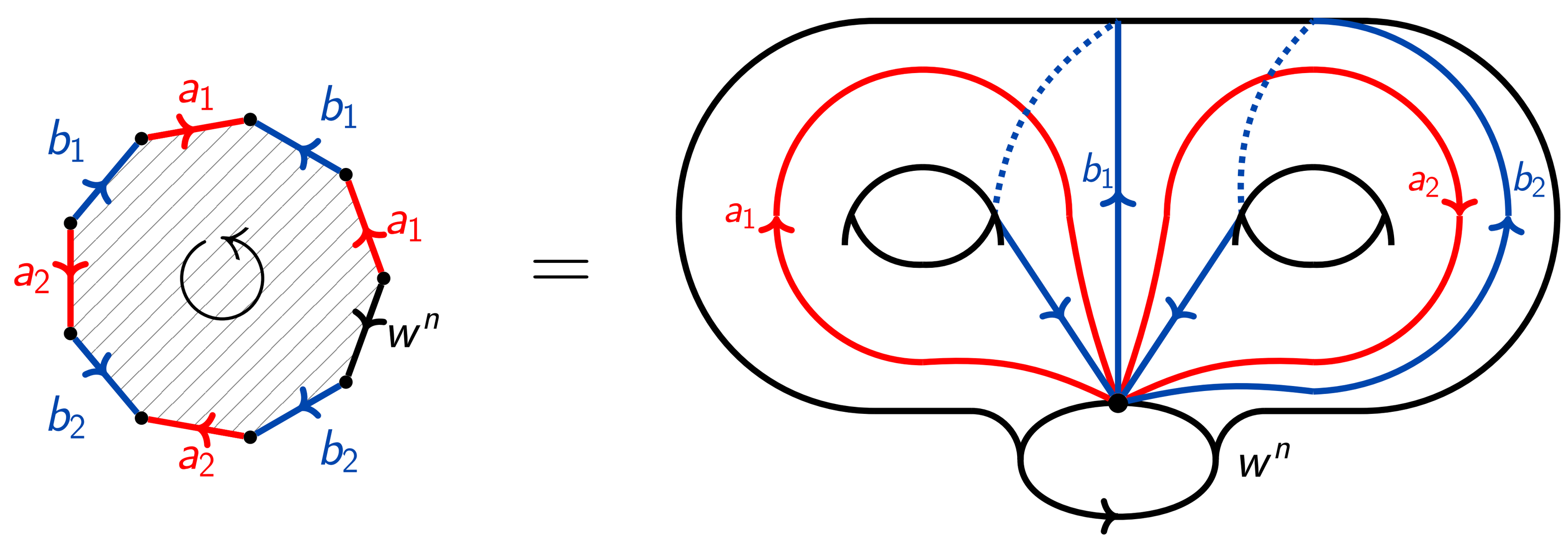
$$\mathrm{scl}_G(w) = \inf\left\{\frac{-\chi^-(\Sigma)}{2\deg(\partial\phi)} \,\middle|\, \phi : (\Sigma, \partial\Sigma) \to (X, \gamma)\right\},$$

where $\phi : (\Sigma, \partial\Sigma) \to (X, \gamma)$ denotes the data of Σ an oriented compact surface, $\phi : \Sigma \to X$, and $\partial\phi : \partial\Sigma \to S^1$, such that

$$\begin{array}{ccc} \partial\Sigma & \longrightarrow & \Sigma \\ {\scriptstyle\partial\phi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\phi} \\ S^1 & \xrightarrow[\gamma]{} & X \end{array}$$

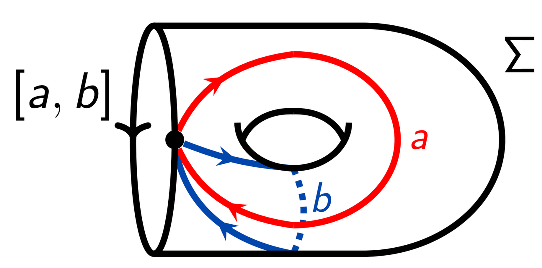
and $\partial\phi_*[\partial\Sigma] = \underbrace{\deg(\partial\phi)}_{>0}[S^1]$ in $H_1(S^1)$.

Key picture. An equality $w^n = [a_1, b_1][a_2, b_2]$ yields a Dehn diagram:



Example 1. Given $a, b \in G$,

$$\mathrm{scl}_G([a, b]) \leq -\frac{1}{2}\chi^-\left(\ \cdots\ \right) = \frac{1}{2}.$$



In fact, $\mathrm{cl}_G([a, b]^n) \leq \lfloor n/2 \rfloor + 1$ (!)

Example 2. If $H = \langle a, b \mid [a, [a, b]] = [b, [a, b]] = 1\rangle$, then

$$[a, b]^n = [a^n, b],$$

so $\mathrm{scl}_H([a, b]) = 0$.
In fact, whenever G is **amenable**, we have $\mathrm{scl}_G(w) = 0$ for all $w \in [G, G]$.

Quasimorphisms. A **quasimorphism** is a map $\phi : G \to \mathbb{R}$ such that

$$D(\phi) := \sup_{a, b \in G} |\phi(ab) - \phi(a) - \phi(b)| < \infty.$$

We say that ϕ is **homogeneous** if $\forall w \in G$, $\forall n \in \mathbb{Z}$, $\phi(w^n) = n \cdot \phi(w)$.
We denote $Q(G) := \{\text{homogeneous quasimorphisms } G \to \mathbb{R}\} \leq \mathbb{R}^G$.

Bavard Duality. $\mathrm{scl}_G(w) = \sup\left\{\frac{\phi(w)}{2D(\phi)} \,\middle|\, \phi \in Q(G),\ D(\phi) \neq 0\right\}$.

Spectral gaps

Definition. A group G is said to have an **ε-spectral gap** for scl if

$$\forall w \in G,\ \mathrm{scl}_G(w) \in \{0\} \cup [\varepsilon, +\infty].$$

Moreover, G has an **ε-strong spectral gap** for scl if

$$\forall w \in G \smallsetminus \{1\},\ \mathrm{scl}_G(w) \geq \varepsilon.$$

Theorem. Hyperbolic groups (Calegari–Fujiwara '10), MCGs (Bestvina–Bromberg–Fujiwara '16), 3-manifold groups (Chen–Heuer) have spectral gaps.

General philosophy. Negative curvature leads to large values of scl.

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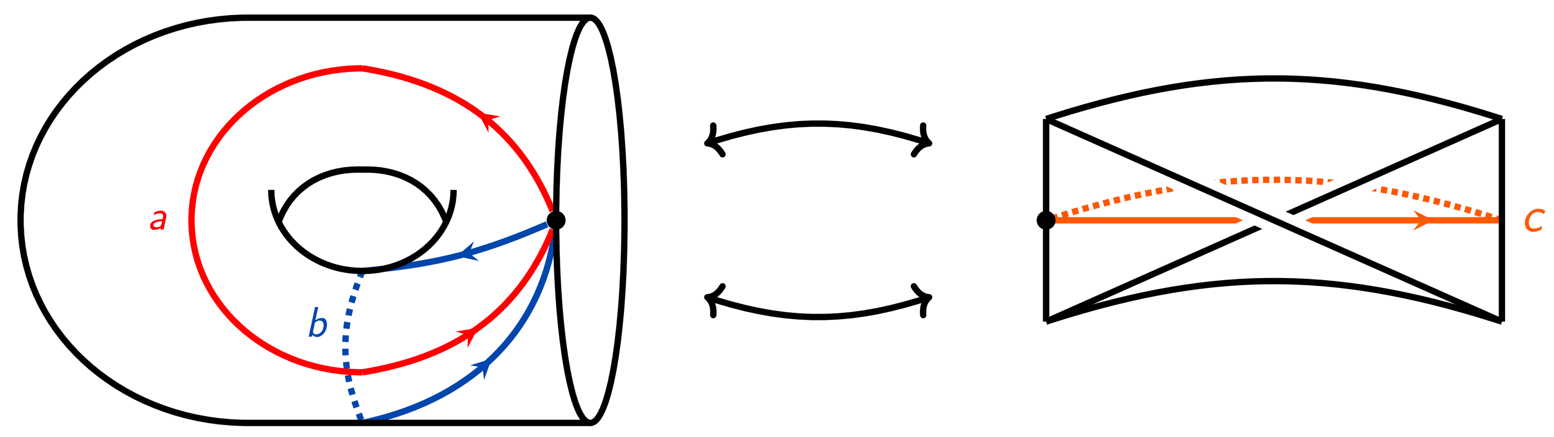
Strong spectral gap theorems (Duncan–Howie, Heuer, M.)

*The following groups have a $\frac{1}{2}$-**strong spectral gap**:*

- *Free groups (Duncan–Howie), and more generally*
- *Right-angled Artin groups (Heuer, M.).*

Observation. Strong spectral gaps pass to subgroups. Therefore, a group G with an element w satisfying $\mathrm{scl}_G(w) \in (0, 1/2)$ cannot embed in a RAAG.

Example. Consider Dyck's surface $\Delta := (\mathbb{RP}^2)^{\#3}$:



We have $\pi_1\Delta = \langle a, b, c \mid c^2 = [a, b]\rangle$, and $\mathrm{scl}_{\pi_1\Delta}(c) = 1/4$.
Therefore, $\pi_1\Delta \not\hookrightarrow$ RAAG.

Geometric method for spectral gaps (M.). Work with the topological interpretation of scl; fix $\gamma : S^1 \to X$ and start with $f : (\Sigma, \partial\Sigma) \to (X, \gamma)$.

1. Equip Σ with a certain geometric structure called an **angle structure**. This structure comes with a notion of **curvature**.
2. Estimate the curvature of Σ locally.
3. Sum up the curvatures and use a **Gauß–Bonnet formula** to deduce an estimate of $\chi(\Sigma)$.

Rationality

Theorem. Scl is computable and has rational values in

- Free groups (Calegari '09),
- Graphs of groups with infinite cyclic vertex and edge groups (Chen '20).

Question. What about closed surface groups?

Definition. Let $\alpha \in H_2(X, \gamma)$. The **relative Gromov seminorm** of α is

$$\|\alpha\|_1 := \inf\left\{\frac{-2\chi^-(\Sigma)}{\deg(\phi)} \,\middle|\, \phi : (\Sigma, \partial\Sigma) \to (X, \gamma)\right\},$$

where we now require $\phi_*[\Sigma, \partial\Sigma] = (\deg\phi)\alpha$ in $H_2(X, \gamma)$.

Proposition. We have

$$4\,\mathrm{scl}_G(w) = \inf\{\|\alpha\|_1 \mid \alpha \in H_2(X, \gamma),\ \partial\alpha = [\gamma]\}.$$

In particular, if $X = S$ is a compact surface with $\partial S \neq \varnothing$, then $4\,\mathrm{scl}_{\pi_1 S}(w) = \|\alpha\|_1$ for $\alpha \in H_2(S, \gamma)$ the unique class with $\partial\alpha = [\gamma]$.
If however S is closed, there are many classes with $\partial\alpha = [\gamma]$.

Strategy for rationality in closed surface groups (M.)

Let S be an oriented closed connected surface, $\gamma : S^1 \to S$, and $w = [\gamma] \in \pi_1 S$.

1. *Compute $\|\alpha\|_1$ for every $\alpha \in H_2(S, \gamma)$ with $\partial\alpha = [\gamma]$. This relies on the fact that $\|\cdot\|_1$ in (closed) surfaces often behaves like scl in free groups.*
2. *Minimise $\|\alpha\|_1$ over all α as above.*

This will give $\mathrm{scl}_{\pi_1 S}(w)$ by the above proposition.

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