# Hyperbolic Geometry
## — Problem sheet. Hyperbolic space in higher dimensions

In this problem sheet, you will develop the basics of $n$-dimensional hyperbolic geometry, including the Riemannian structure, geodesics, isometries and their classification, and discrete groups. Only the beginning will be general, and then we will focus on dimension 3, which will be of most interest to us.

### The upper half-space in all dimensions

The *one-point compactification* of $\mathbb{R}^n$ is the topological space $\hat{\mathbb{R}}^n$ obtained from $\mathbb{R}^n$ by adding one point $\infty$, whose open neighbourhoods are all sets of the form $\{\infty\} \cup (\mathbb{R}^n \smallsetminus K)$ with $K \subseteq \mathbb{R}^n$ compact.

**Exercise $\mathcal{H}$.A**

*Prove that $\hat{\mathbb{R}}^n$ is homeomorphic to the $n$-sphere $S^n$.*
*Hint: Think about the stereographic projection.*

**Definition $\mathcal{H}$.1.** The *upper half-space model* for hyperbolic $n$-space is the Riemannian manifold

$$\mathbb{H}^n \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_n > 0\} \subseteq \mathbb{R}^n,$$

with the Riemannian metric

$$\frac{1}{x_n^2} \sum_{i=1}^{n} \mathrm{d}x_i^2.$$

The *boundary* of $\mathbb{H}^n$ is $\partial\mathbb{H}^n \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_n = 0\} \cup \{\infty\} \subseteq \hat{\mathbb{R}}^n$; we also set $\bar{\mathbb{H}}^n \coloneqq \mathbb{H}^n \cup \partial\mathbb{H}^n \subseteq \hat{\mathbb{R}}^n$.

Let $\mathcal{S}$ be a generalised sphere (i.e. a sphere or a hyperplane) in $\hat{\mathbb{R}}^n$ orthogonal to $\partial\mathbb{H}^n$; in other words, $\mathcal{S}$ is either

- A sphere with centre on the hyperplane $\{x_n = 0\}$, or
- An affine hyperplane containing an affine line of direction $(0, \dots, 0, 1)$.

Denote by $\iota_{\mathcal{S}} : \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ the inversion in $\mathcal{S}$.

**Exercise $\mathcal{H}$.B**

(i) *Prove that $\iota_{\mathcal{S}}$ stabilises $\mathbb{H}^n$ and induces a Riemannian isometry $\mathbb{H}^n \to \mathbb{H}^n$.*
(ii) *Deduce that there is an embedding* $\mathrm{Möb}(n-1) \hookrightarrow \mathrm{Isom}(\mathbb{H}^n)$.
(iii) *What is the image of this embedding when $n = 2$?*

Deadline: **Friday 27 November 2026**.
Hand in your work in class, or send it by email to amarchand@impan.pl. Questions, corrections, and comments are very helpful; please send them to the same address or speak to me in class.

## Geodesics and geodesically convex subspaces

From now on, we assume that $n \geq 3$. Our next goal is to understand shortest paths and geodesics of $\mathbb{H}^n$.

**Exercise $\mathcal{H}$.C**

(i) *Prove that all vertical rays* $(p_0 + \mathbb{R}_{>0}(0, \dots, 0, 1))$ *are shortest paths of* $\mathbb{H}^n$.
(ii) *Using the action* $\mathrm{Möb}(n-1) \curvearrowright \mathbb{H}^n$, *describe all shortest paths of* $\mathbb{H}^n$.
(iii) *Show that* $\mathrm{inj}(\mathbb{H}^n) = \infty$ *and that shortest paths coincide with geodesics.*

Let $\mathcal{P}$ be the intersection with $\mathbb{H}^n$ of a generalised 2-sphere in $\hat{\mathbb{R}}^n$ orthogonal to $\partial\mathbb{H}^n$ (i.e. a 2-sphere with centre on $\{x_n = 0\}$ or a 2-dimensional affine subspace of $\mathbb{R}^n$ containing an affine line of direction $(0, \dots, 0, 1)$).

**Exercise $\mathcal{H}$.D**

(i) *Prove that there is a Riemannian isometric embedding* $\mathbb{H}^2 \hookrightarrow \mathbb{H}^n$ *with image* $\mathcal{P}$.
(ii) *Prove that* $\mathcal{P}$ *is* geodesically convex *in the sense that every geodesic of* $\mathbb{H}^n$ *with endpoints in* $\mathcal{P}$ *is entirely contained in* $\mathcal{P}$.

## Isometries of $\mathbb{H}^3$

We now focus on dimension $n = 3$, which will be the case that interests us the most throughout the rest of the course. We have found an embedding $\mathrm{Möb}(2) \hookrightarrow \mathrm{Isom}(\mathbb{H}^3)$; we will now show that it is surjective. Recall that $\mathrm{PGL}(2, \mathbb{C})$ is an index-2 subgroup of $\mathrm{Möb}(\mathbb{C}) = \mathrm{Möb}(2)$ via its action on $\hat{\mathbb{C}}$ by homographies.

We first extend the action $\mathrm{Isom}(\mathbb{H}^3)$ to $\bar{\mathbb{H}}^3$. To do so, let $\xi \in \partial\mathbb{H}^3$. Pick a geodesic ray $\gamma : [0, \infty) \to \mathbb{H}^3$ with $\gamma(t) \to \xi$ as $t \to \infty$. Given $f \in \mathrm{Isom}(\mathbb{H}^3)$, the composition $f\gamma$ is a geodesic ray; we set

$$f(\xi) \coloneqq \lim_{t \to \infty} f\gamma(t) \in \partial\mathbb{H}^3.$$

We admit that this gives an action $\mathrm{Isom}(\mathbb{H}^3) \curvearrowright \bar{\mathbb{H}}^3$ by homeomorphisms.

**Exercise $\mathcal{H}$.E**

(i) *Prove that the action* $\mathrm{Isom}^+(\mathbb{H}^3) \curvearrowright \partial\mathbb{H}^3$ *is simply transitive on triples of distinct points, in the following sense: given two triples of distinct points* $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ *and* $\underline{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ *in* $\partial\mathbb{H}^3$, *there is a unique* $f \in \mathrm{Isom}^+(\mathbb{H}^3)$ *such that* $f(\underline{\xi}) = \underline{\zeta}$.
*Hint: Three distinct points in* $\partial\mathbb{H}^3$ *span a geodesically convex copy of* $\mathbb{H}^2$ *in* $\mathbb{H}^3$.
(ii) *Prove that, under the morphism* $\mathrm{Möb}(2) \hookrightarrow \mathrm{Isom}(\mathbb{H}^3)$, *the image of* $\mathrm{PGL}(2, \mathbb{C})$ *is equal to* $\mathrm{Isom}^+(\mathbb{H}^3)$.
(iii) *Deduce that the morphism* $\mathrm{Möb}(2) \hookrightarrow \mathrm{Isom}(\mathbb{H}^3)$ *is surjective.*

The following summarises what we have proved so far.

**Theorem $\mathcal{H}$.2.** *The action* $\mathrm{Möb}(2) \curvearrowright \mathbb{H}^3$ *described in Exercise* $\mathcal{H}$*.B induces isomorphisms*

$$\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{Möb}(2) \qquad \textit{and} \qquad \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{Möb}^+(2) \cong \mathrm{PGL}(2,\mathbb{C}). \qquad \square$$

### Dynamical classification of isometries

Analogously to what we did in $\mathbb{H}^2$, we now work towards classifying isometries of $\mathbb{H}^3$.

Given $f \in \mathrm{Isom}^+(\mathbb{H}^3)$, an *axis* for $f$ is a geodesic $\gamma : \mathbb{R} \to \mathbb{H}^3$ for which there exists $\tau \in \mathbb{R}$ such that

$$\forall t \in \mathbb{R},\ f\gamma(t) = \gamma(t+\tau).$$

In other words, $f$ preserves the image of $\gamma$ and acts on it by translation.

**Exercise $\mathcal{H}$.F**

*Let $f \in \mathrm{Isom}^+(\mathbb{H}^3) \smallsetminus \{\mathrm{id}\}$. Suppose that $\gamma, \gamma'$ are two axes for $f$.*

(i) *If $f$ fixes both $\gamma$ and $\gamma'$ pointwise, show that $\gamma = \gamma'$.*

(ii) *If $f$ translates $\gamma$ by a non-zero constant $\tau$, show that $\gamma$ and $\gamma'$ remain at bounded distance from each other. Deduce that $\gamma = \gamma'$.*

(iii) *If we had defined an axis to be a geodesic preserved (setwise) by $f$, show that there would have been examples of non-trivial orientation-preserving isometries with several distinct axes.*

**Exercise $\mathcal{H}$.G**

(i) *For $f \in \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PGL}(2,\mathbb{C})$, prove that exactly one of the following assertions holds:*

  (a) *$f$ has an axis that is fixed pointwise; we say that $f$ is* elliptic.

  (b) *$f$ has an axis that is translated by a non-zero constant $\tau$; we say that $f$ is* hyperbolic *or* loxodromic.

  (c) *$f$ does not have any axis; we say that $f$ is* parabolic.

  *Find a quantity from which (analogously to $|\operatorname{tr}|$ in dimension 2) one can read the type of $f$.*

(ii) *For each type of isometry, describe the fixed-point sets in $\mathbb{H}^3$ and in $\partial\mathbb{H}^3$, draw a picture of a well-chosen conjugate of $f$ and depict the orbits of points of $\mathbb{H}^3$ under $\langle f \rangle$ (depending on their position).*

*Remark* $\mathcal{H}$.3. The result of $\mathcal{H}$.G(i) holds verbatim for orientation-preserving isometries of $\mathbb{H}^n$, but requires a different proof because we do not have access to $\mathrm{PGL}(2,\mathbb{C})$ anymore. For a general $n \geq 1$, there is an isomorphism $\mathrm{Isom}^+(\mathbb{H}^n) \cong \mathrm{SO}^+(n,1)$ (via the hyperboloid model, see Exercise 3.B for the $n = 2$ case), and different types of isometries correspond to different Jordan forms for elements of $\mathrm{SO}^+(n,1)$.

We summarise the classification of isometries in $\mathbb{H}^n$ below for future use.

**Theorem $\mathcal{H}$.4.** *Let $f \in \mathrm{Isom}^+(\mathbb{H}^n)$. Then exactly one of the following assertions holds:*

(i) *$f$ has an axis that is fixed pointwise; we say that $f$ is* elliptic.
(ii) *$f$ has an axis that is translated by a non-zero constant $\tau$; we say that $f$ is* hyperbolic *or* loxodromic.
(iii) *$f$ does not have any axis; we say that $f$ is* parabolic.

Given $f \in \mathrm{Isom}(\mathbb{H}^3)$, the *translation distance* of $f$ is the function

$$\begin{aligned} d_f : \mathbb{H}^3 &\to \mathbb{H}^3 \\ x &\mapsto d(x, f(x)). \end{aligned}$$

The *translation length* of $f$ is

$$\tau_f \coloneqq \inf_{x \in \mathbb{H}^3} d_f(x).$$

**Exercise $\mathcal{H}$.H**

*Let $f \in \mathrm{Isom}^+(\mathbb{H}^3)$. Compute $\tau_f$ and describe the set $\{x \in \mathbb{H}^3 : d_f(x) = \tau_f\}$, depending on the type (elliptic, hyperbolic, parabolic) of $f$.*

## Poincaré's Theorem and examples of Kleinian groups

We now turn to discrete groups of isometries. As in dimension 2, $\mathrm{Isom}^+(\mathbb{H}^3)$ is equipped with the compact-open topology, which coincides with the usual topology on $\mathrm{PGL}(2, \mathbb{C})$.

**Definition $\mathcal{H}$.5.** A *Kleinian group* is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PGL}(2, \mathbb{C})$.

As in the 2-dimensional case, many examples of Kleinian groups can be constructed using Poincaré's Theorem on polygons with side-pairings. We now have to replace polygons with polyhedra and sides with faces. We focus on the case with reflections for simplicity; the groups that we construct are not Kleinian because they contain orientation-reversing elements, but they have index-2 subgroups which are Kleinian.

**Exercise $\mathcal{H}$.I**

(i) *Write a statement analogous to Poincaré's Theorem with reflections (III.28) for $\mathbb{H}^3$. We admit that this statement holds.*
(ii) *Draw an octahedron in $\mathbb{H}^3$ with all vertices ideal, and all dihedral angles equal to $\pi/2$. What is the Coxeter group generated by reflections in its faces?*
(iii) *Describe all the Coxeter groups that can be obtained from Poincaré's Theorem with reflections applied to tetrahedra with all vertices ideal.*