Hyperbolic Geometry — Exercises from Lecture 6

Exercise 6.A

(i) *Verify that the first four axioms of Euclid hold in (your favourite model of) the hyperbolic plane (replacing the word "straight line" with "geodesic").*
(ii) *Two geodesics are called* parallel *if they are disjoint. Show the following strong negation to the parallel postulate in the hyperbolic plane: given γ a geodesic, and $p \notin \gamma$, there are infinitely many geodesics through p that are parallel to γ.*

Exercise 6.B

(i) *Compute the circumference $\mathscr{C}_{\text{hyp}}(r)$ of a hyperbolic circle of radius r.*
(ii) *Compute the area $\mathscr{A}_{\text{hyp}}(r)$ of a hyperbolic disc of radius r.*
(iii) *What is the asymptotic behaviour of $\mathscr{C}_{\text{hyp}}(r)$ and $\mathscr{A}_{\text{hyp}}(r)$ as $r \to \infty$? Compare with the Euclidean case.*

Exercise 6.C

Let Δ be an ideal triangle in $\mathbb{H}^2$ with one vertex at ∞, and the other two on the geodesic with endpoints ± 1. Compute explictly $\text{Area}(\Delta)$ *and prove that Δ satisfies the Gauß–Bonnet Formula.*

Exercise 6.D

Given $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\sum_i \alpha_i < \pi$, prove that there exists a hyperbolic triangle with vertex angles $\alpha_1, \alpha_2, \alpha_3$, and that it is unique up to isometry.

Exercise 6.E

(i) *Prove that there exists a constant $\delta > 0$ such that, for every hyperbolic triangle with vertices p_1, p_2, p_3, every point of the geodesic $[p_1, p_2]$ is at distance at most δ from some point of $[p_1, p_3] \cup [p_3, p_2]$.*
Hint: Use Exercise 6.B.
The existence of such a δ is called Gromov-hyperbolicity *(we also say that $\mathbb{H}^2$ is δ-*hyperbolic*).*
(ii) *Is the Euclidean plane Gromov-hyperbolic?*
($\star$) *Can you think of other metric spaces which are Gromov-hyperbolic?*

Exercise 6.F

Let γ be a geodesic in $\mathbb{H}^2$.
(i) *Given $p \in \mathbb{H}^2$, show that there is a unique $w_0 \in \gamma$ such that*

$$d_{\mathbb{H}^2}(p, w_0) = \min_{w \in \gamma} d_{\mathbb{H}^2}(p, w).$$

Show that w_0 is characterised by the property that $\gamma \perp (p, w_0)$.
Hence, we have a map $\pi_\gamma : \mathbb{H}^2 \to \gamma$ given by $p \mapsto w_0$.

(ii) *If γ_1, γ_2 are two geodesics in $\mathbb{H}^2$ which are disjoint in $\bar{\mathbb{H}}^2$, prove that there is a unique couple $(w_1, w_2) \in \gamma_1 \times \gamma_2$ such that*

$$d_{\mathbb{H}^2}(w_1, w_2) = \min_{(z_1,z_2)\in\gamma_1\times\gamma_2} d_{\mathbb{H}^2}(z_1, z_2).$$

Show that w_1 and w_2 are characterised by the fact that (w_1, w_2) is the unique common perpendicular to γ_1 and γ_2.

(iii) *Prove that the map $\pi_\gamma : \mathbb{H}^2 \to \gamma$ is 1-Lipschitz (for $d_{\mathbb{H}^2}$).*