Hyperbolic Geometry — Exercises from Lecture 4

Exercise 4.A

Let $\iota : z \mapsto z/|z|^2$ be the standard inversion in $\hat{\mathbb{C}}$. Let C be the circle of centre $p \in \mathbb{C}$ and radius $R \in \mathbb{R}_{>0} \smallsetminus \{|p|\}$. Prove that $\iota(C)$ is a circle and determine its centre and radius.

Exercise 4.B

(In this exercise, do not use $\mathrm{PGL}(2,\mathbb{C})$ and its action by homographies.) Prove that $\mathrm{Möb}(\mathbb{C})$ contains reflections along lines, translations, rotations, and homotheties. Deduce that the action $\mathrm{Möb}(\mathbb{C}) \curvearrowright \hat{\mathbb{C}}$ is transitive.

Exercise 4.C

Let $f \in \mathrm{PGL}(2,\mathbb{C})$.

(i) *Assume that $f \neq \mathrm{id}$. Under the action $\mathrm{PGL}(2,\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, prove that exactly one of the following assertions hold:*

- *f has two fixed points z_1, z_2 satisfying $z_2 = -1/\bar{z}_1$, and f is conjugate (in $\mathrm{PGL}(2,\mathbb{C})$) to a rotation.*
- *f has two fixed points z_1, z_2 satisfying $z_2 = -1/\bar{z}_1$, and f is conjugate (in $\mathrm{PGL}(2,\mathbb{C})$) to a complex homothety; moreover, for all $z \in \mathbb{P}^1(\mathbb{C}) \smallsetminus \{z_2\}$, $f^n(z) \to z_1$ as $n \to \infty$ (we say that z_1 is* attractive *and z_2 is* repulsive*).*
- *f has a single fixed point, and f is conjugate (in $\mathrm{PGL}(2,\mathbb{C})$) to a translation.*

Hint: Conjugate by an element of $\mathrm{PGL}(2,\mathbb{C})$ to send the fixed points to well-chosen points.

(ii) *Let $n \geq 2$. An n-th* root *of f is an element $h \in \mathrm{PGL}(2,\mathbb{C})$ such that $h^n = f$. How many n-th roots does f have?*

Exercise 4.D

Let $f \in \mathrm{PGL}(2,\mathbb{C})$. Prove that, for all $z \in \mathbb{P}^1(\mathbb{C})$ with $fz \neq z$, we have

$$[z, fz, f^2z, f^3z] = \frac{(\operatorname{tr} f)^2}{\det f}.$$

N.B. *For $A \in GL(2,\mathbb{C})$ and $\lambda \in \mathbb{C}^*$, we have $(\operatorname{tr}(\lambda A))^2/\det(\lambda A) = (\operatorname{tr} A)^2/\det A$. Consequently, the function $(\operatorname{tr}(-))^2/\det(-) : \mathrm{GL}(2,\mathbb{C}) \to \mathbb{C}$ descends to the quotient $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^* \,\mathrm{id}$.*

Exercise 4.E

Prove that $\mathrm{PSL}(2,\mathbb{R})$ is the intersection with $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{Möb}^+(\mathbb{C})$ of the subgroup of $\mathrm{Möb}(\mathbb{C})$ generated by the horizontal translation $z \mapsto z+1$, and by inversions in circles orthogonal to the real line and containing i.

Exercise 4.F

Prove that the action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *is transitive.*

Exercise 4.G

(In this exercise, do not use the hyperboloid model, and in particular Exercise 3.B.) Given $p \in \mathbb{H}^2$*, identify its stabiliser* $\mathrm{Stab}(p)$ *under the action* $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *as a well-known group.*