Hyperbolic Geometry — Exercises from Lecture 7

Exercise 7.A

(i) *Prove that a subgroup* Γ *of* $\mathbb{R}^n$ *is discrete if and only if* $\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$ *for some vectors* $v_1, \ldots, v_k \in \mathbb{R}^n$ *which are linearly independent over* $\mathbb{R}$.

(ii) *Let* $k \in \mathbb{C}^\times$. *When is the subgroup generated by* k *in* $\mathbb{C}^\times$ *discrete?*

Exercise 7.B

Prove the equivalence of the four conditions in the definition of properly discontinuous actions (Definition III.2).

Exercise 7.C

Prove that, under the isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ *of Theorem II.11, the quotient topology on* $\mathrm{PSL}(2,\mathbb{R})$ *induced by the standard topology on* $\mathrm{SL}(2,\mathbb{R})$ *coincides with the compact-open topology on* $\mathrm{Isom}(\mathbb{H}^2)$.

Exercise 7.D

Let $f \in \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$. *Depending on the type of* f *(elliptic, parabolic, or hyperbolic), give a fundamental domain for the subgroup* $\langle f \rangle$ *of* $\mathrm{Isom}(\mathbb{H}^2)$. *Draw it for a chosen conjugate of* f, *in the model of your choice.*

Exercise 7.E

Prove that the Dirichlet domain $\mathscr{D}(x)$ *of a discrete group* Γ *at at point* $x \in \mathbb{X}^2$ *is a* locally finite fundamental domain, *in the sense that, for every compact set* $K \subseteq \mathbb{X}^2$, *the set* $\{\gamma \in \Gamma \,:\, \gamma\mathscr{D}(x) \cap K \neq \varnothing\}$ *is finite.*

Exercise 7.F

If Γ *is a discrete subgroup of* $\mathrm{Isom}^+(\mathbb{X}^2)$, *prove that there is a point* $x \in \mathbb{X}^2$ *such that* $\mathrm{Stab}_\Gamma(x) = \{1\}$.

Exercise 7.H

(i) *Show that* $\mathrm{PSL}(2,\mathbb{Z})$ *is a Fuchsian group.*

(ii) *Show that* $\mathrm{PSL}(2,\mathbb{Z})$ *is generated by the elements*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(iii) *Determine the Dirichlet domain of* $\mathrm{PSL}(2,\mathbb{Z})$ *at* $2i$.

Exercise 7.G

Let Γ *be a subgroup of* $\mathrm{Isom}(\mathbb{X}^2)$ *admitting a fundamental domain* $\mathcal{D}$ *which is locally*

finite (see Exercise 7.E). Prove that Γ *is generated by*

$$\left\{\gamma \in \Gamma \,:\, \bar{\mathcal{D}} \cap \gamma\bar{\mathcal{D}} \neq \varnothing\right\}.$$