Hyperbolic geometry, discrete groups, and manifolds

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Winter Semester 2026–2027
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I

The hyperbolic plane

The goal of the first chapter is to introduce the main protagonist of this course, the hyperbolic plane. We will see some historical motivation, and define the hyperbolic plane using several different models. The multitude of available models will enrich our study of hyperbolic geometry, and we will learn to jump from one to the other depending on the situation. The formal definitions of the models for the hyperbolic plane rely on Riemannian geometry; the necessary notions will also be introduced in this chapter.

Recommended reading:

- [Cannon et al., 1997, §1–8]
- [Thurston, 1997, §2.1–3]
- [Beardon, 1983, §7.1–2]
- [Martelli, 2016, §2.1]

Lecture 1 — The Poincaré half-plane

1.1 Non-Euclidean geometry

Our journey into hyperbolic geometry begins with Euclidean geometry, or rather its negation. Euclidean geometry is the geometry of the Euclidean plane (or more generally, of Euclidean n-space). It was axiomatised by Euclid.

Axioms I.1 (Euclid)**.** (i) *Every pair of points can be joined by a unique straight line segment.*

(ii) *Every straight line segment can be extended indefinitely in both directions.*

(iii) *There is a unique circle of a given centre and radius.*

(iv) *All right angles are congruent to one another.*

(v) *Given a line ℓ and a point $p \notin \ell$, there is a unique line ℓ' containing p and parallel to ℓ.*

The fifth axiom is sometimes also called the *parallel postulate*[1]. As the parallel

[1] In fact, Euclid's fifth axiom was a little different, but equivalent to the parallel postulate given the other four. See [Cannon et al., 1997].

postulates sounds more complicated than the other four axioms, mathematicians have spent a lot of time and energy trying to prove that it can be deduced from them, in order to simplify the axioms of geometry (at this stage, there is no need to specify "Euclidean", since this is the only geometry that was known).

In the nineteenth century, several mathematicians (including Gauß, Lobachevskiĭ, Bolyai) had the fecund idea of trying to create a consistent theory of geometry where the parallel postulate does not hold — this gave birth to *hyperbolic geometry*, which was the first non-Euclidean geometry to be discovered and explored. Later, Riemann introduced what is now called *Riemannian geometry*, which provides a very general setup to define and study a very broad array of geometries. The Riemannian setup now gives us a very convenient language to define hyperbolic geometry, and we will take the liberty to follow this anachronism.

Ideas from Riemannian geometry will already appear in this lecture without general definitions; they will be formalised in the second lecture.

1.2 Definition of the Poincaré half-plane

The underlying set of our first model is the complex upper half-plane:

$$\mathbb{H}^2 \coloneqq \{z \in \mathbb{C} \,:\, \Im(z) > 0\}\,.$$

For each point $z \in \mathbb{H}^2$, we define a *local hyperbolic norm* $\|\cdot\|^z_{\mathrm{hyp}}$ via

$$\|v\|^z_{\mathrm{hyp}} \coloneqq \frac{1}{\Im(z)}\|v\|_{\mathrm{eucl}},$$

for all $v \in \mathbb{R}^2$, where $\|\cdot\|_{\mathrm{eucl}}$ is the usual Euclidean norm on $\mathbb{R}^2$. Defining local norms at each point of our space might sound a little strange for now, but this will be clarified when we introduce Riemannian geometry. For now, the only thing we need is that this allows us to define hyperbolic lengths and distances.

Definition I.2. The *hyperbolic length* of a (piecewise) C^1 path $\gamma : [a,b] \to \mathbb{H}^2$ is

$$|\gamma|_{\mathrm{hyp}} \coloneqq \int_a^b \|\gamma'(t)\|^{\gamma(t)}_{\mathrm{hyp}}\,\mathrm{d}t.$$

The *hyperbolic distance* between two points $z, w \in \mathbb{H}^2$ is

$$d_{\mathrm{hyp}}(z,w) \coloneqq \inf\left\{|\gamma|_{\mathrm{hyp}} \,:\, \gamma \in C^1_{\mathrm{pw}}\left([a,b],\mathbb{H}^2\right),\, \gamma(a) = z,\, \gamma(b) = w\right\},$$

where $C^1_{\mathrm{pw}}\left([a,b],\mathbb{H}^2\right)$ is the set of piecewise C^1 paths $[a,b] \to \mathbb{H}^2$.

Exercise 1.A

Let γ be a path of the form

$$\gamma : t \mapsto x_0 + iy(t)$$

for some $x_0 \in \mathbb{R}$ and some strictly increasing piecewise C^1 function $y : [a,b] \to (0,+\infty)$. Compute $|\gamma|_{\mathrm{hyp}}$.

Exercise 1.B

Let $\gamma : [a,b] \to \mathbb{H}^2$ be a piecewise C^1 path, and let $\phi : [a',b'] \to [a,b]$ be a strictly increasing C^1 function. Prove that $|\gamma \circ \phi|_{\mathrm{hyp}} = |\gamma|_{\mathrm{hyp}}$.

Proposition I.3. *d_{hyp} defines a metric on $\mathbb{H}^2$.*

Proof. Symmetry and the triangle inequality are clear (reverse and concatenate paths). For separation, fix $z \neq w \in \mathbb{H}^2$, and let $\gamma : [a,b] \to \mathbb{H}^2$ be a piecewise C^1 path from z to w. Decompose γ as $\gamma = x + iy$, with $x, y : [a,b] \to \mathbb{R}$, and $y(t) > 0$ for all $t \in [a,b]$.

First assume that $y(t) \leq 2y(a)$ for all $t \in [a,b]$. Then we have

$$|\gamma|_{\mathrm{hyp}} = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,\mathrm{d}t \geq \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{2y(a)}\,\mathrm{d}t = \frac{1}{2\Im(z)}|\gamma|_{\mathrm{eucl}},$$

where $|\gamma|_{\mathrm{eucl}}$ denotes the *Euclidean length* of γ (given by $|\gamma|_{\mathrm{eucl}} \coloneqq \int_a^b \|\gamma'(t)\|_{\mathrm{eucl}}\,\mathrm{d}t$). But the Euclidean length of a piecewise C^1 path is a at least equal to the Euclidean distance between its endpoints, so that

$$|\gamma|_{\mathrm{hyp}} \geq \frac{d_{\mathrm{eucl}}(z,w)}{2\Im(z)}.$$

Otherwise, let $t_0 \coloneqq \min\{t \in [a,b] \,:\, y(t) \geq 2y(a)\}$, and let $\gamma' \coloneqq \gamma|_{[a,t_0]}$. The previous computation gives

$$|\gamma|_{\mathrm{hyp}} \geq |\gamma'|_{\mathrm{hyp}} \geq \frac{1}{2\Im(z)}|\gamma'|_{\mathrm{eucl}}.$$

But the Euclidean length of γ' is at least equal to the Euclidean distance between z and the line $\{u \in \mathbb{C} \,:\, \Im(u) = 2\Im(z)\}$. The latter distance is equal to $\Im(z)$, yielding $|\gamma|_{\mathrm{hyp}} \geq \frac{1}{2}$ in this case.

In general, we have proved that

$$d_{\mathrm{hyp}}(z,w) \geq \frac{1}{2}\min\left\{1, \frac{d_{\mathrm{eucl}}(z,w)}{\Im(z)}\right\} > 0. \qquad \square$$

The metric space $(\mathbb{H}^2, d_{\mathrm{hyp}})$ is called the *Poincaré half-plane*.

Exercise 1.C

What is the topology on $\mathbb{H}^2$ induced by d_{hyp}?

1.3 First examples of shortest paths and isometries

The following will give us shortest paths in $\mathbb{H}^2$ in a very specific case.

Lemma I.4. *Consider the orthogonal projection $p : \mathbb{H}^2 \to \mathbb{H}^2$ given by $z \mapsto z - \Re(z)$. Given a piecewise C^1 path $\gamma : [a, b] \to \mathbb{H}^2$, we have*

$$|p \circ \gamma|_{\text{hyp}} \leq |\gamma|_{\text{hyp}},$$

with equality if and only if $\Re(\gamma)$ is constant.

Proof. Write $\gamma = x + iy$, where $x, y : [a, b] \to \mathbb{R}$, with $y(t) > 0$ for all $t \in [a, b]$. We have

$$|p \circ \gamma|_{\text{hyp}} = \int_a^b \frac{|y'(t)|}{y(t)}\,\mathrm{d}t \leq \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,\mathrm{d}t = |\gamma|_{\text{hyp}},$$

with equality if and only if $x'(t) = 0$ for all $t \in [a, b]$. □

Proposition I.5. *Let $z, w \in \mathbb{H}^2$.*

(i) *We have $d_{\text{hyp}}(z, w) \geq |\log \Im(z) - \log \Im(w)|$.*

(ii) *If $\Re(z) = \Re(w)$, then the vertical segment is the unique (up to reparametrisation) shortest path between z and w.*

Proof. By Lemma I.4, it suffices to prove (i) when $\Re(z) = \Re(w)$. Moreover, it suffices to consider piecewise C^1 paths γ from z to w for which $\Re(\gamma)$ is constant. The computation of Exercise 1.A shows that the length of such a path is at least $|\log \Im(z) - \log \Im(w)|$, with equality if and only if $\Im(\gamma)$ is strictly increasing or decreasing. Hence, $d_{\text{hyp}}(z, w) \geq |\log \Im(z) - \log \Im(w)|$, and this is attained exactly by reparametrisations of the vertical segment from z to w. □

Isometries will be key in many geometric arguments throughout the course. The following observation allows us to easily verify that certain maps are isometries; the hypothesis will turn out to be the definition of a Riemannian isometry, which we will see in the next lecture.

Observation I.6. Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be a C^1 diffeomorphism such that, for every $z \in \mathbb{H}^2$ and for every $v \in \mathbb{R}^2$,

$$\|\mathrm{d}f_z(v)\|_{\text{hyp}}^{f(z)} = \|v\|_{\text{hyp}}^z.$$

Then f is an isometry: for all $z, w \in \mathbb{H}^2$,

$$d_{\text{hyp}}\left(f(z), f(w)\right) = d_{\text{hyp}}(z, w).$$

Proof. It is easy to verify that $|f \circ \gamma|_{\text{hyp}} = |\gamma|_{\text{hyp}}$ for every piecewise C^1 path γ; it follows that f preserves d_{hyp}. □

Exercise 1.D

Prove that the following maps are isometries of $\mathbb{H}^2$:

(i) *Horizontal translations,*
(ii) *Reflections along vertical lines,*
(iii) *Homotheties.*
Deduce that the action of the isometry group $\mathrm{Isom}(\mathbb{H}^2) \curvearrowright \mathbb{H}^2$ *is transitive.*

In fact, we will see later that the action $\mathrm{Isom}(\mathbb{H}^2) \curvearrowright \mathbb{H}^2$ is transitive in a stronger sense: $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on pairs of points at a given distance. Together with Proposition I.5, this will allow us to deduce that any two points in $\mathbb{H}^2$ are connected by a unique shortest path. This is a typical example of the use of isometries to deduce geometric properties of the hyperbolic plane by reducing to an easier case; we will see many arguments of this kind.

Exercise 1.E

The hyperbolic line *is the set* $\mathbb{H}^1 := (0, +\infty)$*, with local hyperbolic norm given by* $\|v\|^y_{\mathrm{hyp}} = \frac{|v|}{y}$ *for* $y \in \mathbb{H}^1$ *and* $v \in \mathbb{R}$*. The hyperbolic length of a (piecewise)* C^1 *path and the hyperbolic distance are defined analogously to the 2-dimensional case.*
(i) *Compute* $d_{\mathrm{hyp}}(y_1, y_2)$ *for* $y_1, y_2 \in \mathbb{R}$*, and deduce that* $\mathbb{H}^1$ *admits an isometric embedding into* $\mathbb{H}^2$*.*
(ii) *Show that* $\mathbb{H}^1$ *is isometric to* $\mathbb{R}$ *with the usual norm.*
($\star$) *Is* $\mathbb{H}^2$ *isometric to* $\mathbb{R}^2$ *with the Euclidean norm?*

Lecture 2 — Riemannian geometry, the Poincaré disc

This lecture introduces the basics of Riemannian geometry that will be useful for defining our different models of the hyperbolic plane; many propositions and theorems here will be left unproved. We work in the smooth category throughout.

2.1 Definition of a Riemannian manifold

Let M be a (smooth) d-manifold ($d \geq 1$). Recall that the *tangent bundle* of M is a $2d$-manifold decomposing as

$$TM = \coprod_{p \in M} T_pM,$$

where, for each $p \in M$, the subspace T_pM is the tangent space at p, and carries the extra structure of a real vector space of dimension d. The tangent bundle is equipped with a smooth projection $\pi : TM \to M$.

The 2-*covariant tensor bundle* of M is a manifold decomposing as

$$(TM^*)^{\otimes 2} = \coprod_{p \in M} T_pM^* \otimes T_pM^*,$$

where $T_pM^* := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$ is the (linear) dual of T_pM. For each $p \in M$, we will identify elements of $T_pM^* \otimes T_pM^*$ with bilinear forms $T_pM \times T_pM \to \mathbb{R}$. Again, $(TM^*)^{\otimes 2}$ is equipped with a smooth projection $\pi : (TM^*)^{\otimes 2} \to M$.

Definition I.7. A *Riemannian manifold* is the data (M, g) of a (smooth) manifold M together with a map $g : M \to (TM^*)^{\otimes 2}$ which

- is a section of $\pi : (TM^*)^{\otimes 2} \to M$ — i.e. $\pi \circ g = \mathrm{id}_M$ — and
- induces a scalar product at each point — i.e. for each $p \in M$, the bilinear form $g(p) : T_pM \times T_pM \to \mathbb{R}$ is a scalar product.

The map g is called a *Riemannian metric* (or Riemannian structure) on M.

We have not defined the smooth structures on the tangent and tensor bundles, and we will not really need the above technical language. The important idea to keep in mind is that a Riemannian metric on a manifold M is the data of a scalar product $g(p)$ on each tangent space T_pM, such that $g(p)$ varies smoothly with p. We will denote the scalar product $g(p)$ on T_pM by $\langle \cdot, \cdot \rangle_g^p$, and the associated norm by $\| \cdot \|_g^p$: for each tangent vector $v \in T_pM$,

$$\|v\|_g^p := \sqrt{\langle v, v \rangle_g^p}.$$

Example I.8. Let M be an embedded d-submanifold of $\mathbb{R}^N$. Equip $\mathbb{R}^N$ with its standard Euclidean structure. For each $x \in M$, the tangent space T_pM can be seen as a linear subspace of $\mathbb{R}^N$; let $g(p)$ be the restriction of the scalar product on $\mathbb{R}^N$ to T_pM. Then g defines a Riemannian metric on M.

More generally, an embedded submanifold M of a Riemannian manifold M_0 inherits a Riemannian structure from M_0, called the *induced Riemannian structure.*

Let $(x_1, \dots, x_d)$ be local coordinates for the d-manifold M at a point p. We normally denote by $(\mathrm{d}x_1, \dots, \mathrm{d}x_d)$ the basis of T_pM^* which is dual to the basis $(e_1, \dots, e_d)$ of T_pM induced by the local coordinates. The unique scalar product $\langle \cdot, \cdot \rangle$ on T_pM making $(e_1, \dots, e_d)$ an orthonormal basis is then denoted by

$$\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2.$$

These notations will be convenient for defining Riemannian metrics, and we will use them repeatedly in this chapter.

Example I.9. The Poincaré half-plane is the Riemannian surface (2-manifold) whose underlying manifold is $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ with its usual smooth structure, with the Riemannian metric

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

Compare with §1.2.

Exercise 2.A

Consider the sphere

$$\mathbb{S}^2 := \left\{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1\right\} \subseteq \mathbb{R}^3.$$

Write the Riemannian metric on $\mathbb{S}^2$ induced by the standard Riemannian structure of $\mathbb{R}^3$ in the local coordinates given by stereographic projection.

2.2 Lengths of paths, distances, and isometries

For the rest of this lecture, we fix a (connected) Riemannian manifold (M, g).

Definition I.10. Let $\gamma : [a, b] \to M$ be a (piecewise) C^1 path. The *length* of γ is

$$|\gamma|_g := \int_a^b \|\gamma'(t)\|_g^{\gamma(t)} \, \mathrm{d}t.$$

The *distance* between two points $p, q \in M$ is

$$d_g(p, q) := \inf \left\{ |\gamma|_g \ : \ \gamma \in C^1_{\mathrm{pw}}([a, b], M) \, , \ \gamma(a) = p, \ \gamma(b) = q \right\}.$$

Remark I.11. The function $d_g : M \times M \to [0, +\infty)$ defines a metric on M (as for Proposition I.3, separation is not obvious but will follow from the results appearing later in this lecture). The topology on M induced by d_g coincides with the topology on M as a manifold; the proof is the same as that of Exercise 1.C.

Definition I.12. Let (M, g) and (N, h) be two Riemannian manifolds. A *Riemannian isometric embedding* $f : (M, g) \hookrightarrow (N, h)$ is a smooth embedding $f : M \hookrightarrow N$ which preserves the Riemannian metrics in the sense that, for all $p \in M$ and $v, w \in T_pM$, we have

$$\langle \mathrm{d}f_p(v), \mathrm{d}f_p(w) \rangle_h^{f(p)} = \langle v, w \rangle_g^p \, .$$

A *Riemannian isometry* is a Riemannian isometric embedding which is in addition a diffeomorphism.

Proposition I.13. *If $f : (M, g) \to (N, h)$ is a Riemannian isometry, then f is also an isometry of metric spaces $(M, d_g) \to (N, d_h)$.*

Proof. See Observation I.6. □

Example I.14. The d-torus $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ has a Riemannian structure defined in the local coordinates $(x_1, \ldots, x_d)$ coming from $\mathbb{R}^d$ by $\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2$. This is called the *flat structure* on $\mathbb{T}^d$.

For $d = 2$, note that there is no (smooth) Riemannian isometric embedding $\mathbb{T}^2 \hookrightarrow \mathbb{R}^3$; in particular, the above-defined Riemannian structure on $\mathbb{T}^2$ is different from any metric obtained by smoothly embedding $\mathbb{T}^2$ (as a manifold) into $\mathbb{R}^3$.

Exercise 2.B

Prove that the natural projection $\mathbb{R}^d \to \mathbb{T}^d$ is a local Riemannian isometric embedding for the flat structure on $\mathbb{T}^d$. Is it an isometric embedding of metric spaces?

The following theorem is far beyond the scope of this couse, but we mention the result.

Theorem I.15 (Nash Embedding)**.** *If (M, g) is a d-dimensional Riemannian manifold, then there exist $D \leq \frac{1}{2} d(d+1)(3d+11)$, as well as a Riemannian isometric embedding $M \hookrightarrow \mathbb{R}^D$.*

2.3 Geodesics

Geodesics are important paths in Riemannian manifolds. In a course on Riemannian geometry, they would usually be defined as local minimisers of the *energy functional*

$$\mathscr{E} : \gamma \mapsto \int_a^b \left(\|\gamma'(t)\|_g^{\gamma(t)} \right)^2 \, \mathrm{d}t.$$

Such local minimisers can be characterised by a differential equation depending on the Riemannian metric; this is called the *equation of geodesics*. For our purpose, the following equivalent definition will however suffice.

Definition I.16. A *geodesic* on (M, g) is a C^1 path $\gamma : I \to M$ (where I is an interval) for which there is a constant $K > 0$ satisfying:

(i) (Constant speed) $\|\gamma'(t)\|_g^{\gamma(t)} = K$ for all $t \in I$,

(ii) (Local length minimisation) For each $t_1 \in I$, there is an open subinterval J of I containing t_1 such that, for all $s, t \in J$,

$$d_g\left(\gamma(s), \gamma(t)\right) = \left|\gamma|_{[s,t]}\right|_g = K|s - t|.$$

Proving the following theorem would require going via the equation of geodesics; we will merely remember the result.

Theorem I.17. *Let (M, g) be a Riemannian manifold. For every $p \in M$ and $v \in T_pM$, there is an open interval I containing 0 and a geodesic $\gamma : I \to M$ such that*

$$\gamma(0) = p \quad \textit{and} \quad \gamma'(0) = v.$$

Such a geodesic with the interval I maximal for inclusion is unique.

It now follows from Proposition I.5(ii) that, in $\mathbb{H}^2$, the vertical ray $i\mathbb{R}_{>0}$ is a geodesic (with an appropriate parametrisation).

2.4 The exponential map

We now turn to the last notion we will need from Riemannian geometry: the exponential map.

Definition I.18. Given $p \in M$, let U_p be the set of tangent vectors $v \in T_pM$ for which the maximal geodesic $\gamma_{(p,v)} : I_{(p,v)} \to M$ satisfies $1 \in I_{(p,v)}$. The *exponential map* at p is the smooth map

$$\begin{aligned} \exp_p : U_p &\longrightarrow M \\ v &\longmapsto \gamma_{(p,v)}(1). \end{aligned}$$

Proposition/Definition I.19. *Let (M, g) be a Riemannian manifold, and let $p \in M$. There is $\varepsilon > 0$ such the map*

$$\exp_p : B(0, \varepsilon) \to M$$

is injective (and in fact, is a diffeomorphism onto its image). Here, we write $B(0, \varepsilon) \coloneqq \{v \in T_pM \,:\, \|v\|_g^p < \varepsilon\}$.

The supremum of such ε is called the injectivity radius *of M at p, and denoted by $\mathrm{inj}_p(M)$. We also define the (global) injectivity radius of M by*

$$\mathrm{inj}(M) \coloneqq \inf_{p \in M} \mathrm{inj}_p(M).$$

Exercise 2.C

If $\gamma : [0, 1] \to M$ is a geodesic with $\gamma(0) = \gamma(1) = p$, prove that

$$\mathrm{inj}_p(M) \leq \frac{1}{2}|\gamma|_g.$$

Exercise 2.D

Let (M, g) be a Riemannian manifold.

(i) *Let $\gamma : [a, b] \to M$ be a piecewise C^1 path with constant speed (i.e. there is a constant K such that $\|\gamma'(t)\|_g^{\gamma(t)} = K$ for all $t \in [a, b]$). Suppose that for every C^1 path $\eta : [a, b] \to M$ with $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$, we have $|\gamma|_g \leq |\eta|_g$. Prove that γ is a geodesic.*

(ii) *Prove that the converse does not hold.*

($\star$) *Prove the following partial converse: let $p \in M$, and consider $B \coloneqq \{v \in T_pM \,:\, \|v\|_g^p < \mathrm{inj}_p(M)\}$. If $\gamma : [0, 1] \to \exp_p(B)$ is a geodesic with $\gamma(0) = p$, and $c : [0, 1] \to M$ is a piecewise C^1 path with $c(0) = \gamma(0)$ and $c(1) = \gamma(1)$, then*

$$|c|_g \geq |\gamma|_g.$$

2.5 Completeness and the Hopf–Rinow Theorem

We finish by mentioning the following important theorem.

Theorem I.20 (Hopf–Rinow)**.** *Given (M, g) a Riemnnian manifold and $p \in M$, the following assertions are equivalent:*

(i) $\exp_p$ *is defined on all of T_pM.*

(ii) *Closed and bounded subsets of M are compact.*

(iii) *M is Cauchy-complete.*

(iv) *M is* geodesically complete*: every maximal geodesic is defined over $\mathbb{R}$.*

Moreover, the above conditions imply:

(v) *For all $q \in M$, there is a geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, and $|\gamma|_{0,1}|_g = d_g(p, q)$.*

Example I.21. The Euclidean space $\mathbb{R}^d$, the d-torus $\mathbb{T}^d$, the d-sphere $\mathbb{S}^d$, are all geodesically complete.

Remark I.22. The implication (v) ⇒ (i) in Theorem I.20 is false: for instance, consider the open disc $D^2 := \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2\}$, with the Euclidean metric $\mathrm{d}x^2 + \mathrm{d}y^2$.

2.6 The Poincaré disc

Definition I.23. The *Poincaré disc* is the Riemannian surface $\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, with Riemannian metric given by

$$\left(\frac{2}{1 - x^2 - y^2}\right)^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$$

Proposition I.24. *Rotations centred at zero, as well as reflections along lines passing through zero, are Riemannian isometries of $\mathbb{D}^2$.*

Of all the isometries connecting the different models, the following is the most important to remember; it is an example of a Möbius transformation, and those will turn out to play a major role.

Theorem I.25. *Identify $\mathbb{D}^2$ with $\{z \in \mathbb{C} \,:\, |z|^2 < 1\}$, and consider the* Cayley transform

$$\begin{aligned} \phi : \mathbb{H}^2 &\longrightarrow \mathbb{D}^2 \\ z &\longmapsto \frac{z - i}{z + i}. \end{aligned}$$

Then ϕ is a Riemannian isometry from $\mathbb{H}^2$ to $\mathbb{D}^2$.

Proof. To check that $\phi(\mathbb{H}^2) \subseteq \mathbb{D}^2$, note that, for $z \in \mathbb{H}^2$,

$$|\phi(z)|^2 = \frac{z - i}{z + i} \cdot \frac{\bar{z} + i}{\bar{z} - i} = \frac{|z|^2 + 1 - 2\Im(z)}{|z|^2 + 1 + 2\Im(z)} < 1,$$

since $\Im(z) > 0$. Computing $\mathrm{d}\phi$ corresponds to computing the complex derivative ϕ', namely

$$\phi'(z) = \frac{2i}{(z + i)^2}.$$

Now fix $z \in \mathbb{H}$. Given a tangent vector v at z seen as an element of $\mathbb{C}$, we want to check that $\|v\|^z_{\mathbb{H}^2} = \|\phi'(z)v\|^{\phi(z)}_{\mathbb{D}^2}$. Hence, we want to check the equality

$$\frac{1}{\Im(z)} = \frac{2\,|\phi'(z)|}{1 - |\phi(z)|^2},$$

which follows from the above computations.

Finally, note that ϕ has an inverse given by

$$\begin{aligned} \psi : \mathbb{D}^2 &\longrightarrow \mathbb{H}^2 \\ w &\longmapsto i\frac{1+w}{1-w}, \end{aligned}$$

showing that ϕ is surjective. Hence, ϕ is a diffeomorphism $\mathbb{H}^2 \to \mathbb{D}^2$ respecting the Riemannian metrics, so it is a Riemannian isometry. □

The existence of an isometry $\mathbb{D}^2 \cong \mathbb{H}^2$ is fundamental: it allows us to translate any geometric statement known in $\mathbb{D}^2$ to $\mathbb{H}^2$, and vice versa. Sometimes, we will even move back and forth between the two models within a single proof.

Remark I.26. The image under the Cayley transform of the vertical ray $\{0\} \times (0, +\infty) \subseteq \mathbb{H}^2$ is the horizontal diameter $(-1, +1) \times \{0\} \subseteq \mathbb{D}^2$. In particular, the horizontal diameter is a shortest path of $\mathbb{D}^2$ by Proposition I.5(ii); more than that, it is the unique shortest path between any two points on it.

Proof. For $y > 0$, note that

$$\phi(iy) = \frac{y-1}{y+1}.$$

□

At the end of Lecture 1, we mentioned the fact that $\mathrm{Isom}(\mathbb{H}^2)$ acts transitively on pairs of points at a given distance, without proving it. Looking at the hyperbolic plane from the point of view of the Poincaré disc now gives us the right tools to prove this fact.

Proposition I.27. *Let $z_1, w_1, z_2, w_2 \in \mathbb{D}^2$ with $d(z_1, w_1) = d(z_2, w_2)$. Then there exists $f \in \mathrm{Isom}(\mathbb{D}^2)$ such that $f(z_1) = z_2$ and $f(w_1) = w_2$.*

Proof. We already know that the action $\mathrm{Isom}(\mathbb{D}^2) \curvearrowright \mathbb{D}^2$ is transitive by Exercise 1.D and the fact that $\mathbb{D}^2 \cong \mathbb{H}^2$. Hence, we can assume that $z_1 = z_2 = 0$. Now let f_1 (respectively f_2) be the unique rotation centred at 0 sending w_1 (respectively w_2) to the radius $[0, 1) \times \{0\} \subseteq \mathbb{D}^2$. Note that $f_1, f_2 \in \mathrm{Isom}(\mathbb{D}^2)$ by Proposition I.24. Since the radius $[0, 1) \times \{0\} \subseteq \mathbb{D}^2$ is a shortest path (see Remark I.26), and since $d(0, f_1(w_1)) = d(0, w_1) = d(0, w_2) = d(0, f_2(w_2))$, we must have $f_1(w_1) = f_2(w_2)$. Now the isometry $f \coloneqq f_2^{-1} \circ f_1$ satisfies $f(z_1) = z_2$ and $f(w_1) = w_2$. □

Corollary I.28. *Every pair of points in $\mathbb{D}^2$ is joined by a unique shortest path.*

Proof. Let $z, w \in \mathbb{H}^2$. By Proposition I.27, there is $f \in \mathrm{Isom}(\mathbb{D}^2)$ such that $f(z) = 0$ and $f(w) \in [0, 1) \times \{0\}$. Hence, $f(z)$ and $f(w)$ are joined by a unique shortest path (namely the straight line segment — see Remark I.26), and therefore so are z and w. □

Lecture 3 — The hyperboloid and Klein disc models

In this lecture, we introduce our last two models of the hyperbolic plane. We will use them less frequently than the first two, but they give us different perspectives on hyperbolic geometry.

3.1 The hyperboloid model

Consider the *Lorentzian form*, i.e. the standard quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ of signature $(2, 1)$:

$$q(x, y, z) \coloneqq x^2 + y^2 - z^2.$$

Definition I.29. The *hyperboloid model* is the Riemannian surface with underlying set

$$\mathscr{H}^2 \coloneqq \left\{(x, y, z) \in \mathbb{R}^3 \ : \ q(x, y, z) = -1 \text{ and } z > 0\right\},$$

and with Riemannian metric

$$\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2.$$

Warning I.30. The set $\mathscr{H}^2$ is an embedded submanifold of $\mathbb{R}^3$, but we do not equip it with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^3$!

Remark I.31. For each $p \in \mathscr{H}^2$, the tangent space $T_p\mathscr{H}^2$ is $p^\perp$ (where orthogonality is understood in terms of q), and the scalar product on $T_p\mathscr{H}^2$ is the restriction of q to $T_p\mathscr{H}^2$, which is positive definite.

Proof. Consider the bilinear form $Q : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ corresponding to q:

$$Q\left((x_1, y_1, z_1), (x_2, y_2, z_2)\right) = x_1x_2 + y_1y_2 - z_1z_2.$$

For $p = (x, y, z) \in \mathscr{H}^2$, we have

$$T_p\mathscr{H}^2 = \operatorname{Ker} \mathrm{d}q_p = \operatorname{Ker} \begin{pmatrix} x & y & -z \end{pmatrix} = \left\{v \in \mathbb{R}^3 \ : \ Q(p, v) = 0\right\} = p^\perp.$$

It is apparent from the definition that the scalar product on $T_p\mathscr{H}^2$ is the restriction of q to $T_p\mathscr{H}^2$. To check that it is positive definite, set

$$\begin{aligned} u &\coloneqq (y, -x, 0), \\ v &\coloneqq \left(x, y, \frac{1}{z} - z\right). \end{aligned}$$

Note that (u, v) is a q-orthogonal basis of $T_p\mathscr{H}^2$, and that $q(u) = x^2 + y^2 > 0$ and $q(v) = (x^2 + y^2)/z^2 > 0$; it follows that the restriction of q to $T_p\mathscr{H}^2$ is positive definite. □

Exercise 3.A

Show that the stereographic projection centred at $(0,0,-1)$ on the plane $\{z=0\}$ induces a Riemannian isometry $\mathscr{H}^2 \to \mathbb{D}^2$.

Consider the group

$$\mathrm{O}(2,1) \coloneqq \left\{ A \in \mathrm{GL}(3,\mathbb{R}) \ : \ \forall v \in \mathbb{R}^3,\ q(Av) = q(v) \right\}.$$

Note that $\mathrm{O}(2,1)$ preserves the hyperboloid $\{q(p) = -1\}$; since the latter has two sheets, one of which is $\mathscr{H}^2$, the subgroup

$$\mathrm{O}^+(2,1) \coloneqq \left\{ A \in \mathrm{O}(2,1) \ : \ A(\mathscr{H}^2) = \mathscr{H}^2 \right\}$$

has index 2 in $\mathrm{O}(2,1)$. Elements of $\mathrm{O}(2,1)$ are called *Lorentzian transformations* and elements of $\mathrm{O}^+(2,1)$ are called *positive Lorentzian transformations*.

Warning I.32. $\mathrm{O}^+(2,1)$ is not equal to $\mathrm{O}(2,1) \cap \mathrm{SL}(3,\mathbb{R})$; it does contain orientation-reversing elements.

Observation I.33. There is an inclusion $\mathrm{O}^+(2,1) \leq \mathrm{Isom}(\mathscr{H}^2)$.

Exercise 3.B

Let $p, q \in \mathscr{H}^2$.

(i) *Show that there is $g \in \mathrm{O}^+(2,1)$ such that $gp = q$ and $gq = p$.*
Hint: Consider $(p-q)^\perp$.

(ii) *Show that any isometry of $T_p\mathscr{H}^2$ fixing 0 is induced by an element of $\mathrm{O}^+(2,1)$ fixing p.*

(iii) *Using (i) and (ii), deduce that $\mathrm{O}^+(2,1) = \mathrm{Isom}(\mathscr{H}^2)$.*

Property (i) is transitivity *of the action $\mathrm{O}^+(2,1) \curvearrowright \mathscr{H}^2$, and (ii) is its* isotropy.

3.2 First notions of projective geometry

Our final model is not defined via Riemannian geometry. Before introducing it, we will need some notions from projective geometry, which we will develop more in the next chapter.

Let k be a field; for us, k will always be $\mathbb{R}$ or $\mathbb{C}$.

Definition I.34. Let V be a k-vector space, equipped with the action $k^* \curvearrowright V$. The *projective space* of V is

$$\mathbb{P}(V) \coloneqq (V \smallsetminus \{0\})/k^*.$$

Note that $\mathbb{P}(V)$ can be identified with the set of lines in V.

Given two real vector spaces V, W, a map $\mathbb{P}(V) \to \mathbb{P}(W)$ is called *projective* if it is induced by an injective linear map $V \to W$. This gives in particular an action

$\mathrm{GL}(V) \curvearrowright \mathbb{P}(V)$ by projective maps, with kernel $k^* \,\mathrm{id}$. The *projective linear group* of V is

$$\mathrm{PGL}(V) \coloneqq \mathrm{GL}(V)/k^* \,\mathrm{id},$$

so that there is an induced action $\mathrm{PGL}(V) \curvearrowright \mathbb{P}(V)$.

We write in particular $\mathbb{P}^{n-1}(k) \coloneqq \mathbb{P}(k^n)$[2]. The image of $(x_1, \dots, x_n) \in k^n \smallsetminus \{0\}$ under the natural projection $k^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}(k)$ is denoted by $[x_1 : \cdots : x_n]$. Hence, we have

$$[\lambda x_1 : \cdots : \lambda x_n] = [x_1 : \cdots : x_n] \qquad \text{for all } \lambda \in k^*.$$

The above notation is called *homogeneous coordinates*. We set $\mathrm{PGL}(n,k) \coloneqq \mathrm{PGL}(k^n)$; equivalently,

$$\mathrm{PGL}(n,k) = \mathrm{GL}(n,k)/k^* \,\mathrm{id}\,.$$

The equivalence class in $\mathrm{PGL}(n,k)$ of a matrix in $\mathrm{GL}(n,k)$ will be written as a matrix with square brackets. We also denote by $\mathrm{PSL}(n,k)$ the image in $\mathrm{PGL}(n,k)$ of $\mathrm{SL}(n,k)$.

Let L be an affine line in V with $L \not\ni 0$, and let F be the unique 2-dimensional subspace of V containing L. The *projective line* $\mathbb{P}(L)$ is the image of $F \smallsetminus \{0\}$ under $V \smallsetminus \{0\} \to \mathbb{P}(V)$. Note that there is an induced bijection $\mathbb{P}(F) \to \mathbb{P}(L)$. Hence, every projective line admits an identification with $\mathbb{P}^1(k)$; this identification is non-canonical but any two choices differ by an element of $\mathrm{PGL}(2,k)$.

Exercise 3.C

Fix a k-vector space V.

(i) *Given two distinct points $p \neq q \in \mathbb{P}(V)$, prove that there is a unique projective line through p and q.*

(ii) *Show that two distinct projective lines in $\mathbb{P}(V)$ intersect in a unique point.*

(iii) *Let $\omega \in \mathbb{P}(V)$, and let $\mathbb{P}(L)$ be a projective line in $\mathbb{P}(V)$ not containing ω. The* projection *on $\mathbb{P}(L)$ based at ω is the map $\pi : \mathbb{P}(V) \smallsetminus \{\omega\} \to \mathbb{P}(L)$ defined as follows: given $p \in \mathbb{P}(V) \smallsetminus \{\omega\}$, let (p, ω) be the unique projective line through p and ω, and let $\pi(p)$ be the unique point of $(p, \omega) \cap \mathbb{P}(L)$. Given a projective line $\mathbb{P}(L')$ not containing ω, show that the restriction $\pi|_{\mathbb{P}(L')} : \mathbb{P}(L') \to \mathbb{P}(L)$ is a projective isomorphism (i.e. is the projective map induced by a linear isomorphism). Deduce that π preserves the cross-ratio.*

We now introduce the cross-ratio. To do so, we first need the following.

Lemma I.35. $\mathrm{PGL}(2,k)$ *acts simply transitively on the set of triples of distinct points in $\mathbb{P}^1(k)$.*

Proof. Let p_1, p_2, p_3 be three distinct points in $\mathbb{P}^1(k)$, and let v_1, v_2, v_3 be respective preimages of those points under $k^2 \to \mathbb{P}^1(k)$. We want to find $g \in \mathrm{PGL}(2,k)$ such that $gp_1 = [0:1]$, $gp_2 = [1:1]$, and $gp_3 = [1:0]$. Note first that, since v_1 and v_3 are linearly independent (as $p_1 \neq p_3$), there exists $\bar{g}_1 \in \mathrm{GL}(2,k)$ such that $\bar{g}_1(v_1) = (0,1)$

[2]The change of exponent is to ensure that $\mathbb{P}^{n-1}(\mathbb{R})$ is a manifold of dimension $n-1$.

and $\bar{g}_1(v_3) = (1,0)$. Write $\bar{g}_1(v_2) = (a,b) \in k^2$, and note that we must have $a \neq 0$ and $b \neq 0$ (since $p_1 \neq p_2 \neq p_3$). Set

$$\bar{g}_2 \coloneqq \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \in \mathrm{GL}(2,k),$$

and let g be the image of $\bar{g}_2\bar{g}_1$ in $\mathrm{PGL}(2,k)$. Now we have $gp_1 = [0:1]$, $gp_2 = [1:1]$, and $gp_3 = [1:0]$ as wanted.

For uniqueness, it suffices to prove that any $g \in \mathrm{PGL}(2,k)$ fixing $[0:1]$, $[1:1]$, and $[1:0]$ must be equal to the identity. To see this, consider a representative $\bar{g}$ of g in $\mathrm{GL}(2,k)$, and write

$$\bar{g} = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \in \mathrm{GL}(2,k).$$

The assumption on g means that $\bar{g}(1,0) \in k^*(1,0)$, $\bar{g}(1,1) \in k^*(1,1)$, and $\bar{g}(0,1) \in k^*(0,1)$. It easily follows that $\bar{g} \in k^* \,\mathrm{id}$, so $g = 1$ in $\mathrm{PGL}(2,k)$. □

We will often identify $\mathbb{P}^1(k)$ with $k \cup \{\infty\}$ via $x \in k \mapsto [x:1] \in \mathbb{P}^1(k)$ and $\infty \mapsto [1:0] \in \mathbb{P}^1(k)$.

Definition I.36. The *cross-ratio* of four distinct points $a,b,c,d \in \mathbb{P}^1(k)$ is the unique $z \in k$ for which there is $g \in \mathrm{PGL}(2,k)$ such that

$$g(a,b,c,d) = (0,1,z,\infty).$$

Remark I.37. The action $\mathrm{PGL}(2,k) \curvearrowright \mathbb{P}^1(k)$ respects the cross-ratio.

Let $\mathbb{P}(L)$ be a projective line in a projective space $\mathbb{P}(V)$, and let a,b,c,d be four distinct points on $\mathbb{P}(L)$. Recall that $\mathbb{P}(L)$ can be identified with $\mathbb{P}^1(k)$, and that this identification is unique up to the action of $\mathrm{PGL}(2,k)$. Since $\mathrm{PGL}(2,k)$ preserves the cross-ratio by Remark I.37, this allows us to define $[a,b,c,d]$.

Exercise 3.D

Given a,b,c,d four distinct elements of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, prove that

$$[a,b,c,d] = \frac{c-a}{b-a} \cdot \frac{b-d}{c-d}.$$

Exercise 3.E

Let a,b,c,d,e be five distinct points in $\mathbb{P}^1(k)$.

(i) *Express the following cross-ratios in terms of $[a,b,c,d]$:*

$$(\alpha) \quad [a,c,b,d] \qquad (\beta) \quad [d,b,c,a] \qquad (\gamma) \quad [b,a,c,d].$$

(ii) *Prove the equality $[a,b,d,e] = [a,b,c,e] \cdot [a,c,d,e]$.*

(iii) *Assume that* $k = \mathbb{R}$. *Suppose that* a, a', b, c, d', d *occur in this order in* $\mathbb{P}^1(\mathbb{R})$. *Show that* $[a', b, c, d'] \geq [a, b, c, d]$.

Exercise 3.F

Let z_1, z_2, z_3, z_4 *be four distinct points in* $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

(i) *Prove that the circle or line* $\mathscr{C}(z_1, z_2, z_4)$ *through* z_1, z_2, z_4 *is orthogonal to the circle or line* $\mathscr{C}(z_1, z_3, z_4)$ *through* z_1, z_3, z_4 *if and only if*

$$\Re\,[z_1, z_2, z_3, z_4] = 0.$$

N.B. *If* z_1, z_2, z_4 *are colinear, or if one of them is* ∞, *then* $\mathscr{C}(z_1, z_2, z_4)$ *is a line; in all other cases, it is a circle.*

(ii) *Generalise* (i) *to characterise the property that* $\mathscr{C}(z_1, z_2, z_4)$ *and* $\mathscr{C}(z_1, z_3, z_4)$ *meet at an angle* ϑ.

3.3 The Klein disc

We are now ready to introduce our fourth and last model of the hyperbolic plane.

Definition I.38. The *projective disc*, or *Klein disc*, is the set

$$\mathbb{K}^2 := \left\{ [x_1 : x_2 : x_3] \in \mathbb{P}^2(\mathbb{R}) \ : \ x_1^2 + x_2^2 - x_3^2 < 0 \right\}.$$

Note that $\mathbb{K}^2$ is the image of $\mathscr{H}^2$ under the projection $\mathbb{R}^3 \smallsetminus \{0\} \to \mathbb{P}^2(\mathbb{R})$. Moreover, there is a bijection $\mathbb{D}^2 \to \mathbb{K}^2$ given by

$$(x, y) \in \mathbb{D}^2 \longmapsto [x : y : 1] \in \mathbb{K}^2.$$

Hence, $\mathbb{K}^2$ can be identified with the unit disc; this is the *standard affine chart* of $\mathbb{K}^2$.

The *boundary* of $\mathbb{K}^2$ is

$$\partial\mathbb{K}^2 := \left\{ [x : y : z] \in \mathbb{P}^2(\mathbb{R}) \ : \ x_1^2 + x_2^2 - x_3^2 = 0 \right\}.$$

The above bijection $\mathbb{D}^2 \to \mathbb{K}^2$ extends to a bijection $\bar{\mathbb{D}}^2 \to \bar{\mathbb{K}}^2 := \mathbb{K}^2 \cup \partial\mathbb{K}^2$; the boundary of the unit disc is identified with the boundary of $\mathbb{K}^2$.

Warning I.39. The above bijection $\mathbb{D}^2 \to \mathbb{K}^2$ is not an isometry for the metric structure that we are about to define on $\mathbb{K}^2$.

Definition I.40. Let $p \neq q \in \mathbb{K}^2$. Consider the projective line $\mathbb{P}(L)$ through p and q, and let α, β be the two intersection points of $\mathbb{P}(L)$ with $\partial\mathbb{K}^2$, so that α, p, q, β occur in this order along $\mathbb{P}(L)$. The *projective distance* from p to q is defined by

$$d_{\text{proj}}(p, q) := \frac{1}{2} \log\,[\alpha, p, q, \beta]\,,$$

where the cross-ratio is computed under the identification $\mathbb{P}(L) \cong \mathbb{P}^1(\mathbb{R})$.

We also set $d_{\text{proj}}(p,p) \coloneqq 0$.

Proposition I.41. *d_{proj} defines a metric on $\mathbb{K}^2$.*

Proof. Separation is clear from the definition of the cross-ration; symmetry should follow from your answers to Exercise 3.E; it remains to prove the triangle inequality. To do so, consider three distinct points $p, q, r \in \mathbb{K}^2$.

First case: $q \in [p,r]$ (i.e. q lies on the projective line (p,r) through p and r, between p and r). Hence, α, p, q, r, β occur in this order along (p,r) (where $\{\alpha,\beta\} = (p,r) \cap \partial\mathbb{K}^2$). Then Exercise 3.E(ii) shows that

$$d_{\text{proj}}(p,r) = d_{\text{proj}}(p,q) + d_{\text{proj}}(q,r).$$

Second case: $q \in (p,r) \smallsetminus [p,r]$. Then either $p \in [q,r]$ or $r \in [p,q]$; we only treat the case where $p \in [q,r]$, as the other one is symmetric. The previous argument gives

$$d_{\text{proj}}(p,r) = -d_{\text{proj}}(p,q) + d_{\text{proj}}(q,r) \leq d_{\text{proj}}(p,q) + d_{\text{proj}}(q,r).$$

Third case: $q \notin (p,r)$. Let α, β be the intersection points of the projective line (p,r) with $\partial\mathbb{K}^2$ (such that α, p, r, β occur in this order on (p,r)). Consider the lines L and Λ tangent to $\partial\mathbb{K}^2$ at α and β respectively; they intersect at some point $\omega \in \mathbb{P}^2(\mathbb{R})$. Let α_1, β_1 be the intersection points of the projective line (p,q) with $\partial\mathbb{K}^2$ (such that α_1, p, q, β_1 occur in this order on (p,q)). The projective lines (α_1,ω) and (β_1,ω) meet $(\alpha,\beta) = (p,r)$ at some points α_1' and β_1' respectively, why (q,ω) meets (α,β) at some point q'. Now we have

$$\begin{aligned}
d_{\text{proj}}(p,q) &= \frac{1}{2}\log[\alpha_1, p, q, \beta_1] && \text{by definition} \\
&= \frac{1}{2}\log[\alpha_1', p, q', \beta_1'] && \text{by Exercise 3.C(iii).} \\
&\geq \frac{1}{2}\log[\alpha, p, q', \beta] && \text{by Exercise 3.E(iii)} \\
&= d_{\text{proj}}(p,q') && \text{by definition.}
\end{aligned}$$

Similarly, $d_{\text{proj}}(q,r) \geq d_{\text{proj}}(q',r)$. But $q' \in (p,r)$, so we deduce from the first and second cases that

$$d_{\text{proj}}(p,r) \leq d_{\text{proj}}(p,q') + d_{\text{proj}}(q',r) \leq d_{\text{proj}}(p,q) + d_{\text{proj}}(q,r). \qquad \square$$

Remark I.42. The proof of Proposition I.41 shows that, if three points $p, q, r \in \mathbb{K}^2$ satisfy $q \in [p,r]$ (in the projective sense, or equivalently, in the standard affine chart), then

$$d_{\text{proj}}(p,r) = d_{\text{proj}}(p,q) + d_{\text{proj}}(q,r).$$

It follows that affine lines in the standard affine chart (with an appropriate parametrisation) are isometrically embedded in $\mathbb{K}^2$.

Warning I.43. Straight lines are not isometrically embedded in $\mathbb{D}^2$!

Exercise 3.G

(i) *Show that the projection* $\mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2(\mathbb{R})$ *restricts to an isometry (in the sense of metric spaces)* $\mathscr{H}^2 \to \mathbb{K}^2$.
(ii) *Determine the geodesics of* $\mathscr{H}^2$.

II

$\mathrm{PSL}(2,\mathbb{R})$, isometries, and geodesics

This chapter is a more in-depth exploration of the hyperbolic plane, with a main focus on its isometries. More tools from projective geometry will allow us to understand the full isometry group of $\mathbb{H}^2$. This will interact with the study of geometric properties of $\mathbb{H}^2$, including in particular geodesics.

Recommended reading:

- [Cannon et al., 1997, §9–13]
- [Thurston, 1997, §2.4–5]
- [Beardon, 1983, Chapters 3, 4, 7]
- [Martelli, 2016, Chapter 2]

Lecture 4 — Möbius transformations and homographies

The main protagonist of this lecture is $\mathrm{PGL}(2,\mathbb{C})$, which first appeared in the previous lecture, and now returns under a different disguise.

4.1 Inversions and Möbius transformations

We write $\hat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$.

Definition II.1. Let $a \in \mathbb{R}^n$, $r > 0$, and let $S(a,r) := \{x \in \mathbb{R}^n \,:\, \|x-a\|_{\mathrm{eucl}} = r\}$ be the (Euclidean) sphere of centre a and radius r. The *inversion* in $S(a,r)$ is the map

$$\begin{aligned} \hat{\mathbb{R}}^n &\longrightarrow \hat{\mathbb{R}}^n \\ x &\longmapsto a + \left(\frac{r}{\|x-a\|_{\mathrm{eucl}}}\right)^2 (x-a) \qquad \text{for } x \in \mathbb{R}^n \smallsetminus \{a\} \\ a &\longmapsto \infty \\ \infty &\longmapsto a. \end{aligned}$$

Observation II.2. Every inversion defines an involution $\hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$.

The *Möbius group* Möb(n) is the subgroup of Bij$(\hat{\mathbb{R}}^n)$ (the group of self-bijections of $\hat{\mathbb{R}}^n$) generated by all inversions. Elements of Möb(n) are called *Möbius transformations.* We will also denote by Möb$(\mathbb{C})$ the group Möb(2) when we want to see its elements as self-maps of $\hat{\mathbb{C}} \coloneqq \mathbb{C} \cup \{\infty\}$.

Proposition II.3. *Let* $f \in$ Möb$(\mathbb{C})$.

(i) *If* C *is a* generalised circle *in* $\hat{\mathbb{C}}$ *(i.e. a circle in* $\mathbb{C}$*, or a line with a point at* ∞ *— viewed as a circle of infinite radius), then* $f(C)$ *is a generalised circle.*

(ii) *The map* f *is* conformal*, i.e. it preserves angles.*

In fact, these properties also hold in Möb(n)*, for* $n \geq 2$.

Proof. It suffices to prove both points when f is an inversion. Moreover, any inversion can be obtained from the *standard inversion* ι in the circle $S(0,1)$ of centre 0 and radius 1 by conjugating by a translation and a homothety. Since translations and homotheties clearly preserve generalised circles and are conformal, it suffices to prove both results for ι. In complex coordinates, note that ι is given by

$$\iota : z \mapsto \frac{z}{|z|^2}.$$

(i) Consider first a line through 0:

$$L \coloneqq \{\lambda w_1 \ : \ \lambda \in \mathbb{R}\} \cup \{\infty\},$$

with $w_1 \in \mathbb{C}$ such that $|w_1| = 1$. Then we have

$$\iota(L) = \{\iota(\lambda w_1) \ : \ \lambda \in \mathbb{R} \cup \{\infty\}\} = \left\{\frac{1}{\lambda} w_1 \ : \ \lambda \in \mathbb{R} \cup \{\infty\}\right\} = L.$$

Next, consider an affine line not containing 0:

$$L \coloneqq \{w_0 + \lambda w_1 \ : \ \lambda \in \mathbb{R}\} \cup \{\infty\},$$

with $w_0, w_1 \in \mathbb{C}$ and $|w_1| = 1$. We can assume that w_0 is the point of L of minimal modulus; equivalently, w_0 is orthogonal to w_1, i.e. $\Re(w_0 \bar{w}_1) = 0$. Since $0 \notin L$, we must have $w_0 \neq 0$. We claim that $\iota(L)$ is the circle C of centre $\frac{1}{2}\iota(w_0)$ and radius $1/(2|w_0|)$. To do so, we compute, for $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\left|\iota(w_0 + \lambda w_1) - \frac{1}{2}\iota(w_0)\right|^2 &= \left|\frac{w_0 + w_1\lambda}{|w_0|^2 + \lambda^2} - \frac{w_0}{2|w_0|^2}\right|^2 \\
&= \left|w_0 \cdot \frac{|w_0|^2 - \lambda^2}{2|w_0|^2\,(|w_0|^2 + \lambda^2)} + w_1 \cdot \frac{\lambda}{|w_0|^2 + \lambda^2}\right|^2 \\
&= \left(\frac{1}{2|w_0|\,(|w_0|^2 + \lambda^2)}\right)^2 \left(\left(|w_0|^2 - \lambda^2\right)^2 + 4\lambda^2|w_0|^2\right) \\
&= \left(\frac{1}{2|w_0|\,(|w_0|^2 + \lambda^2)}\right)^2 \left(|w_0|^2 + \lambda^2\right)^2 \\
&= \left(\frac{1}{2|w_0|}\right)^2.
\end{aligned}$$

In the above computation, we repeatedly use the fact that $|aw_0+bw_1|^2 = a^2|w_0|^2+b^2$ for all $a, b \in \mathbb{R}$, since w_0 and w_1 are orthogonal and $|w_1| = 1$. This proves that

$$\iota(L) \subseteq C.$$

To prove the reverse inclusion, consider the half-plane consisting of vectors with non-negative scalar product with w_1:

$$A^+ \coloneqq \{z \in \mathbb{C} \,:\, \Re(z\bar{w}_1) \geq 0\} \subseteq \mathbb{C}.$$

Note that A^+ is preserved by ι; in particular, $\iota(L \cap A^+) \subseteq C \cap A^+$, and $C \cap A^+$ is a half-circle with endpoints 0 and $\iota(w_0)$. But it is clear that $\iota(w_0) \in \iota(L \cap A^+)$; moreover, $\iota(w_0 + \lambda w_1) \to 0$ as $t \to +\infty$, so $\iota(L \cap A^+)$ contains points arbitrarily close to 0. Since $L \cap A^+$ is connected and ι is continuous, $\iota(L \cap A^+)$ is also connected; it follows that

$$\iota(L \cap A^+) = C \cap A^+ \smallsetminus \{0\}.$$

If A^- is the half-plane consisting of vectors with non-positive scalar product with w_1, then the same argument shows that $\iota(L \cap A^-) = C \cap A^- \smallsetminus \{0\}$. Now it follows that

$$\iota(L) = C.$$

Next, we treat the case of a circle C through 0. Given p the centre of C, consider

$$w_0 \coloneqq \iota(2p),$$

and let $w_1 \in \mathbb{C}$ with $|w_1| = 1$ and $\Re(w_0\bar{w}_1) = 0$. If $L \coloneqq \{w_0 + \lambda w_1 \,:\, \lambda \in \mathbb{R}\} \cup \{\infty\}$, then the above shows that $\iota(L) = C$; since ι is involutive, it follows immediately that

$$\iota(C) = L.$$

The case of a circle not containing 0 is Exercise 4.A below.

(ii) The standard inversion ι is the composition of $z \mapsto 1/z$, which is holomorphic hence conformal, and of complex conjugation $z \mapsto \bar{z}$, which is also conformal. Therefore, ι itself is conformal. □

Exercise 4.A

Let $\iota : z \mapsto z/|z|^2$ be the standard inversion in $\hat{\mathbb{C}}$. Let C be the circle of centre $p \in \mathbb{C}$ and radius $R \in \mathbb{R}_{>0} \smallsetminus \{|p|\}$. Prove that $\iota(C)$ is a circle and determine its centre and radius.

Exercise 4.B

(In this exercise, do not use PGL$(2,\mathbb{C})$ *and its action by homographies.) Prove that* Möb$(\mathbb{C})$ *contains reflections along lines, translations, rotations, and homotheties. Deduce that the action* Möb$(\mathbb{C}) \curvearrowright \hat{\mathbb{C}}$ *is transitive.*

4.2 Action of $\mathrm{PGL}(2,\mathbb{C})$ by homographies

As previously explained, we identify $\mathbb{P}^1(\mathbb{C})$ with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via $z \in \mathbb{C} \mapsto [z:1]$ and $\infty \mapsto [1:0]$.

Proposition II.4. *There is an action* $\mathrm{PGL}(2,\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ *given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}.$$

This is called the action of $\mathrm{PGL}(2,\mathbb{C})$ *by* homographies.

Proof. The above formula simply translates to $\mathbb{C} \cup \{\infty\}$ the action $\mathrm{PGL}(2,\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ by projective transformations. □

Exercise 4.C

Let $f \in \mathrm{PGL}(2,\mathbb{C})$.

(i) *Assume that* $f \neq \mathrm{id}$. *Under the action* $\mathrm{PGL}(2,\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, *prove that exactly one of the following assertions hold:*

- f *has two fixed points* z_1, z_2, *and* f *is conjugate (in* $\mathrm{PGL}(2,\mathbb{C})$*) to a rotation.*
- f *has two fixed points* z_1, z_2, *and* f *is conjugate (in* $\mathrm{PGL}(2,\mathbb{C})$*) to a complex homothety; moreover, for all* $z \in \mathbb{P}^1(\mathbb{C}) \smallsetminus \{z_2\}$, $f^n(z) \to z_1$ *as* $n \to \infty$ *(we say that* z_1 *is* attractive *and* z_2 *is* repulsive*).*
- f *has a single fixed point, and* f *is conjugate (in* $\mathrm{PGL}(2,\mathbb{C})$*) to a translation.*

Hint: Conjugate by an element of $\mathrm{PGL}(2,\mathbb{C})$ *to send the fixed points to well-chosen points.*

(ii) *Let* $n \geq 2$. *An* n-th root *of* f *is an element* $h \in \mathrm{PGL}(2,\mathbb{C})$ *such that* $h^n = f$. *How many* n*-th roots does* f *have?*

Exercise 4.D

Let $f \in \mathrm{PGL}(2,\mathbb{C})$. *Prove that, for all* $z \in \mathbb{P}^1(\mathbb{C})$ *with* $fz \neq z$, *we have*

$$[z, fz, f^2z, f^3z] = \frac{(\operatorname{tr} f)^2}{\det f}.$$

N.B. *For* $A \in GL(2,\mathbb{C})$ *and* $\lambda \in \mathbb{C}^*$, *we have* $(\operatorname{tr}(\lambda A))^2/\det(\lambda A) = (\operatorname{tr} A)^2/\det A$. *Consequently, the function* $(\operatorname{tr}(-))^2/\det(-) : \mathrm{GL}(2,\mathbb{C}) \to \mathbb{C}$ *descends to the quotient* $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*\,\mathrm{id}$.

Proposition II.5. $\mathrm{PGL}(2,\mathbb{C})$ *is an index-*2 *subgroup of* $\mathrm{Möb}(\mathbb{C})$*, with coset representatives* id *and* $\tau : z \mapsto \bar{z}$.

Proof. It follows from Exercise 4.B that Möb(ℂ) contains PGL(2, ℂ) (acting on $\hat{\mathbb{C}}$ by homographies) and τ. Let $f \in \text{Möb}(\mathbb{C})$; we want to show that $f \in \text{PGL}(2,\mathbb{C}) \cup \text{PGL}(2,\mathbb{C})\tau$. After possibly composing with τ, we can assume that f preserves the orientation of ℂ. Moreover, Lemma I.35 allows us, after possibly composing with an element of PGL(2, ℂ), to assume that $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. In particular, f preserves the horizontal line $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}}$. Now f sends generalised circles to generalised circles (by Proposition II.3(i)) and fixes ∞, so f sends affine lines to affine lines. Since f is conformal (by Proposition II.3(ii)) and fixes 0 and 1, it actually preserves each individual line containing either 0 or 1. But every point of $\hat{\mathbb{C}}$ is realised as the unique intersection point of a pair of lines, one containing 0 and the other containing 1; this proves that $f = \text{id}$. Hence, we have shown that

$$\text{Möb}(\mathbb{C}) = \text{PGL}(2,\mathbb{C}) \cup (\text{PGL}(2,\mathbb{C}) \circ \tau)\,. \qquad \square$$

Remark II.6. The above shows that PGL(2, ℂ) is exactly the index-2 subgroup of Möb(ℂ) consisting of orientation-preserving elements, sometimes denoted by $\text{Möb}^+(\mathbb{C})$. Its elements are called *positive Möbius transformations*.

Example II.7. The Cayley transform (see Theorem I.25) is the positive Möbius transformation defined by the matrix

$$\begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix} \in \text{PGL}(2,\mathbb{C}).$$

It sends the horizontal line $\{\Re(z) = 0\}$ to the unit circle.

In particular, by Proposition II.3, the Cayley transform is conformal and sends generalised circles to generalised circles.

4.3 Action of PSL(2, ℝ) on $\mathbb{H}^2$

We see $\hat{\mathbb{R}}$ as the subset of $\hat{\mathbb{C}}$ given by the real line $\{\Im(z) = 0\}$, together with the point ∞. Hence, $\hat{\mathbb{R}}$ is the topological boundary of $\mathbb{H}^2$ in $\hat{\mathbb{C}}$.

Homographies act on $\hat{\mathbb{C}}$; we are interested in homographies preserving $\mathbb{H}^2$. Given a field k, recall that the *special projective group* PSL(2, k) is the image under $\text{GL}(2,k) \to \text{PGL}(2,k)$ of the *special linear group* SL(2, k). For $k = \mathbb{R}$, PSL(2, ℝ) is an index-2 subgroup of PGL(2, ℝ).

Proposition II.8. *Under the action* $\text{PGL}(2,\mathbb{C}) \curvearrowright \hat{\mathbb{C}}$ *by homographies,*

(i) *The stabiliser of* $\hat{\mathbb{R}}$ *is* PGL(2, ℝ)*, and*
(ii) *The stabiliser of* $\mathbb{H}^2$ *is* PSL(2, ℝ)*.*

Proof. It is clear that $\text{PGL}(2,\mathbb{R}) \subseteq \text{Stab}(\hat{\mathbb{R}})$. Conversely, let $f \in \text{Stab}(\hat{\mathbb{R}}) \subseteq \text{PGL}(2,\mathbb{C})$. The existence part of Lemma I.35 (for $k = \mathbb{R}$) implies the existence of $g \in \text{PGL}(2,\mathbb{R})$ such that gf fixes 0, 1, and ∞. The uniqueness part of Lemma I.35 (for $k = \mathbb{C}$) then gives $gf = \text{id}$; this shows that $f \in \text{PGL}(2,\mathbb{R})$ and completes the proof of (i).

For (ii), note that $\mathrm{PSL}(2,\mathbb{R})$ stabilises $\hat{\mathbb{R}}$ by (i); moreover, every element of $\mathrm{PSL}(2,\mathbb{R})$ sends i to a point of $\mathbb{H}^2$ since

$$\Im\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot i\right) = \Im\left(\frac{b+ia}{d+ic}\right) = \frac{ad-bc}{c^2+d^2} = \frac{1}{c^2+d^2} > 0.$$

It follows by continuity that $\mathrm{PSL}(2,\mathbb{R})$ stabilises $\mathbb{H}^2$. Hence, we have

$$\mathrm{PSL}(2,\mathbb{R}) \leq \mathrm{Stab}(\mathbb{H}^2) \leq \mathrm{Stab}\left(\hat{\mathbb{R}}\right) = \mathrm{PGL}(2,\mathbb{R}).$$

Since $[\mathrm{PGL}(2,\mathbb{R}) : \mathrm{PSL}(2,\mathbb{R})] = 2 = [\mathrm{Stab}(\hat{\mathbb{R}}) : \mathrm{Stab}(\mathbb{H}^2)]$, it follows that $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Stab}(\mathbb{H}^2)$ as wanted. □

Exercise 4.E

Prove that $\mathrm{PSL}(2,\mathbb{R})$ *is the intersection with* $\mathrm{PGL}(2,\mathbb{C}) = \mathrm{Möb}^+(\mathbb{C})$ *of the subgroup of* $\mathrm{Möb}(\mathbb{C})$ *generated by horizontal translations* $z \mapsto z+\mu$ $(\mu \in \mathbb{R})$*, and by inversions in circles orthogonal to the real line and containing* i*.*

The following proposition brings us back to hyperbolic geometry.

Proposition II.9. *The action* $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *is by isometries.*

Proof. By Exercise 4.E, it suffices to show that horizontal translations and inversions in circles orthogonal to the real line and containing i are isometries. We have already seen it for horizontal translations (Exercise 1.D). If C is a circle containing i and orthogonal to the real line, then the Cayley transform $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ sends C to a circle or a line (since ϕ is a Möbius transformation) containing 0 (since $\phi(i) = 0$), and orthogonal to the unit circle (since ϕ is conformal, and sends the real line to the unit circle). Hence, $\phi(C)$ must be a line through 0, and the inversion in C is conjugate under ϕ to a reflection in a line through 0. But we have seen that such maps preserve the metric of $\mathbb{D}^2$ (Proposition I.24). Since ϕ is an isometry, it follows that the inversion in C preserves the metric of $\mathbb{H}^2$. □

Exercise 4.F

Prove that the action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *is transitive.*

Exercise 4.G

(In this exercise, do not use the hyperboloid model, and in particular Exercise 3.B.) Given $p \in \mathbb{H}^2$*, identify its stabiliser* $\mathrm{Stab}(p)$ *under the action* $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *as a well-known group.*

Lecture 5 — Isometries of the hyperbolic plane

The goal of this lecture is to understand and classify isometries of the hyperbolic plane. Using what we have learned about inversions, homographies, and Möbius transformations, we will be able to make explicit the isometry group of $\mathbb{H}^2$, and we will give a dynamical classification of isometries. Both will be useful for later, when we consider groups acting isometrically on $\mathbb{H}^2$.

5.1 The full isometry group

We have seen in Proposition II.9 that $\mathrm{PSL}(2,\mathbb{R})$ acts isometrically on $\mathbb{H}^2$, or in other words that there is a group homomorphism $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^2)$. It follows for example from Exercise 4.C that this morphism is injective (i.e. the action is faithful). Its image is contained in the index-2 subgroup $\mathrm{Isom}^+(\mathbb{H}^2)$ of orientation-preserving isometries of $\mathbb{H}^2$ (see Remark II.6). In fact, this will give us all the orientation-preserving isometries.

We first need a preparatory lemma.

Lemma II.10. *Let C be a circle for the hyperbolic metric on $\mathbb{H}^2$ or $\mathbb{D}^2$.*

(i) *C is a Euclidean circle (with possibly different centre and radius).*

(ii) *In the case of the Poincaré disc $\mathbb{D}^2$, if the hyperbolic centre of C is at 0, then so is its Euclidean centre.*

(iii) *In the case of the Poincaré half-plane $\mathbb{H}^2$, if the hyperbolic centre of C lies on $i\mathbb{R}_{>0}$, then so does its Euclidean centre.*

Proof. Let C be a hyperbolic circle centred at 0 in $\mathbb{D}^2$. We have seen in Proposition I.24 that rotations centred at 0 are isometries of $\mathbb{D}^2$; since they fix 0, they must preserve C. It follows that C is a union of Euclidean circles centred at 0. Now any diameter of the unit disc is a shortest path (see Remark I.26), so it can meet C in at most two points; therefore, C is a single Euclidean circle centred at 0; this proves (i) for hyperbolic circles centred at 0 in $\mathbb{D}^2$, as well as (ii).

Since the action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{D}^2$ obtained by conjugating $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ by the Cayley transform $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ is transitive (see Exercise 4.F) and sends circles to circles (since ϕ and its inverse do: ϕ is a Möbius transformation, and the only circles that ϕ^{-1} sends to lines are those containing 1, but they are not contained in $\mathbb{D}^2$), (i) is proved for hyperbolic circles centred at any point of $\mathbb{D}^2$. Furthermore, the fact that ϕ and ϕ^{-1} send circles to circles immediately implies that (i) holds in $\mathbb{H}^2$.

For (iii), recall that reflection in $i\mathbb{R}_{>0}$ is an isometry of $\mathbb{H}^2$, so it must preserve hyperbolic circles centred on $i\mathbb{R}_{>0}$; it follows that their Euclidean centres must also lie on $i\mathbb{R}_{>0}$. □

Exercise 5.A

Let $r > 0$. Compute the Euclidean centre and radius of the hyperbolic circle of centre i and radius r in $\mathbb{H}^2$.

We can now fully determine the isometry group of the hyperbolic plane.

Theorem II.11. *The action* $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ *induces an isomorphism*

$$\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2,\mathbb{R}).$$

Proof. We have already seen that the action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$ induces an embedding $\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$; it remains to show that this is surjective. Let $f \in \mathrm{Isom}^+(\mathbb{H}^2)$. Using Exercises 4.F and 4.G, we can assume after composing by an element of $\mathrm{PSL}(2,\mathbb{R})$ that $f(i) = i$ and f preserves the vertical ray $i\mathbb{R}_{>0}$. We can assume after possibly composing by the hyperbolic rotation of angle π around i (i.e. the homography $z \mapsto -1/z$) that $f(0) = 0$ and $f(\infty) = \infty$. It follows from the fact that f is an isometry that it fixes $i\mathbb{R}_{>0}$ pointwise.

We now claim that $f = \mathrm{id}$. To prove this, let $p \in \mathbb{H}^2$. Choose two distinct points $\omega_1 \neq \omega_2 \in i\mathbb{R}_{>0} \smallsetminus \{p\}$, and let C_1, C_2 be the hyperbolic circles through p centred at ω_1 and ω_2 respectively. Lemma II.10(iii) tells us that C_1 and C_2 are (distinct) Euclidean circles centred at points of $i\mathbb{R}_{>0}$. By choice, they intersect at p, so we must have

$$C_1 \cap C_2 = \{p, \tau(p)\},$$

where $\tau : z \mapsto -\bar{z}$ is the reflection through $i\mathbb{R}_{>0}$. But f fixes $i\mathbb{R}_{>0}$ pointwise, and is an isometry, so f must preserve C_1 and C_2 setwise; it follows that $f(p) \in \{p, \tau(p)\}$. This holds for all $p \in \mathbb{H}^2$, and f is continuous, so $f \in \{\mathrm{id}, \tau\}$. But f is orientation-preserving, so $f = \mathrm{id}$ as claimed. □

Corollary II.12. $\mathrm{Isom}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R}) \cup (\mathrm{PSL}(2,\mathbb{R}) \circ \tau)$*, where* $\tau : z \mapsto -\bar{z}$ *is the reflection through* $i\mathbb{R}_{>0}$*.* □

5.2 Dynamical classification of isometries

The boundary of the hyperbolic plane has been somewhat present in several arguments so far; it plays a major role in the dynamical classification of isometries and we now define it formally.

Definition II.13.

- Seeing $\mathbb{D}^2$ as a subset of $\mathbb{C}$, the *boundary* of $\mathbb{D}^2$ is

  $$\partial\mathbb{D}^2 := \{z \in \mathbb{C} \,:\, |z| = 1\},$$

  and the *closure* of $\mathbb{D}^2$ is $\bar{\mathbb{D}}^2 := \mathbb{D}^2 \cup \partial\mathbb{D}^2$; this coincides with the topological closure of $\mathbb{D}^2$ as a subset of $\mathbb{C}$.
- Seeing $\mathbb{H}^2$ as a subset of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, the *boundary* of $\mathbb{H}^2$ is

  $$\partial\mathbb{H}^2 := \mathbb{P}^1(\mathbb{R}) = \{z \in \mathbb{C} \,:\, \Im(z) = 0\} \cup \{\infty\},$$

  and the *closure* of $\mathbb{H}^2$ is $\bar{\mathbb{H}}^2 := \mathbb{H}^2 \cup \partial\mathbb{H}^2$; this coincides with the topological closure of $\mathbb{H}^2$ as a subset of $\mathbb{P}^1(\mathbb{C})$.

Observation II.14. Since isometries of $\mathbb{H}^2$ are elements of $\mathrm{PSL}(2,\mathbb{R})$ possibly composed with a reflection (by Corollary II.12), they extend to continuous maps $\bar{\mathbb{H}}^2 \to \bar{\mathbb{H}}^2$.

We now focus on the orientation-preserving case; the following gives a dynamical classification of elements of $\mathrm{Isom}^+(\mathbb{H}^2)$. Note that, since $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{id}\}$, every $g \in \mathrm{PSL}(2,\mathbb{R})$ has a well-defined trace up to a sign, and $|\operatorname{tr} g|$ is well-defined.

Proposition/Definition II.15. *Let $g \in \mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$, with $g \neq \mathrm{id}$.*

(i) *If $|\operatorname{tr} g| < 2$, then g has exactly one fixed point in $\mathbb{H}^2$, and none in $\partial\mathbb{H}^2$. It is conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to*

$$\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix},$$

for some $\vartheta \in (0,\pi)$; its homography action is a (hyperbolic) rotation of angle -2ϑ.
We then say that g is elliptic.

(ii) *If $|\operatorname{tr} g| = 2$, then g has no fixed point in $\mathbb{H}^2$, and exactly one in $\partial\mathbb{H}^2$. It is conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to*

$$\begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix};$$

its homography action is a horizontal translation.
We then say that g is parabolic.

(iii) *If $|\operatorname{tr} g| > 2$, then g has no fixed point in $\mathbb{H}^2$, and exactly two in $\partial\mathbb{H}^2$. It is conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix},$$

for some $\lambda \in (0,+\infty) \smallsetminus \{1\}$; its homography action is a homothety.
We then say that g is hyperbolic, *or* loxodromic.

Proof. Write

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R}).$$

Since $g \neq \mathrm{id}$, we can assume after conjugating by an element of $\mathrm{PSL}(2,\mathbb{R})$ that ∞ is not a fixed point, i.e. $c \neq 0$. For $z \in \mathbb{C}$, we have

$$gz = z \iff az + b = z(cz + d) \iff cz^2 + z(d-a) - b = 0.$$

Hence, finding fixed points of g amounts to solving a quadratic equation (recall that $c \neq 0$), with discriminant

$$\Delta = (d-a)^2 + 4bc = (d+a)^2 - 4(ad-bc) = |\operatorname{tr} g|^2 - 4.$$

(i) If $|\operatorname{tr} g| < 2$, then $\Delta < 0$, so g has two conjugate non-real fixed points in $\mathbb{C}$, exactly one of which lies in $\mathbb{H}^2$. Moreover, we assumed that $g\infty \neq \infty$, so g has no fixed point in $\partial\mathbb{H}^2$. Since $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$, we can assume after conjugating g that i is the unique fixed point of g. A simple computation shows that g is now of the desired form. The geometric interpretation of g is Exercise 5.B below.

(ii) If $|\operatorname{tr} g| = 2$, then $\Delta = 0$, so g has a double fixed point in $\mathbb{R}$, and none other in $\mathbb{C}$. In addition, ∞ is not a fixed point, confirming that g has exactly one fixed point in $\partial\mathbb{H}^2$ and none in $\mathbb{H}^2$. Since the action $\mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{R}\cup\{\infty\}$ is transitive, we can conjugate g again by an element of $\mathrm{PSL}(2,\mathbb{R})$ so that $g\infty = \infty$ (note that we are possibly conjugating by the inverse of the element that we previously used to ensure that ∞ was not a fixed point). A simple computation shows that g is conjugate to an element of the desired form; it is clear that these are horizontal translations of $\mathbb{H}^2$.

(iii) If $|\operatorname{tr} g| > 2$, then $\Delta > 0$, so g has two fixed points in $\mathbb{R} \subseteq \partial\mathbb{H}^2$, and none in $\mathbb{H}^2$. Again, we conjugate g so that those two fixed points are 0 and ∞; this means that $b = c = 0$, so that we have $ad = 1$ and $|a+d| > 1$; it follows that g is of the desired form; it is clear that those are homotheties of $\mathbb{H}^2$. □

Remark II.16. The proof of Proposition/Definition II.15 could have been seen in more linear algebraic terms by viewing elements of $\mathrm{PSL}(2,\mathbb{R})$ through their action on $\mathbb{P}^1(\mathbb{C}) = (\mathbb{C}^2\smallsetminus\{0\})/\mathbb{C}^*$. In this language, finding fixed points for an element of $\mathrm{PSL}(2,\mathbb{R})$ amounts to finding eigenvectors for a representative matrix.

Exercise 5.B

Consider

$$g \coloneqq \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \curvearrowright \mathbb{H}^2.$$

Compute the conjugate of g under the Cayley transform $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ and interpret it geometrically.

Exercise 5.C

Let $g \in \mathrm{Isom}^+(\mathbb{H}^2)$. Depending on the type of g (elliptic, parabolic, hyperbolic),

(i) *Describe the dynamics of the action of g on $\bar{\mathbb{H}}^2$: depending on $p \in \bar{\mathbb{H}}^2$, what is the asymptotic behaviour of $g^n(p)$ as $n \to \infty$?*

(ii) *Describe geometrically the orbits of g: for each $p \in \mathbb{H}^2$, find a "nice" subset of $\mathbb{H}^2$ containing the set $\langle g\rangle\, p$.*

Hint: "Nice" can be interpreted as meaning: a circle or a line in an appropriate model of the hyperbolic plane.

(iii) *When does there exist a geodesic in $\mathbb{H}^2$ which is stabilised (setwise) by g?*

Exercise 5.D

(i) *Let $f, g \in \mathrm{Isom}(\mathbb{H}^2)$ and assume that f and $g^{-1}fg$ have the same set of fixed points P in $\partial\mathbb{H}^2$. Prove that P is stabilised (setwise) by g.*
(ii) *Let $f, g \in \mathrm{Isom}(\mathbb{H}^2)$. Suppose that f and g are non-elliptic and commute. Prove that f and g are either both hyperbolic or both parabolic, and have the same fixed points in $\partial\mathbb{H}^2$.*

Exercise 5.E

In this exercise, we admit that the hyperbolic metric satisfies the following convexity property: given a geodesic $\gamma : [0,1] \to \mathbb{H}^2$, and a point $p \in \mathbb{H}^2$, we have

$$d_{\mathbb{H}^2}\left(p, \gamma\left(\frac{1}{2}\right)\right) \leq \frac{1}{2} d_{\mathbb{H}^2}\left(p, \gamma(0)\right) + \frac{1}{2} d_{\mathbb{H}^2}\left(p, \gamma(1)\right).$$

(i) *Let $(p_i)_{i\in I}$ be a finite collection of points in $\mathbb{H}^2$. Associate to each p_i a mass $\mu_i \in \mathbb{R}_{>0}$. Consider the function*

$$\begin{aligned} M_2 : \mathbb{H}^2 &\longrightarrow [0, +\infty) \\ x &\longmapsto \sum_{i\in I} \mu_i \cdot d_{\mathbb{H}^2}(x, p_i)^2. \end{aligned}$$

Prove that M_2 has a minimum, which is attained at a unique point of $\mathbb{H}^2$; this point is called the hyperbolic centre of mass *of the weighted family $\{(p_i, \mu_i)\}_{i\in I}$.*
(ii) *Deduce that any finite subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ has a global fixed point.*
(⋆) *Let $\{(p_i, \mu_i)\}_{i\in I}$ be as in (i), and consider*

$$\begin{aligned} M_1 : \mathbb{H}^2 &\longrightarrow [0, +\infty) \\ x &\longmapsto \sum_{i\in I} \mu_i \cdot d_{\mathbb{H}^2}(x, p_i). \end{aligned}$$

Does M_1 have a minimum? If so, is it attained at a unique point?
Points attaining the minimum of M_1 are called medians *of $\{(p_i, \mu_i)\}_{i\in I}$. In the special case where $|I| = 3$ and $\mu_i = 1$ for all $i \in I$, give a geometric characterisation of the median(s).*

5.3 Isometries in the Poincaré disc

We now translate what we know about isometries in the half-plane model to the Poincaré disc model.

We introduce the matrix group

$$\mathrm{U}(1,1) := \left\{ \begin{pmatrix} \alpha & e^{i\theta}\beta \\ \bar{\beta} & e^{i\theta}\bar{\alpha} \end{pmatrix} \,:\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1,\ \theta \in \mathbb{R} \right\} \leq \mathrm{GL}(2, \mathbb{C}).$$

Note that $\mathrm{U}(1,1)$ is exactly the subgroup of $\mathrm{GL}(2,\mathbb{C})$ preserving the standard Hermitian

form on $\mathbb{C}^2$ of signature $(1,1)$ (given by $h(z,w) \coloneqq z_1\bar{w}_1 - z_2\bar{w}_2$). We also set

$$\mathrm{SU}(1,1) \coloneqq \mathrm{U}(1,1) \cap \mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,:\, \alpha,\beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\} \leq \mathrm{GL}(2,\mathbb{C}).$$

As is now usual, we denote by $\mathrm{PU}(1,1)$ and $\mathrm{PSU}(1,1)$ the respective images of $\mathrm{U}(1,1)$ and $\mathrm{SU}(1,1)$ in $\mathrm{PGL}(2,\mathbb{C})$.

Remark II.17. $\mathrm{PSU}(1,1) = \mathrm{PU}(1,1)$.

Proof. This is because (as usual, we use square brackets to denote equivalence classes in $\mathrm{PGL}(2,\mathbb{C})$)

$$\begin{bmatrix} \alpha & e^{i\theta}\beta \\ \bar{\beta} & e^{i\theta}\bar{\alpha} \end{bmatrix} = \begin{bmatrix} e^{-i\theta/2}\alpha & e^{i\theta/2}\beta \\ e^{i\theta/2}\bar{\beta} & e^{-i\theta/2}\bar{\alpha} \end{bmatrix} \in \mathrm{PSU}(1,1). \qquad \square$$

The group $\mathrm{PU}(1,1)$ now plays the same role for the Poincaré disc as $\mathrm{PSL}(2,\mathbb{R})$ does for the half-plane; this is the following exercise.

Exercise 5.F

(i) *We see* $\mathbb{D}^2$ *as a subset of* $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$*. Prove that*

$$\mathrm{Stab}_{\mathrm{PGL}(2,\mathbb{C})}(\mathbb{D}^2) = \mathrm{PU}(1,1),$$

and that the action $\mathrm{PU}(1,1) \curvearrowright \mathbb{D}^2$ *induces an isomorphism*

$$\mathrm{Isom}^+(\mathbb{D}^2) \cong \mathrm{PU}(1,1).$$

(ii) *Consider* $h(z,w) \coloneqq z_1\bar{w}_1 - z_2\bar{w}_2$ *the standard Hermitian form of signature* $(1,1)$*. Give a direct proof that* $\mathrm{PU}(1,1)$ *stabilises* $\mathbb{D}^2$ *by identifying* $\mathbb{D}^2$ *with* $\mathbb{P}(S) \subseteq \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1(\mathbb{C})$ *for a certain subset* S *of* $\mathbb{C}^2$ *defined from* h*.*

Lecture 6 — Geodesics and their geometry

In this lecture, we use our understanding of hyperbolic isometries to determine geodesics in the Poincaré half-plane and Poincaré disc (we had already determined them in the Klein disc and hyperboloid models). We then study the geometry of the hyperbolic plane in more details, focusing in particular on triangles; we will see some specificities of hyperbolic geometry compared to Euclidean geometry.

6.1 Geodesics in $\mathbb{H}^2$ and $\mathbb{D}^2$

Before going any further, let us clarify the relation between geodesics (i.e. locally length-minimising paths, see Definition I.16) and shortest paths (i.e. globally length-minimising paths) in the hyperbolic plane.

Proposition II.18. (i) $\mathbb{H}^2$ *and* $\mathbb{D}^2$ *have infinite injectivity radii.*

(ii) *In* $\mathbb{H}^2$ *and* $\mathbb{D}^2$*, geodesics coincide with shortest paths.*

Proof. Recall that shortest paths are geodesics in any Riemannian manifold; this is Exercise 2.D(i). Moreover, the converse is true if the injectivity radius is infinite, by Exercise 2.D(ii). Hence, it suffices to prove (i). We will prove that $\mathrm{inj}_0(\mathbb{D}^2) = \infty$, and it will follow by transitivity of the action $\mathrm{Isom}(\mathbb{D}^2) \curvearrowright \mathbb{D}^2$ (Exercise 1.D) that $\mathrm{inj}(\mathbb{D}^2) = \infty$, and that $\mathrm{inj}(\mathbb{H}^2) = \infty$ since $\mathbb{H}^2$ and $\mathbb{D}^2$ are isometric.

We know that the horizontal diameter is a shortest path, and hence a geodesic, in $\mathbb{D}^2$. This means that there exists a bijective C^1 function $\theta : \mathbb{R} \to (-1,+1)$ with $\theta(0) = 0$, such that the path $\eta := (\theta, 0) : \mathbb{R} \to \mathbb{D}^2$ is a geodesic, which we assume to have unit speed: $\|\eta'(t)\|_{\mathrm{hyp}}^{\eta(t)} = 1$ for all t. Now the tangent space $T_0\mathbb{D}^2$ of $\mathbb{D}^2$ at 0 has a natural identification with $\mathbb{R}^2$, and the above implies that, for $v = (x, 0) \in \mathbb{R} \times \{0\} \subseteq T_0\mathbb{D}^2 = \mathbb{R}^2$, the maximal geodesic $\gamma_{(0,v)}$ with initial position 0 and initial speed v is defined on $\mathbb{R}$ and satisfies $\gamma_{(0,v)}(1) = \eta(x)$; in other words, $\exp_0(v) = \eta(x)$. Moreover, rotations around 0 are isometries of $\mathbb{D}^2$, so for each $A \in \mathrm{SO}(2,\mathbb{R})$, we have

$$\exp_0(A(x,0)) = A \cdot \eta(x).$$

Since $\eta : \mathbb{R} \to (-1,+1)$ is a bijection, it easily follows that $\exp_0$ is injective on $\mathbb{R}^2$, i.e. $\mathrm{inj}_0(\mathbb{D}^2) = \infty$. □

Now let us summarise what we know about geodesics in the hyperbolic plane.

- The vertical ray $i\mathbb{R}_{>0}$ is a geodesic in $\mathbb{H}^2$ (Proposition I.5), and the horizontal diameter $(-1,1) \times \{0\}$ is a geodesic in $\mathbb{D}^2$ (Remark I.26).
- Every pair of points in $\mathbb{H}^2$ and $\mathbb{D}^2$ is joined by a unique geodesic (since $\mathrm{inj}(\mathbb{H}^2) = \mathrm{inj}(\mathbb{D}^2) = \infty$).

Using what we know about isometries of the hyperbolic plane, we can now determine all geodesics.

Theorem II.19. (i) *Geodesics of* $\mathbb{H}^2$ *are circular arcs and vertical rays orthogonal to the real line.*
(ii) *Geodesics of* $\mathbb{D}^2$ *are circular arcs and segments orthogonal to the unit circle.*

Proof. We already know that the horizontal diameter is a geodesic in $\mathbb{D}^2$; hence, so is every diameter of the unit disc since rotations around 0 are isometries of $\mathbb{D}^2$. Moreover, these are all geodesics through 0 by uniqueness of the geodesic connecting a pair of points. Since the Cayley transform $\phi : \mathbb{H}^2 \to \mathbb{D}^2$ and its inverse are Möbius transformations, they send generalised circles (circles and lines) to generalised circles, and are conformal (in particular, they preserve orthogonality). It follows that all circular arcs and vertical rays through i and orthogonal to the real line are geodesics of $\mathbb{H}^2$; moreover, these are all geodesics through i. Using homotheties and horizontal translations (which are isometries of $\mathbb{H}^2$ by Exercise 1.D), we deduce that circular arcs and vertical rays orthogonal to the real line are all geodesics of $\mathbb{H}^2$. Applying the Cayley transform again, it follows that circular arcs orthogonal to the unit circle are all geodesics of $\mathbb{D}^2$. □

Exercise 6.A

(i) *Verify that the first four axioms of Euclid hold in (your favourite model of) the hyperbolic plane (replacing the word "straight line" with "geodesic").*
(ii) *Two geodesics are called* parallel *if they are disjoint. Show the following strong negation to the parallel postulate in the hyperbolic plane: given γ a geodesic, and $p \notin \gamma$, there are infinitely many geodesics through p that are parallel to γ.*

6.2 Angles and areas

A fundamental result will be the Gauß–Bonnet Formula, which connects the area of a triangle with its angles. But before stating it, we need to define both areas and angles.

Definition II.20. Let (M, g) be a Riemannian manifold, and let $\gamma, \eta : I \to M$ be two C^1 paths in M, defined on an interval $I \subseteq \mathbb{R}$. Let $t_0 \in I$ such that $\gamma(t_0) = \eta(t_0) =: p$. Then we define the *(unoriented) angle* of γ and η at t_0 by

$$\angle_g^{t_0}(\gamma, \eta) := \arccos\left(\frac{\langle \gamma'(t_0), \eta'(t_0)\rangle_g^p}{\|\gamma'(t_0)\|_g^p \cdot \|\eta'(t_0)\|_g^p}\right).$$

The following observation says that angles are very easy to read from the half-plane or the Poincaré disc model — to some extent, this is why we often work with those.

Observation II.21. Let γ, η be two C^1 paths in $\mathbb{H}^2$ with $\gamma(0) = \eta(0)$. Then the angles of γ and η at 0 for the hyperbolic metric (i.e. the metric of $\mathbb{H}^2$) and the Euclidean metric (seeing $\mathbb{H}^2$ as a subset of the Euclidean plane) coincide:

$$\angle_{\mathbb{H}^2}^0(\gamma, \eta) = \angle_{\text{eucl}}^0(\gamma, \eta).$$

This is because, at every point of $\mathbb{H}^2$, the Riemannian metric of $\mathbb{H}^2$ is a multiple of the Euclidean metric (it is equal to $\frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$, and $\mathrm{d}x^2 + \mathrm{d}y^2$ is the Euclidean metric).

The same thing holds in $\mathbb{D}^2$; we say that $\mathbb{H}^2$ and $\mathbb{D}^2$ are *conformal models*.

Warning II.22. Neither $\mathbb{K}^2$ nor $\mathscr{H}^2$ is a conformal model.

For simplicity, we define areas directly in $\mathbb{H}^2$ and $\mathbb{D}^2$, rather than in a general Riemannian surface.

Definition II.23. Let D be a Lebesgue-measurable subset of $\mathbb{H}^2$ or $\mathbb{D}^2$

- If $D \subseteq \mathbb{H}^2$, we set
$$\text{Area}_{\mathbb{H}^2}(D) := \int_D \frac{1}{y^2}\,\mathrm{d}x\,\mathrm{d}y.$$
- If $D \subseteq \mathbb{D}^2$, we set
$$\text{Area}_{\mathbb{D}^2}(D) := \int_D \left(\frac{2}{1-r^2}\right)^2 r\,\mathrm{d}r\,\mathrm{d}\theta.$$

A simple computation shows that areas are preserved by isometries.

Exercise 6.B

(i) *Compute the circumference* $\mathscr{C}_{\text{hyp}}(r)$ *of a hyperbolic circle of radius* r.
(ii) *Compute the area* $\mathscr{A}_{\text{hyp}}(r)$ *of a hyperbolic disc of radius* r.
(iii) *What is the asymptotic behaviour of* $\mathscr{C}_{\text{hyp}}(r)$ *and* $\mathscr{A}_{\text{hyp}}(r)$ *as* $r \to \infty$*? Compare with the Euclidean case.*

6.3 Hyperbolic triangles

By a *hyperbolic triangle*, we mean three distinct points in $\mathbb{H}^2$ (or any other model of the hyperbolic plane), called the *vertices*, together with three geodesics connecting them pairwise, called the *edges*. The *angle* at a vertex is the angle between the two geodesics at this vertex. An *ideal triangle* is a triangle for which vertices are allowed to lie on $\bar{\mathbb{H}}^2$. A vertex on $\partial\mathbb{H}^2$ is called an *ideal vertex*. The angle at an ideal vertex is defined to be 0. The area of a hyperbolic triangle is the area of its convex hull.

Theorem II.24 (Gauß–Bonnet)**.** *Let* Δ *be a (possibly ideal) triangle in* $\mathbb{H}^2$ *with vertex angles* $\alpha_1, \alpha_2, \alpha_3$*. Then*

$$\text{Area}(\Delta) = \pi - \sum_{i=1}^{3} \alpha_i.$$

Proof. We denote by p_1, p_1, p_3 the three vertices of Δ, with α_i being the angle at p_i. Moreover, let ξ_2, ξ_3 be the endpoints on $\partial\mathbb{H}^2$ of the geodesic (p_2, p_3), chosen so that ξ_2, p_2, p_3, ξ_3 occur in this order. We assume that the triangle is non-degenerate (i.e. there is no geodesic containing all of p_1, p_2, p_3), otherwise two of the angles are 0, one is π, the area is 0, and the result holds.

First case: Δ has at least one ideal vertex, say p_1. Since Δ is non-degenerate, the ideal points p_1, ξ_2, ξ_3 are all distinct; hence, by Lemma I.35, there is $g \in \text{PGL}(2, \mathbb{R})$ such that $gp_1 = \infty$, $g\xi_2 = -1$, and $g\xi_3 = +1$. Moreover, we can assume that $g \in \text{PSL}(2, \mathbb{R})$ after possibly swapping p_2 and p_3 (and hence ξ_2 and ξ_3). The fact that $g\Delta$ satisfies the Gauß–Bonnet Formula is Exercise 6.C below; since $\text{PSL}(2, \mathbb{R})$ preserves angles and areas, it follows that Δ also satisfies the Gauß–Bonnet Formula.

Second case: Similar to the first case, we can assume after applying an element of $\text{PSL}(2, \mathbb{R})$ that $\xi_2 = -1$, $\xi_3 = +1$, and that p_1 lies on the geodesic between p_2 and ∞. Now consider the triangles

- Δ_∞ with vertices ∞, p_2, p_3, and with angles $0, \alpha_2, \alpha_3 + \theta$ — where θ is the angle at p_3 formed by the geodesics $[p_3, p_1]$ and $[p_3, \infty]$,
- Δ_c with vertices ∞, p_1, p_3, and with angles $0, \pi - \alpha_1, \theta$.

Note that $\Delta_\infty = \Delta \cup \Delta_c$, and $\Delta \cap \Delta_c = [p_1, p_3]$ (as convex hulls); hence

$$\text{Area}_{\mathbb{H}^2}(\Delta) = \text{Area}_{\mathbb{H}^2}(\Delta_\infty) - \text{Area}_{\mathbb{H}^2}(\Delta_c).$$

But the areas of Δ_∞ and Δ_c can be computed thanks to the first case above; this yields

$$\begin{aligned}\mathrm{Area}_{\mathbb{H}^2}(\Delta) &= (\pi - \alpha_2 - (\alpha_3 + \theta)) - (\pi - (\pi - \alpha_1) - \theta) \\ &= \pi - \alpha_1 - \alpha_2 - \alpha_3.\end{aligned}$$

□

Exercise 6.C

Let Δ be an ideal triangle in $\mathbb{H}^2$ with one vertex at ∞, and the other two on the geodesic with endpoints ± 1. Compute explictly Area(Δ) *and prove that Δ satisfies the Gauß–Bonnet Formula.*

Exercise 6.D

Given $\alpha_1, \alpha_2, \alpha_3 \geq 0$ with $\sum_i \alpha_i < \pi$, prove that there exists a hyperbolic triangle with vertex angles $\alpha_1, \alpha_2, \alpha_3$, and that it is unique up to isometry.

Exercise 6.E

(i) *Prove that there exists a constant $\delta > 0$ such that, for every hyperbolic triangle with vertices p_1, p_2, p_3, every point of the geodesic $[p_1, p_2]$ is at distance at most δ from some point of $[p_1, p_3] \cup [p_3, p_2]$.*
Hint: Use Exercise 6.B.
The existence of such a δ is called Gromov-hyperbolicity *(we also say that $\mathbb{H}^2$ is δ-*hyperbolic*).*

(ii) *Is the Euclidean plane Gromov-hyperbolic?*

(⋆) *Can you think of other metric spaces which are Gromov-hyperbolic?*

6.4 Miscellaneous geometric properties

Proposition II.25. *Let $p \neq q \in \mathbb{H}^2$, and let γ be the unique bi-infinite geodesic through p and q. Denote by ξ_- and ξ_+ the endpoints of γ on $\partial\mathbb{H}^2$, so that ξ_-, p, q, ξ_+ occur in this order along γ.*

Seeing $\bar{\mathbb{H}}^2$ as a subset of $\mathbb{P}^1(\mathbb{C})$, the cross-ratio $[\xi_-, p, q, \xi_+]$ is a positive real number, and

$$d_{\mathbb{H}^2}(p, q) = \log[\xi_-, p, q, \xi_+].$$

Proof. Choose $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $g\xi_- = 0$ and $g\xi_+ = \infty$; hence, $gp = i\lambda$ and $gq = i\mu$ for some $0 < \lambda < \mu < \infty$. Now we have

$$\begin{aligned}
d_{\mathbb{H}^2}(p, q) &= d_{\mathbb{H}^2}(i\lambda, i\mu) && \text{since } \mathrm{PSL}(2, \mathbb{R}) \text{ acts isometrically on } \mathbb{H}^2 \\
&= \log\left(\frac{\mu}{\lambda}\right) && \text{by Exercise 1.A and since } (0, \infty) \text{ is a geodesic} \\
&= \log[0, i\lambda, i\mu, \infty] && \text{by Exercise 3.D} \\
&= \log[\xi_-, p, q, \xi_+] && \text{since } \mathrm{PSL}(2, \mathbb{R}) \text{ preserves the cross-ratio.}
\end{aligned}$$

□

Exercise 6.F

Let γ be a geodesic in $\mathbb{H}^2$.

(i) *Given $p \in \mathbb{H}^2$, show that there is a unique $w_0 \in \gamma$ such that*

$$d_{\mathbb{H}^2}(p, w_0) = \min_{w \in \gamma} d_{\mathbb{H}^2}(p, w).$$

If $p \notin \gamma$, show that w_0 is characterised by the property that $\gamma \perp (p, w_0)$.

Hence, we have a map $\pi_\gamma : \mathbb{H}^2 \to \gamma$ given by $p \mapsto w_0$.

(ii) *If γ_1, γ_2 are two geodesics in $\mathbb{H}^2$ which are disjoint in $\bar{\mathbb{H}}^2$, prove that there is a unique couple $(w_1, w_2) \in \gamma_1 \times \gamma_2$ such that*

$$d_{\mathbb{H}^2}(w_1, w_2) = \min_{(z_1, z_2) \in \gamma_1 \times \gamma_2} d_{\mathbb{H}^2}(z_1, z_2).$$

Show that w_1 and w_2 are characterised by the fact that (w_1, w_2) is the unique common perpendicular to γ_1 and γ_2.

(iii) *Prove that the map $\pi_\gamma : \mathbb{H}^2 \to \gamma$ is 1-Lipschitz (for $d_{\mathbb{H}^2}$).*

III

Discrete groups, tilings, and surfaces

In the previous chapter, we have learnt that the group of orientation-preserving isometries of the hyperbolic plane is $\mathrm{PSL}(2,\mathbb{R})$, acting on $\mathbb{H}^2$ by homographies. We are now interested in understanding discrete subgroups of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ and their geometry. The same theory also applies to discrete subgroups of $\mathrm{Isom}^+(\mathbb{E}^2)$, where $\mathbb{E}^2$ denotes the Euclidean plane, and this will enlarge our source of examples (even in the more comfortable world of Euclidean geometry).

Recommended reading:

- [Beardon, 1983, Chapters 8–10]
- [Ratcliffe, 2019, Chapter 6]
- [Martelli, 2016, Chapters 3, 6]

Lecture 7 — Discrete groups & their actions

In this lecture, we are interested in discrete groups, a notion at the crossroads of group theory and topology. We will see how discreteness can be characterised in terms of certain actions.

7.1 Discrete topological groups and properly discontinuous actions

Recall that a *topological group* is the data of a group G together with a topology on the underlying set of G such that the group operation and inversion are continuous.

Definition III.1. A topological group Γ is called *discrete* if it satisfies one of the following equivalent conditions:

(i) The topology on Γ is discrete (i.e. all sets are open).

(ii) The trivial subgroup $\{1\}$ is open in Γ.

A *discrete subgroup* of a topological group G is a subgroup Γ of G which is discrete when equipped with the topology induced by that of G.

Exercise 7.A

(i) *Prove that a subgroup Γ of $\mathbb{R}^n$ is discrete if and only if $\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$ for some vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ which are linearly independent over $\mathbb{R}$.*

(ii) *Let $k \in \mathbb{C}^\times$. When is the subgroup generated by k in $\mathbb{C}^\times$ discrete?*

A metric space X is *proper* if closed balls in X are compact. Given a proper metric space X, our goal is to understand which subgroups of $\mathrm{Isom}(X)$ are discrete, in terms of their action on X.

Definition III.2. Let X be a proper metric space. An action $\Gamma \curvearrowright X$ by isometries is *properly discontinuous* if it satisfies one of the following equivalent conditions:

(i) Every $x \in X$ has a neighbourhood N_x in X such that

$$\#\{\gamma \in \Gamma \,:\, \gamma N_x \cap N_x \neq \varnothing\} < \infty.$$

(ii) For every compact subset $K \subseteq X$, we have

$$\#\{\gamma \in \Gamma \,:\, \gamma K \cap K \neq \varnothing\} < \infty.$$

(iii) For every $x \in X$ and for every compact subset $K \subseteq X$, we have

$$\#\{\gamma \in \Gamma \,:\, \gamma x \in K\} < \infty.$$

(iv) For every $x \in X$, the orbit Γx is closed and discrete in X and the stabiliser $\mathrm{Stab}_\Gamma(x)$ is finite.

Exercise 7.B

Prove the equivalence of the four conditions in the definition of properly discontinuous actions (Definition III.2).

Warning III.3. For general topological spaces, the four conditions in the definition of properly discontinuous actions (Definition III.2) might not be equivalent. In fact, there are several notions of properly discontinuous actions in the literature, sometimes (but not always) coinciding with one of these four conditions, and they are not equivalent to each other.

7.2 Characterisation of discrete groups of isometries

Given a proper metric space X, we equip $\mathrm{Isom}(X)$ with the *compact-open topology*, i.e. the topology generated by the sets

$$\mathcal{O}_{K,U} \coloneqq \{f \in \mathrm{Isom}(X) \,:\, f(K) \subseteq U\}$$

for all compact subsets K and open subsets U of X. This is the topology of uniform convergence on compact subsets.

Remark III.4. If X is a proper metric space, then the compact-open topology on $\mathrm{Isom}(X)$ is metrisable: it is induced by the metric d_{Isom} given by

$$d_{\mathrm{Isom}}(f_1, f_2) \coloneqq \sum_{k=0}^{\infty} \frac{1}{2^k} \sup_{x \in \bar{B}(x,k)} d\left(f_1(x), f_2(x)\right)$$

for $f_1, f_2 \in \mathrm{Isom}(X)$. In particular, the compact-open topology on $\mathrm{Isom}(X)$ is Hausdorff.

Exercise 7.C

Prove that, under the isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ *of Theorem II.11, the quotient topology on* $\mathrm{PSL}(2,\mathbb{R})$ *induced by the standard topology on* $\mathrm{SL}(2,\mathbb{R})$ *coincides with the compact-open topology on* $\mathrm{Isom}(\mathbb{H}^2)$.

Proposition III.5. *Let X be a proper metric space, and let Γ be a subgroup of* $\mathrm{Isom}(X)$. *The following are equivalent:*

(i) *Γ is discrete for the compact-open topology on* $\mathrm{Isom}(X)$.

(ii) *The action $\Gamma \curvearrowright X$ is properly discontinuous.*

Proof. (ii) $\Rightarrow$ (i) Pick a point $x \in X$. By the characterisation (iv) of proper discontinuity in Definition III.2, the orbit Γx is (closed and) discrete and the stabiliser $\mathrm{Stab}_\Gamma(x)$ is finite. Now consider the orbit map $\omega_x : \Gamma \to \Gamma x$ given by $\gamma \mapsto \gamma x$; it is clearly continuous for the compact-open topology. Since the orbit Γx is discrete, the singleton $\{x\}$ is open in Γx, so its preimage under ω_x is open in Γ. But $\omega_x^{-1}(\{x\}) = \mathrm{Stab}_\Gamma(x)$. Therefore, $\mathrm{Stab}_\Gamma(x)$ is a finite open subgroup of Γ. Since singletons are closed (as the topology on Γ is Hausdorff, see Remark III.4), it follows that the trivial subgroup $\{1\}$ is open in Γ, so Γ is discrete.

(i) $\Rightarrow$ (ii) We will prove that the action $\Gamma \curvearrowright X$ satisfies the characterisation (iii) of proper discontinuity in Definition III.2. To do so, let $x \in X$, and let $K \subseteq X$ be a compact subset. Note that

$$\underbrace{\{\gamma \in \Gamma \,:\, \gamma x \in K\}}_{=:\Gamma_K} = \Gamma \cap \underbrace{\{f \in \mathrm{Isom}(X) \,:\, fx \in K\}}_{=:G_K}.$$

We observe that

- Γ is discrete by assumption, hence closed in $\mathrm{Isom}(X)$. Indeed, since $\mathrm{Isom}(X)$ is metrisable (see Remark III.4), it suffices to prove that Γ is sequentially closed. But consider $(f_n)_{n\geq 1}$ a sequence in Γ which converges to some $f_\infty \in \mathrm{Isom}(X)$. Then $f_{n+1}^{-1} f_n \to 1$ in $\mathrm{Isom}(X)$. But $\{1\}$ is an open set, so there is some $n_0 \geq 1$ such that $f_{n+1} = f_n$ for all $n \geq n_0$; thus $f_\infty = f_{n_0} \in \Gamma$.
- G_K is compact; this is an application of the Arzelà–Ascoli Theorem.

Hence, Γ_K is compact; it is discrete since it is contained in Γ; therefore Γ_K is finite. □

7.3 Discrete groups of isometries

Until the end of the course, we let $\mathbb{X}^2 \coloneqq \mathbb{S}^2$ or $\mathbb{E}^2$ or $\mathbb{H}^2$:

- $\mathbb{S}^2$ is the sphere, embedded in $\mathbb{R}^3$ as the unit sphere and equipped with the Riemannian metric induced by that of $\mathbb{R}^3$.
- $\mathbb{E}^2$ is the Euclidean plane.
- $\mathbb{H}^2$ is the hyperbolic plane (realised as the Poincaré half-plane, but this really does not matter.)

Up to rescaling, the above three spaces are the only simply-connected surfaces of constant Gaussian curvature (we do not define Gaussian curvature, but the respective Gaussian curvatures of $\mathbb{S}^2$, $\mathbb{E}^2$, and $\mathbb{H}^2$, are $+1$, 0, and -1).

We will study discrete subgroups of $\mathrm{Isom}^+(\mathbb{X}^2)$, which are exactly the subgroups acting properly discontinuously on $\mathbb{X}^2$ by Proposition III.5.

The hyperbolic case is particularly rich and has its own name.

Definition III.6. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$.

Remark III.7. Consider the projection

$$\mathrm{SL}(2,\mathbb{R}) \xrightarrow{\pi} \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{id}\}$$

Then, by definition of the quotient topology, a subgroup $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ is discrete if and only if $\pi^{-1}(\Gamma)$ is discrete.

7.4 Fundamental domains

A key idea is that we can understand a discrete group of isometries by considering a fundamental domain for it.

Definition III.8. Given X a proper metric space and $\Gamma \leq \mathrm{Isom}(X)$, a *fundamental domain* for Γ in X is a subset $\mathcal{D} \subseteq X$ whose translates *tessellate* X in the following sense:

(i) $X = \bigcup_\gamma \gamma\bar{\mathcal{D}}$, and
(ii) $\gamma\mathring{\mathcal{D}} \cap \mathring{\mathcal{D}} = \varnothing$ for all $\gamma \in \Gamma \smallsetminus \{1\}$.

Example III.9. $[0,1]^2$ is a fundamental domain for the action $\mathbb{Z}^2 \curvearrowright \mathbb{E}^2$ by translations.

Exercise 7.D

Let $f \in \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$. Depending on the type of f (elliptic, parabolic, or hyperbolic), give a fundamental domain for the subgroup $\langle f \rangle$ of $\mathrm{Isom}(\mathbb{H}^2)$. Draw it for a chosen conjugate of f, in the model of your choice.

It turns out that the subgroups of $\mathrm{Isom}^+(\mathbb{X}^2)$ admitting open fundamental domains are exactly discrete groups. First, we prove one direction of this statement.

Proposition III.10. *Let Γ be a subgroup of $\mathrm{Isom}^+(\mathbb{X}^2)$ admitting an open fundamental domain for its action on $\mathbb{X}^2$. Then Γ is discrete.*

Proof. Let $\mathcal{D} \subseteq \mathbb{X}^2$ be a non-empty open fundamental domain for Γ. Take $x \in \mathcal{D}$, and note that:

- The stabiliser $\mathrm{Stab}_\Gamma(x)$ is finite (in fact, trivial): if $\gamma \in \mathrm{Stab}_\Gamma(x)$, then in particular $\gamma\mathcal{D} \cap \mathcal{D} \neq \varnothing$, so $\gamma = 1$ by condition (ii) in Definition III.8.
- The orbit Γx is discrete: we have $\Gamma x \cap \mathcal{D} = \{x\}$, again by condition (ii) in Definition III.8. Since $\mathcal{D}$ is open in $\mathbb{X}^2$, the singleton $\{x\}$ is therefore open in Γx. The same is true for every singleton in Γx, so the orbit Γx is discrete.

Similar to the proof of the implication (ii) $\Rightarrow$ (i) of Proposition III.5, Γ is discrete since there exists $x \in \mathbb{X}^2$ such that Γx is discrete and $\mathrm{Stab}_\Gamma(x)$ is finite. □

Remark III.11. In the proof of Proposition III.10, we have only used condition (ii) of the definition of fundamental domains (III.8).

Theorem III.12. *Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{X}^2)$, and let $x \in \mathbb{X}^2$ be a point with $\mathrm{Stab}_\Gamma(x) = \{1\}$. Consider the set*

$$\mathscr{D}(x) := \left\{ y \in \mathbb{X}^2 \ : \ \forall \gamma \in \Gamma \smallsetminus \{1\},\ d_{\mathbb{X}^2}(y,x) < d_{\mathbb{X}^2}(y,\gamma x) \right\} \subseteq \mathbb{X}^2.$$

Then $\mathscr{D}(x)$ is an open convex fundamental domain for Γ in $\mathbb{X}^2$, called the Dirichlet domain *of Γ at x.*

Remark III.13. The Dirichlet domain $\mathscr{D}(x)$ is

$$\mathscr{D}(x) = \bigcap_{\gamma \in \Gamma \smallsetminus \{1\}} \mathscr{D}_\gamma(x),$$

with $\mathscr{D}_\gamma(x) := \{y \in \mathbb{X}^2 \ : \ d(y,x) < d(y,\gamma x)\}$ for $\gamma \in \Gamma$.

Geometrically, $\mathscr{D}_\gamma(x)$ is a half plane bounded by the perpendicular bissector of the geodesic $[x, \gamma x]$.

Proof of Theorem III.12. Convexity of $\mathscr{D}(x)$ follows from Remark III.13.

We claim that the closure of $\mathscr{D}(x)$ is

$$\bar{\mathscr{D}}(x) = \bigcap_{\gamma \in \Gamma} \left\{ y \in \mathbb{X}^2 \ : \ d(y,x) \leq d(y,\gamma x) \right\} \subseteq \mathbb{X}^2. \qquad (*)$$

It is clear that the right-hand side is a closed subset of $\mathbb{X}^2$ containing $\mathscr{D}(x)$; hence, it does contain $\bar{\mathscr{D}}(x)$. Conversely, let $y \in \bigcap_\gamma \{y \in \mathbb{X}^2 \ : \ d(y,x) \leq d(y,\gamma x)\}$. Then we have $[x,y) \subseteq \bigcap_\gamma \mathscr{D}_\gamma(x) = \mathscr{D}(x)$ (we are using the notations of Remark III.13), and therefore $y \in \bar{\mathscr{D}}(x)$. This proves (∗).

Let us deduce that the translates of $\bar{\mathscr{D}}(x)$ cover $\mathbb{X}^2$ (condition (i) in Definition III.8): given $y \in \mathbb{X}^2$, since the orbit Γx is locally finite, there is an element $\gamma_0 \in \Gamma$ such that

$$d\left(y, \gamma_0 x\right) = \min_{\gamma \in \Gamma} d(y, \gamma x).$$

It now follows from (∗) that $\gamma^{-1}y \in \bar{\mathscr{D}}(x)$; this proves that

$$\mathbb{X}^2 = \bigcup_{\gamma\in\Gamma} \gamma\bar{\mathscr{D}}(x).$$

To prove openness of $\mathscr{D}(x)$, we define, for $R > 0$,

$$\begin{aligned}\Gamma_R &\coloneqq \{\gamma \in \Gamma \,:\, d(x,\gamma x) \le 2R\}\\ &\subseteq \left\{\gamma \in \Gamma \,:\, \bar{B}(x,R) \cap \gamma\bar{B}(x,R) \neq \varnothing\right\}.\end{aligned}$$

By discreteness of Γ, the action $\Gamma \curvearrowright \mathbb{X}^2$ is properly discontinuous (Proposition III.5), so the latter set, and hence Γ_R, are finite. We claim that

$$\mathscr{D}(x) \cap B(x,R) = B(x,R) \cap \bigcap_{\gamma\in\Gamma_R\smallsetminus\{1\}} \mathscr{D}_\gamma(x). \tag{†}$$

The inclusion ($\subseteq$) is clear; conversely, let $y \in B(x,R)$ and suppose that $y \in \mathscr{D}_\gamma(x)$ for all $\gamma \in \Gamma_R \smallsetminus \{1\}$. We want to show that $y \in \mathscr{D}_\gamma(x)$ for all $\gamma \in \Gamma \smallsetminus \{1\}$. But given $\gamma \in \Gamma \smallsetminus \Gamma_R$, we have

$$d(y,\gamma x) \ge \underbrace{d(x,\gamma x)}_{>2R} - \underbrace{d(x,y)}_{<R} > R > d(x,y),$$

i.e. $y \in \mathscr{D}_\gamma(x)$. This proves (†). It follows that $\mathscr{D}(x) \cap B(x,R)$ is open as a finite intersection of open sets; therefore $\mathscr{D}(x) = \bigcup_R (\mathscr{D}(x) \cap B(x,R))$ is also open.

Finally, the translates of $\mathring{\mathscr{D}}(x) = \mathscr{D}(x)$ are disjoint (condition (ii) in Definition III.8). Indeed, for $\gamma \in \Gamma \smallsetminus \{1\}$, we have

$$\mathscr{D}(x) \cap \gamma\mathscr{D}(x) = \mathscr{D}(x) \cap \mathscr{D}(\gamma x) = \varnothing. \qquad \square$$

Exercise 7.E

Prove that the Dirichlet domain $\mathscr{D}(x)$ of a discrete group Γ at at point $x \in \mathbb{X}^2$ is a locally finite fundamental domain, *in the sense that, for every compact set $K \subseteq \mathbb{X}^2$, the set $\{\gamma \in \Gamma \,:\, \gamma\mathscr{D}(x) \cap K \neq \varnothing\}$ is finite.*

Exercise 7.F

If Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{X}^2)$, prove that there is a point $x \in \mathbb{X}^2$ such that $\mathrm{Stab}_\Gamma(x) = \{1\}$.

Exercise 7.G

Let Γ be a subgroup of $\mathrm{Isom}(\mathbb{X}^2)$ admitting a fundamental domain $\mathcal{D}$ which is locally finite (see Exercise 7.E). Prove that Γ is generated by

$$\left\{\gamma \in \Gamma \,:\, \bar{\mathcal{D}} \cap \gamma\bar{\mathcal{D}} \neq \varnothing\right\}.$$

Corollary III.14. *A subgroup* $\Gamma \leq \mathrm{Isom}^+(\mathbb{X}^2)$ *is discrete if and only if it admits an open (convex) fundamental domain for its action on* $\mathbb{X}^2$.

Proof. Proposition III.10 gives the implication ($\Leftarrow$). Conversely, let Γ be discrete; by Exercise 7.F, there is a point $x \in \mathbb{X}^2$ with $\mathrm{Stab}_\Gamma(x) = \{1\}$; now Theorem III.12 gives us an open convex fundamental domain for Γ. □

Exercise 7.H

(i) *Show that* $\mathrm{PSL}(2,\mathbb{Z})$ *is a Fuchsian group.*
(ii) *Show that* $\mathrm{PSL}(2,\mathbb{Z})$ *is generated by the elements*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(iii) *Determine the Dirichlet domain of* $\mathrm{PSL}(2,\mathbb{Z})$ *at* $2i$.

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