Hyperbolic Geometry — Exercises from Lecture 5

Exercise 5.A

Let $r > 0$. Compute the Euclidean centre and radius of the hyperbolic circle of centre i and radius r in $\mathbb{H}^2$.

Exercise 5.B

Consider

$$g \coloneqq \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2.$$

Compute the conjugate of g under the Cayley transform $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ and interpret it geometrically.

Exercise 5.C

Let $g \in \mathrm{Isom}^+(\mathbb{H}^2)$. Depending on the type of g (elliptic, parabolic, hyperbolic),

(i) *Describe the dynamics of the action of g on $\bar{\mathbb{H}}^2$: depending on $p \in \bar{\mathbb{H}}^2$, what is the asymptotic behaviour of $g^n(p)$ as $n \to \infty$?*

(ii) *Describe geometrically the orbits of g: for each $p \in \mathbb{H}^2$, find a "nice" subset of $\mathbb{H}^2$ containing the set $\langle g \rangle p$.*

Hint: "Nice" can be interpreted as meaning: a circle or a line in an appropriate model of the hyperbolic plane.

(iii) *When does there exist a geodesic in $\mathbb{H}^2$ which is stabilised (setwise) by g?*

Exercise 5.D

(i) *Let $f, g \in \mathrm{Isom}(\mathbb{H}^2)$ and assume that f and $g^{-1}fg$ have the same set of fixed points P in $\partial\mathbb{H}^2$. Prove that P is also fixed by g.*

(ii) *Let $f, g \in \mathrm{Isom}(\mathbb{H}^2)$. Suppose that f and g are non-elliptic and commute. Prove that f and g are either both hyperbolic or both parabolic, and have the same fixed points in $\partial\mathbb{H}^2$.*

Exercise 5.E

(i) *Let $(p_i)_{i\in I}$ be a finite collection of points in $\mathbb{H}^2$. Associate to each p_i a mass $\mu_i \in \mathbb{R}_{>0}$. Consider the function*

$$\begin{aligned} M_2 : \mathbb{H}^2 &\longrightarrow [0, +\infty) \\ x &\longmapsto \sum_{i\in I} \mu_i \cdot d_{\mathbb{H}^2}(x, p_i)^2. \end{aligned}$$

Prove that M_2 has a minimum, which is attained at a unique point of $\mathbb{H}^2$; this point is called the hyperbolic centre of mass *of the weighted family $\{(p_i, \mu_i)\}_{i\in I}$.*

(ii) *Deduce that any finite subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ has a global fixed point.*

(⋆) *Let* $\{(p_i, \mu_i)\}_{i \in I}$ *be as in* (i)*, and consider*

$$\begin{aligned} M_1 : \mathbb{H}^2 &\longrightarrow [0, +\infty) \\ x &\longmapsto \sum_{i \in I} \mu_i \cdot d_{\mathbb{H}^2}(x, p_i). \end{aligned}$$

Does M_1 *have a minimum? If so, is it attained at a unique point?*
Points attaining the minimum of M_1 *are called* medians *of* $\{(p_i, \mu_i)\}_{i \in I}$*. In the special case where* $|I| = 3$ *and* $\mu_i = 1$ *for all* $i \in I$*, give a geometric characterisation of the median(s).*

Exercise 5.F

(i) *We see* $\mathbb{D}^2$ *as a subset of* $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$*. Prove that*

$$\mathrm{Stab}_{\mathrm{PGL}(2,\mathbb{C})}(\mathbb{D}^2) = \mathrm{PU}(1,1),$$

and that the action $\mathrm{PU}(1,1) \curvearrowright \mathbb{D}^2$ *induces an isomorphism*

$$\mathrm{Isom}^+(\mathbb{D}^2) \cong \mathrm{PU}(1,1).$$

(ii) *Consider* $h(z, w) \coloneqq z_1 \bar{w}_1 - z_2 \bar{w}_2$ *the standard Hermitian form of signature* $(1,1)$*. Give a direct proof that* $\mathrm{PU}(1,1)$ *stabilises* $\mathbb{D}^2$ *by identifying* $\mathbb{D}^2$ *with* $\mathbb{P}(S) \subseteq \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1(\mathbb{C})$ *for a certain subset* S *of* $\mathbb{C}^2$ *defined from* h*.*